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FINITENESS PROPERTY OF DEFORMED REVOLUTION SURFACES IN E^3 (PART I)

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ABSTRACT. The main goal of this paper, is to study the finiteness property of the mean curvature flow for the revolution surfaces in E^3. Also, general example for such property is presented.

1. INTRODUCTION

Algebraic geometry studies varieties which are defined locally as the common zero sets of polynomials. Also, one can define the degree of an algebraic variety by its algebraic structure, where the concept of degree plays a fundamental role. On the other hand, according to Nash's embedding theorem, every Riemannian manifold can be realized as a Riemannian submanifold in some Euclidean space with sufficiently higher codimension. However, one lacks the notion of the degree for Riemannian submanifolds in Euclidean spaces [1].

Inspired by the above simple observation, Bang-Yen Chen introduced in the late 1970's the notions of "order" and "type" for submanifolds of Euclidean spaces and used them to introduce the notion of finite type submanifolds. Just like minimal submanifolds, submanifolds of finite type can be characterized by a spectral variational principle; namely, as critical points of directional deformations [2].

On one hand, the notion of finite type submanifolds provides a very natural way to apply spectral geometry to study submanifolds. On the other hand, one can also apply the theory of finite type submanifolds to investigate the spectral geometry of submanifolds. The first results on submanifolds of finite type were collected in [3, 4]. A list of twelve open problems and three conjectures on submanifolds of finite type was published in [5]. Furthermore, a detailed report of the progress on this theory was presented in [6]. Recently, [7] studied frenet surfaces with pointwise $1-$type Gauss map. Also, the study of finite type submanifolds, in particular, of biharmonic submanifolds, have received a growing attention with many progresses since the beginning of this century. In [1], is provided a detailed account of recent development on the problems and conjectures listed in [5].

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One of the most interesting and profound aspects of classical differential geometry is its interplay with the calculus of variations. The calculus of variations have their roots in the very origins of subject, such as, for instance, in the theory of minimal surfaces. More recently, the variational principles which give rise to the field equations of the general theory of relativity have suggested the systematic investigation of a seemingly new type of variational problem. In the case of the earlier applications one is, at least implicitly, concerned with a multiple integral in the calculus of variations. In additional, the normal variational problem on general surfaces and hyperruled surfaces were studied by some geometers, specifically one may cite [8]-[19].

The mean curvature flow has many physical problems in the nature, starting from the well-known Poisson-Laplace theorem which relates, the pressure and the mean curvature flow of a surface immersed in a liquid until the capillary theory [20].

The main aim in this paper is to study the effectiveness of the normal variation in deferent directions of revolution surfaces in Euclidean $3-$space E^3 for finiteness property. This aim determine whether the property of finiteness for surfaces in E^3 remains the same or not. And we find that the deformation depends on the ϕ function where we deal with some revolution surfaces. Finally, we prove that the variation of surfaces preserves the property of finiteness for some surfaces and does not preserve that property for other surfaces.

2. Basic concepts

In this section, we review some basic definitions and relations. Let a surface $M:\mathbf{X}=\mathbf{X}(u,v)$ in an Euclidean $3-$space E^3. The map $\mathbf{G}:M\rightarrow S^2(1)\subset E^3$ which sends each point of M to the unit normal vector to M at the point is called the Gauss map of a surface M; where $S^2(1)$ denotes the unit sphere of E^3. The standard unit normal vector field $\mathbf{G}$ on the surface M can be defined by:

$$\mathbf{G}=\frac{\mathbf{X}_u\times\mathbf{X}_v}{\mid\mathbf{X}_u\times\mathbf{X}_v\mid}, \tag{1}$$

where $\mathbf{X}_u$ and $\mathbf{X}_v$ are the first partial derivatives with respect to the parameters of $\mathbf{X}$.

Definition 2.1. [21, 22] *Let M be an $n-$dimensional surface. Then the Laplacian Δ operator (or Laplacian-Beitrami operator) associated with the induced metric on M is a mapping which sends any differentiable function f to the function Δf of the form*

$$\Delta=-\frac{1}{\sqrt{g}}\sum_{i,j}\frac{\partial}{\partial x_i}(\sqrt{g}\,g^{ij}\frac{\partial}{\partial x_j}). \tag{2}$$

where x_i is the local coordinate on M, (g_{ij}) is the matrix of the Riemannian metric g on M where $(g^{ij})=(g_{ij})^{-1}$ and $g=\det(g_{ij})$.

The mean curvature H of the surface is defined by

$$H=\frac{1}{2}\sum_{i,j=1}^{2}g^{ij}L_{ij}, \tag{3}$$

where L_{ij} are the coefficients of the second fundamental form.

An isometric immersion $\mathbf{X} : M \to E^3$ of a submanifold M in E^3 is said to be of finite type if $\mathbf{X}$ identified with the position vector field of M in E^3 can be expressed as a finite sum of eigen vectors of the Laplacian Δ of M, that is,

$$\mathbf{X} = X_0 + \sum_{i=1}^{j} X_i, \tag{4}$$

where X_0 is a constant map and $X_1, X_2, \cdots, X_j$ non-constant maps such that

$$\Delta X_i = \lambda_i X_i, \quad \lambda_i \in R, \quad 1 \leq i \leq j. \tag{5}$$

If $\lambda_1, \lambda_2, \ldots, \lambda_j$ are different eigen values, then M is said to be of $j-$type. If in particular, one of λ_i is zero then M is said to be of null $j-$type. If all coordinate function of E^3, restricted to M, are of finite type, then M is said to be of finite type. Otherwise, M is said to be of infinite type. Similarly, a smooth map ϕ of an $2-$dimensional Riemannian manifold M of E^3 is said to be of finite type if ϕ is a finite sum of E^3-valued eigen functions of Δ [3, 4].
Let M be a connected surface in E^3. Then the position vector $\mathbf{X}$ and the mean curvature vector $\mathbf{H}$ of M in E^3 satisfy [4]

$$\Delta \mathbf{X} = -2\,\mathbf{H}, \tag{6}$$

where $\mathbf{H} = H\,\mathbf{G}$. This formula yields the following well-known result: A surface M in E^3 is minimal if and only if all coordinate functions of E^3, restricted to M, are harmonic functions, that is,

$$\Delta \mathbf{X} = 0. \tag{7}$$

We recall theorem of T.Takahashi [23] and [6] which states that a submanifold M of a Euclidean space is of $1-$type, i.e., the position vector field of the submanifold in the Euclidean space satisfies the differential equation

$$\Delta \mathbf{X} = \lambda \mathbf{X}, \tag{8}$$

for some real number λ, if and only if either the submanifold is a minimal submanifold of the Euclidean space $(\lambda = 0)$ or it is a minimal submanifold of a hypersphere of the Euclidean space centered at the origin $(\lambda \neq 0)$.

We will mention the following known result for later use.

Proposition 2.1. [1, 4, 24, 25] *Let M be a $j-$type $(j = 1, 2, \ldots)$ surfaces whose spectral decomposition is given by Eq.* (4)*. If we put* [24]

$$P(T) = \prod_{i=1}^{j} (T - \lambda_i), \tag{9}$$

then

$$P(\Delta)\,(\mathbf{X} - \mathbf{X}_0) = 0. \tag{10}$$

We can rewrite the previous equation as follows

$$\Delta^{j+1}\mathbf{X} + d_1 \Delta^j \mathbf{X} + \cdots + d_j \Delta \mathbf{X} = 0, \tag{11}$$

where $d_1, d_2, \ldots, d_j$ are constants for some $j \geq 1$.
And the monic polynomial P is called the minimal polynomial which plays a very important role to find out whether or nor a surface is of finite type.

Let M be a connected revolution surface which is generated by a plane curve $\alpha(u)$ when it is rotated around a straight line in the same plane. Let the plane be xz and the line be $z-$axis. Then, the parametrization of the plane curve takes the following form [26]:

$$\alpha(u) = \{f(u), h(u)\}. \tag{12}$$

Hence the parametrization of M is usually given by [27]

$$\mathbf{X}(u,v) = \{f(u) \cos v, \ f(u) \sin v, h(u)\}. \tag{13}$$

Definition 2.2. [8, 9, 14] *Let $\mathbf{X} : U \to R^{n+1}$ be a parameterized $n-$surface in R^{n+1}. A variation of $\mathbf{X}$ is a smooth map $\overline{\mathbf{X}} : U \times [0, 1] \to R^{n+1}$ with the property that $\overline{\mathbf{X}}(u^i, 0) = \mathbf{X}(u^i)$ for all $u^i \in U$. Thus a variation surrounds the $n-$surface $\mathbf{X}$ with a family of singular $n-$surface $\overline{\mathbf{X}}_t : U \to R^{n+1}$ defined by*

$$\overline{M} : \overline{\mathbf{X}}_t(u^i) = \overline{\mathbf{X}}(u^i, t) = \mathbf{X}(u^i) + t\,\phi(u^i)\,\mathbf{G}(u^i), \quad i = 1, 2, \ u^i = (u, v). \tag{14}$$

where ϕ is a smooth function along $\mathbf{X}$ and $\mathbf{G}$ is the Gauss map of $\mathbf{X}$, is called a normal variation of $\mathbf{X}$, where t is a parameter where $t \in [0, 1]$. The family of revolution surfaces represented by $\overline{\mathbf{X}}(u^i, t)$ is called a deformable revolution surfaces resulting from $\mathbf{X}(u^i)$ by the normal variation.

3. Finiteness property of the mean curvature flow

In the following, we deal with two cases of revolution surfaces which have worked under the effect of normal variation where mean curvature flow is a term that is used to describe the variation of this surfaces whose function ϕ is given by the mean curvature [20]. Thus, in view of Eq. (14) one can see that $\frac{\partial \overline{\mathbf{X}}}{\partial t} = H\,\mathbf{G}$. Then the finiteness property is studied before and after the deformation and it is noticed that the finiteness property is not affected by the deformation.

Case 4.1. If we put $f(u) = a\,u$ and $h(u) = b\,u$. Then the parametrization of revolution surface in Eq. (13) takes the form

$$\mathbf{X}(u,v) = \{a\,u \cos v, a\,u \sin v, b\,u\}, \tag{15}$$

which is represented a revolution cone and a, b are constants. Hence the unite normal vector field of M is given by

$$\mathbf{G} = \frac{-1}{\sqrt{c_1}}\{b \cos v, b \sin v, -a\}, \tag{16}$$

where $c_1 = a^2 + b^2$.
The metric (g_{ij}) and the contravariant metric (g^{ij}) can be written as

$$(g_{ij}) = diag\,(c_1, a^2 u^2), \quad (g^{ij}) = diag\left(\frac{1}{c_1}, \frac{1}{a^2 u^2}\right), \quad g = c_1 a^2 u^2. \tag{17}$$

Hence, the Laplacian Δ of M can be given as follows

$$\Delta = \frac{-1}{c_1 a^2 u^2}\left(a^2 u \frac{\partial}{\partial u} + a^2 u^2 \frac{\partial^2}{\partial u^2} + c_1 \frac{\partial^2}{\partial v^2}\right). \tag{18}$$

Then, the mean curvature function is given by

$$H = \frac{b}{2\,a\sqrt{c_1}\,u}. \tag{19}$$

Let X_1, X_2, and X_3 be the three components functions of $\mathbf{X}$. Then, we take X_3, where

$$X_3 = b\,u. \tag{20}$$

Hence,

$$\Delta X_3 = -\frac{b}{c_1\,u}. \tag{21}$$

Therefore,

$$\Delta^2 X_3 = \frac{b}{c_1{}^2\,u^3}, \tag{22}$$

and,

$$\Delta^3 X_3 = -\frac{9\,b}{c_1{}^3\,u^5}. \tag{23}$$

Using mathematical induction, we get that

$$\Delta^j X_3 = (-1)^j\,\frac{1^2\,3^2\,5^2\ldots(2j-3)^2\,b}{c_1{}^j\,u^{2j-1}} \quad j \geq 1. \tag{24}$$

Substituting in the decomposition (11). Then one get

$$b\left((-1)^{j+1}\,\frac{1^2\,3^2\,5^2\ldots(2j-1)^2}{c_1{}^{j+1}\,u^{2j+1}} + d_1\,(-1)^j\,\frac{1^2\,3^2\,5^2\ldots(2j-3)^2}{c_1{}^j\,u^{2j-1}} + \cdots - d_j\,\frac{1}{c_1\,u}\right) = 0. \tag{25}$$

But $b \neq 0$, and inside the brackets dose not equal zero. Then the cone surface is infinite type. See [26].

Now, we research what happen for finiteness property after deformation. Let $\overline{M}$ be the surface after variation by mean curvature, i.e., $\phi = H$ in the parametrization (14). Then it can be parameterized by

$$\overline{\mathbf{X}}(u,v,t) = \frac{1}{\delta}\{(a\,\delta\,u - b^2\,t)\cos v\,,\,(a\,\delta\,u - b^2\,t)\,\sin v\,,\,(\delta\,u + a\,t)\,b\}, \tag{26}$$

where, $\delta = 2\,a\,c_1\,u \neq 0$. Then, the unite normal vector field of $\overline{M}$ is given by

$$\overline{\mathbf{G}} = \frac{1}{2\,a\,u\,\sqrt{\delta_1}}\{b\,(t - 2\,a^2\,u^2)\cos v\,, b\,(t - 2\,a^2\,u^2)\sin v\,,\,2\,a^3\,u^2\}, \tag{27}$$

where $\delta_1 = c_1\,a^2\,u^2 - b^2\,t$. Therefore, we have

$$(\overline{g}_{ij}) = diag\left(c_1\,,\,\frac{\delta_1}{c_1}\right), \quad (\overline{g}^{ij}) = diag\left(\frac{1}{c_1}\,,\,\frac{c_1}{\delta_1}\right), \quad \overline{g} = \delta_1. \tag{28}$$

Direct computations, we can find the Laplacian $\overline{\Delta}$ of the deformation $\overline{X}$ as the following

$$\overline{\Delta} = -\frac{1}{c_1\,\delta_1}\left(c_1\,a^2\,u\,\frac{\partial}{\partial u} - \delta_1\,\frac{\partial^2}{\partial u^2} + c_1{}^2\,\frac{\partial^2}{\partial v^2}\right). \tag{29}$$

Let $\overline{X}_1$, $\overline{X}_2$, and $\overline{X}_3$ be the three components functions of $\overline{\mathbf{X}}$. Then, we take

$$\overline{X}_3 = \frac{b}{\delta}\,(\delta\,u + a\,t). \tag{30}$$

Hence,

$$\overline{\Delta}\,\overline{X}_3 = -\frac{a^2\,b}{2\,c_1\,u\,\delta_1}\,(2\,c_1\,u^2 + t). \tag{31}$$

Therefore,

$$\overline{\Delta}^2\,\overline{X}_3 = \frac{a^4\,b\,u}{2\,\delta_1{}^3}\left(2\,c_1\,a^2\,u^2 + (9\,a^2 + 10\,b^2)\,t\right), \tag{32}$$

and

$$\overline{\Delta}^3 \overline{X}_3 = -\frac{a^8\, b\, c_1\, u^3}{2\, \delta_1{}^5} \left(18\, c_1\, a^2\, u^2 + (225\, a^2 + 304\, b^2)\, t\right). \tag{33}$$

The following lemma can be proved by mathematical induction. Here and in the sequel, for convenient, replace deg instead of degree.

Lemma 3.1. *If* $\overline{\eta}$ *is a polynomial in* u, t *and* deg $\overline{\eta} = r$*. Then, we get*

$$\overline{\Delta} \left(\frac{\overline{\eta}\,(u\,,t)}{\delta_1^q} \right) = \frac{\widehat{\overline{\eta}}\,(u\,,t)}{2\, \delta_1^{q+2}}, \tag{34}$$

where $\widehat{\overline{\eta}}$ *is a polynomial in* u, t.

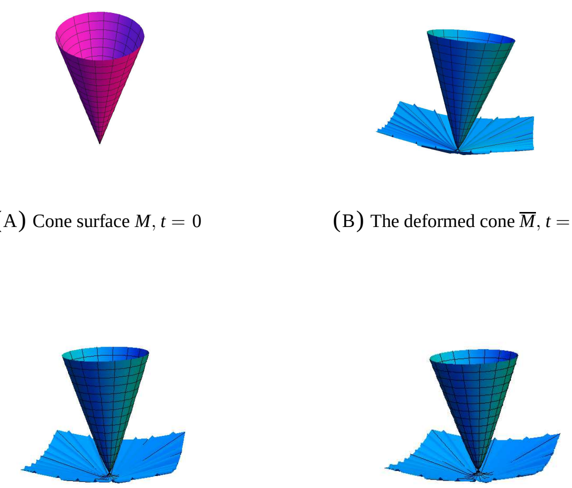
Applying the above Lemma, we get

$$\overline{\Delta}^j\, \overline{X}_3 = \frac{\overline{\eta}_j\,(u\,,t)}{2\, \delta_1^{2j-1}}. \tag{35}$$

Then if j goes up by one, the degree of the numerator of $\overline{\Delta}^j\, \overline{X}_3$ goes up by at most 2, while the degree of the denominator goes up by 4. Hence the decomposition (11) can never be zero. Therefore, $\overline{M}$ is infinite type. And this result agree with the results in paper [24]. From the above result we can easily deduce the following consequence:

Corollary 3.1. *The mean curvature flow of the deformed cone preserves the property of infiniteness.*

See Figure 1.



(A) Cone surface M, $t = 0$

(B) The deformed cone $\overline{M}$, $t = 0.3$

(C) The deformed cone $\overline{M}$, $t = 0.7$

(D) The deformed cone $\overline{M}$, $t = 0.9$

FIGURE 1. The deformed cone : $u \in [0, 2\pi]$, $v \in [-\pi, \pi]$, $a = 1$, $b = 3$
$f(u) = a\,u$, $h(u) = b\,u$, $\phi = H$

Also, we obtain the following result:

Corollary 3.2. *The effect of the mean curvature flow of the deformed cone is very weak* $\forall t > 0$.

Case 4.2. If we put $f(u) = a\cos u$ and $h(u) = a \sin u$ where a is a constant. Then the parametrization of revolution surface in Eq. (13) gives sphere as the following

$$\mathbf{X}(u,v) = \{a \cos u \cos v \,, a \cos u \sin v \,, a \sin u\}. \tag{36}$$

The unite normal vector field of M is

$$\mathbf{G} = -\{\cos u \cos v \,, \cos u \sin v \,, \sin u\}. \tag{37}$$

Therefore, one can obtain

$$(g_{ij}) = a^2\, diag\left(1 \,, \cos^2 u\right), \quad (g^{ij}) = \frac{1}{a^2}\, diag\left(1 \,, \sec^2 u\right), \quad g = a^4 \cos^2 u. \tag{38}$$

Thus, we use Eq. (2) to get the Laplacian Δ of M as the following

$$\Delta = -\frac{\sec u}{a^2}(\cos u\, \frac{\partial^2}{\partial u^2} - \sin u\, \frac{\partial}{\partial u} + \sec u\, \frac{\partial^2}{\partial v^2}). \tag{39}$$

Then, the mean curvature function is given by $H = \frac{1}{a}$. Therefore,

$$\Delta\mathbf{X} = \frac{2}{a^2}\,\mathbf{X}. \tag{40}$$

As we know the sphere is $1-$type [26]. We ask what happens to the finiteness property after variation of sphere by mean curvature flow.

Let $\overline{M}$ be the surface after deformation where $\phi = H$ in Eq. (14). Therefore $\overline{M}$ is parameterized as follows

$$\overline{\mathbf{X}}(u,v,t) = \frac{(a^2 - t)}{a}\,\{\cos u \cos v \,, \cos u \sin v \,, \sin u\}. \tag{41}$$

The unit normal vector field is given as

$$\overline{\mathbf{G}} = \mathbf{G}. \tag{42}$$

Then, we have

$$\begin{aligned}
&(\overline{g}_{ij}) = (a^2 - 2t)\, diag\left(1 \,, \cos^2 u\right), \quad \overline{g} = a^2\, \xi_2 \cos^2 u, \\
&(\overline{g}^{ij}) = \frac{(a^2 - 2\,t)}{a^2\, \xi_2}\, diag\left(1 \,, \sec^2 u\right), \quad \xi_2 = a^2 - 4\,t.
\end{aligned} \tag{43}$$

Thus, we can get the Laplacian $\overline{\Delta}$ of $\overline{M}$ as follows

$$\overline{\Delta} = -\frac{(a^2 - 2t)}{a^2\, \xi_2}\left(\frac{\partial^2}{\partial u^2} - \tan u\, \frac{\partial}{\partial u} + \sec^2 u\, \frac{\partial^2}{\partial v^2}\right). \tag{44}$$

The mean curvature function of $\overline{\mathbf{X}}$ is given by

$$\overline{H} = \frac{a^2 - 3\,t}{a\, \xi_2}. \tag{45}$$

Solving the following equation for $\overline{\lambda}$.

$$\Delta\overline{\mathbf{X}} - \overline{\lambda}\,\overline{\mathbf{X}} = 0. \tag{46}$$

We get

$$\overline{\lambda} = \frac{2\,(a^2 - 3\,t)}{a^2\,(a^2 - 5\,t)}. \tag{47}$$

Then, we conclude that the sphere is $1-$type after variation $\forall t$.

Corollary 3.3. *If we put* $t = 0$, *in* (47) *we have* $\overline{\lambda} = \frac{2}{a^2}$, *which gives the same result of paper* [26], *for original sphere.*

Corollary 3.4. *The mean curvature flow of the deformed sphere preserves the property of finiteness.*

See Figure 2.

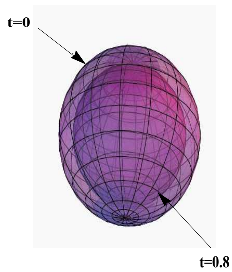


FIGURE 2. The deformed sphere in the original sphere: $u \in [-2, 2]$, $v \in [0, 2\pi]$
$f(u) = a\cos u$, $h(u) = a\sin u$, $a = 2$, $\phi = H$

Corollary 3.5. *The effect of the mean curvature flow of the deformed sphere is very weak* $\forall t > 0$, *where the deformed sphere is still having some geometric properties which were before the deformation.*

4. FINITENESS PROPERTY OF ISOTHERMAL REVOLUTION SURFACE

In this section, we focus on the isothermal revolution surface for finiteness property for the mean curvature flow.

Case 5.1. If we put $f(u) = u$ and $h = h(u)$ in Eq. (13) and for M be isothermal surface ($g_{11} = g_{22}$ and $g_{12} = 0$) then we get

$$1 + h'^2(u) = u^2. \tag{48}$$

Solving the above differential equation gives

$$h(u) = \mp\frac{1}{2}u\sqrt{\omega} \pm \frac{1}{2}\log(u + \sqrt{\omega}) + c, \quad \omega = u^2 - 1, \quad c = \text{constant}. \tag{49}$$

Now, we can rewrite the parametrization of revolution surface in Eq. (13) as the following

$$\mathbf{X}(u,v) = \{u\cos v\,,\, u\sin v\,,\, \frac{1}{2}\left(u\sqrt{\omega} - \log(u + \sqrt{\omega})\right) + c\}, \tag{50}$$

The unite normal vector field of M is

$$\mathbf{G} = -\frac{1}{u}\{\sqrt{\omega}\cos v\,,\, \sqrt{\omega}\sin v\,,\, -1\}. \tag{51}$$

Therefore, we have

$$(g_{ij}) = u^2\, diag\,(1\,,1)\,, \quad (g^{ij}) = \frac{1}{u^2}\, diag\,(1\,,1)\,, \quad g = u^4. \tag{52}$$

In this case the Laplacian Δ takes the formula

$$\Delta = -\frac{1}{u^2}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right). \tag{53}$$

Then, the mean curvature function is given by

$$H = \frac{1}{2\sqrt{\omega}}. \tag{54}$$

Similarly as in the previous cases, we take the first component X_1 of M. Then,

$$X_1 = u \cos v. \tag{55}$$

Hence,

$$\Delta X_1 = \frac{1}{u} \cos v. \tag{56}$$

Therefore,

$$\Delta^2 X_1 = \frac{1}{u^5}(u^2 - 2) \cos v, \tag{57}$$

and

$$\Delta^3 X_1 = \frac{1}{u^9}(u^4 - 14u^2 + 60) \cos v. \tag{58}$$

Lemma 4.1. *If R is a polynomial in u and* $\deg R = r$, *then*

$$\Delta\left(\frac{R}{u^q} \cos v\right) = \frac{\widehat{R}}{u^{q+4}} \cos v, \tag{59}$$

where $\widehat{R}$ is a polynomial in u and $\deg \widehat{R} \leq r + 2$,

Applying the above Lemma, we get

$$\Delta^j X_1 = \frac{R_j}{u^{4j-3}} \cos v, \tag{60}$$

Consequently, if j goes up by one, the degree of the numerator of $\Delta^j X_1$ goes up by at most 2, while the degree of the denominator goes up by 4. Hence the decomposition (11) can never be zero. Therefore, M is infinite type.

Now, M is deformed by mean curvature flow $\phi = H$. Let $\overline{M}$ be a surface after variation. Then, we get the locally parametrization of $\overline{M}$ by

$$\begin{aligned}\overline{\mathbf{X}}(u,v,t) =& \frac{1}{2u\sqrt{\omega}}\{(2u^2 - t)\sqrt{\omega} \cos v,\, (2u^2 - t)\sqrt{\omega} \sin v,\, t + u(u\,\omega + 2c\sqrt{\omega}) \\ & - u\sqrt{\omega} \log(u + \sqrt{\omega})\}.\end{aligned} \tag{61}$$

Consequently,

$$\overline{\mathbf{G}} = \frac{1}{2u\,\omega\sqrt{\omega - t}}\left\{\left((4+t)u^2 - 2u^4 - 2\right) \cos v,\, \left((4+t)u^2 - 2u^4 - 2\right) \sin v,\, 2\,\omega^{\frac{3}{2}}\right\}. \tag{62}$$

Therefore, we get

$$\begin{aligned}(\overline{g}_{ij}) &= -\frac{1}{\omega}\, diag\left(t - u^2\,\omega,\, (t - u^2)\,\omega\right), \quad \overline{g} = \frac{u^4}{\omega}(\omega - t), \\ (\overline{g}^{ij}) &= \frac{1}{u^4(\omega - t)}\, diag\left(\omega\,(u^2 - t),\, u^2\,\omega - t\right).\end{aligned} \tag{63}$$

Direct computations, we can find the Laplacian $\overline{\Delta}$ of $\overline{M}$ is given by

$$\overline{\Delta} = \frac{(t-u^2)\omega}{u^4(\omega-t)}\frac{\partial^2}{\partial u^2} - \frac{t(2-4u^2+u^4)}{u^5(\omega-t)^2}\frac{\partial}{\partial u} - \frac{u^2\omega - t}{u^4(\omega-t)}\frac{\partial^2}{\partial v^2}. \tag{64}$$

Let $\overline{X}_1$, $\overline{X}_2$, and $\overline{X}_3$ be the three components functions of $\overline{\mathbf{X}}$. If we take $\overline{X}_3$, then

$$\overline{\Delta}\,\overline{X}_3 = \frac{\omega^{\frac{-1}{2}}}{2u^5(t-\omega)^2}\left(p_1(u) + t\,q_1(u)\right), \tag{65}$$

where, $p_1(u)$ and $q_1(u)$ are functions of u of degree 8 and 6, respectively. Therefore,

$$\overline{\Delta}^2\,\overline{X}_3 = \frac{\omega^{\frac{1}{2}}}{2u^9(t-\omega)^5}\left(p_2(u) + t\,q_2(u)\right), \tag{66}$$

where $p_2(u)$ and $q_2(u)$ are functions of u of degree 12 and 10, respectively.

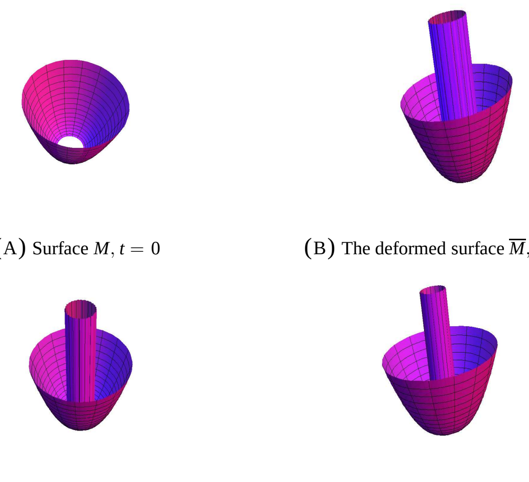
Using mathematical induction, we see

$$\overline{\Delta}^j\,\overline{X}_3 = \frac{\omega^{\frac{2j-3}{2}}}{2u^{4j+1}(t-\omega)^{3j-1}}\left(p_j(u) + t\,q_j(u)\right), \tag{67}$$

where $p_j(u)$ and $q_j(u)$ are functions of u of degree $4j+4$ and $4j+2$, respectively. We note that degree of denominator is larger than degree of numerator. Therefore the decomposition (11) can never be zero. Then $\overline{M}$ is infinite type.

Corollary 4.1. *The mean curvature flow of the deformed isothermal surface preserves the property of infiniteness.*

See Figure 3.



(A) Surface M, $t = 0$

(B) The deformed surface $\overline{M}$, $t = 0.1$

(C) The deformed surface $\overline{M}$, $t = 0.4$

(D) The deformed surface $\overline{M}$, $t = 0.9$

FIGURE 3. The deformed isothermal surface :$u \in [1, \pi]$, $v \in [0, 2\pi]$, $c = 3$
$f(u) = u$, $h(u) = \frac{1}{2}\left(u\sqrt{u^2-1} - \log(u + \sqrt{u^2-1})\right) + c$, $\phi = H$

Corollary 4.2. *The effectiveness of the mean curvature flow of the deformed isothermal surface is very strong* $\forall t > 0$, *i.e., the geometric properties are not hereditary properties.*

Corollary 4.3. *After little computations one can see that the deformed isothermal surface is isothermal if and only if $u = \pm\sqrt{2}$. In other words, the deformed isothermal surface is an isothermal surface at two parametric curves on the surface.*

5. General example

Finally in this section, we study the normal variation under the effect of general function.

Case 6.1. If we put $f(u) = a$ and $h(u) = cu$ where a, c are constants. Then the parametrization of revolution surface in Eq. (13) gives revolution cylinder as the following

$$\mathbf{X}(u,v) = \{a \cos v\,, a \sin v\,, c\,u\}, \quad a, c \neq 0. \tag{68}$$

The unite normal vector field of M is

$$\mathbf{G} = -\{\cos v\,, \sin v\,, 0\}. \tag{69}$$

Thus, we get

$$(g_{ij}) = diag\left(c^2\,, a^2\right), \quad (g^{ij}) = diag\left(\frac{1}{c^2}\,, \frac{1}{a^2}\right), \quad g = a^2 c^2. \tag{70}$$

Then, the mean curvature function is given by $H = \frac{1}{2a}$. Therefore,

$$\Delta\mathbf{X} = -\left(\frac{1}{c^2}\frac{\partial^2}{\partial u^2} + \frac{1}{a^2}\frac{\partial^2}{\partial v^2}\right). \tag{71}$$

Solving the following equation for λ

$$\Delta\mathbf{X} - \lambda\,\mathbf{X} = 0, \tag{72}$$

Then, the eigen values of Δ take the following values.

$$\lambda_1 = \lambda_2 = \frac{1}{a^2}, \quad \lambda_3 = 0. \tag{73}$$

That is, the revolution cylinder is null $2-$type as well know, see [7, 25].
Here, we show the effect of the finiteness property for the deformed revolution cylinder by mean curvature flow.

Let $\overline{M}$ be the surface after variation under the assumption $\phi = \sin(u+v)$. Then $\overline{M}$ has a parametrization as the following

$$\overline{\mathbf{X}}(u,v,t) = \{\cos v\,(a - t\sin\theta), \sin v\,(a - t\sin\theta), c\,u\}, \tag{74}$$

where $\theta = u + v$. The unit normal vector field is given by

$$\overline{\mathbf{G}} = -\frac{1}{c\sqrt{a(a-2t\sin\theta)}}\left\{c\left(a\cos v - t\sin(u+2v)\right), c\left(a\sin v + t\cos(u+2v)\right), at\cos\theta\right\}. \tag{75}$$

Thus, we have

$$(\overline{g}_{ij}) = diag\left(c^2\,, a(a-2t\sin\theta)\right), \quad \overline{g} = a\,c^2\,(a - 2t\sin\theta),$$
$$(\overline{g}^{ij}) = diag\left(\frac{1}{c^2}\,, \frac{1}{a(a-2t\sin\theta)}\right). \tag{76}$$

Direct computations, we can find the Laplacian $\overline{\Delta}$ of $\overline{M}$ is given by

$$\overline{\Delta} = -\frac{1}{c^2\,a\,\theta_1^2}\left(c^2\,t\cos\theta\frac{\partial}{\partial v} - a^2\,t\cos\theta\frac{\partial}{\partial u} + a\,\theta_1^2\frac{\partial^2}{\partial u^2} + c^2\,\theta_1\frac{\partial^2}{\partial v^2}\right), \tag{77}$$

where $\theta_1 = a - 2t \sin \theta$.

The mean curvature function of $\overline{M}$ is given by

$$\overline{H} = \frac{1}{2\,c^2 \sqrt{a}\, \theta_1^{\frac{3}{2}}} \left(c^2 a - \left(3c^2 + a^2\right) t \sin \theta\right). \tag{78}$$

Let $\overline{X}_1$, $\overline{X}_2$, and $\overline{X}_3$ be the three components functions of $\overline{\mathbf{X}}$. Then, we take $\overline{X}_3 = u$, where

$$\overline{\Delta}\, \overline{X}_3 = \frac{t \cos \theta}{c\, \theta_1}. \tag{79}$$

Therefore

$$\overline{\Delta}^2 \overline{X}_3 = \frac{1}{c^3 \theta_1^4} a\,(a^2 + c^2)\, t \cos \theta, \tag{80}$$

and

$$\overline{\Delta}^3 \overline{X}_3 = \frac{1}{c^5 \theta_1^7} a^2 (a^2 + c^2)^2 t \cos \theta. \tag{81}$$

Using mathematical induction we obtain formula

$$\overline{\Delta}^j \overline{X}_3 = \frac{a^{j-1} (a^2 + c^2)^{j-1} t \cos \theta}{c^{2j-1} \theta_1^{3j-2}}. \tag{82}$$

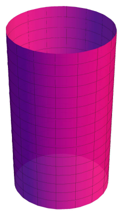
Suppose that the deformed circular cylinder is of finite type. Thus, by applying the decomposition (11), we see

$$t \cos \theta \left(\frac{a^j (a^2 + c^2)^j}{c^{2j+1} \theta_1^{3j+1}} + \frac{a^{j-1} (a^2 + c^2)^{j-1} d_1}{c^{2j-1} \theta_1^{3j-2}} + \cdots + \frac{a\,(a^2 + c^2)\, d_{j-1}}{c^3 \theta_1^4} + \frac{d_j}{c\, \theta_1} \right) = 0. \tag{83}$$

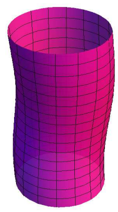
This means $t = 0$ or $\cos \theta = 0$ and it is a contradiction. Then the deformation surface by ϕ is infinite type.

Corollary 5.1. *The deformed revolution cylinder does not preserve the property of finite type.*

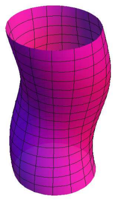
See Figure 4.



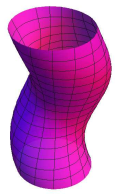
(A) The revolution cylinder M, $t = 0$



(B) The deformed cylinder $\overline{M}$, $t = 0.2$



(C) The deformed cylinder $\overline{M}$, $t = 0.5$



(D) The deformed cylinder $\overline{M}$, $t = 0.9$

FIGURE 4. The deformed revolution cylinder $u \in [0, 2\pi]$, $v \in [-\pi, \pi]$
$f(u) = a$, $h(u) = cu$, $\phi = \sin(u+v)$, $a = 3$, $c = 2$

Corollary 5.2. *The effect of the normal variation for the revolution cylinder is very strong* $\forall t > 0.2$, *i.e., the geometric properties are not hereditary properties.*

6. Conclusion

It is important to remark that the effect of the normal variation in deferent directions of the revolution surfaces of finiteness property is very weak in some cases. In other words, the deformed surfaces are still having some geometric properties which were before the deformation. In other cases, the effect of the normal variation is strong. In other words, the geometric properties of the deformed revolution surfaces are not hereditary properties. In the following, we give a summary of the studied cases previously:

(1) $f(u) = au$, $h(u) = bu$ and $\phi = H$. Hence, M (cone) and its deformed surface $\overline{M}$ are infinite type.
(2) $f(u) = a\cos u$, $h(u) = a\sin u$ and $\phi = H$. Then, M (sphere) and its deformed surface $\overline{M}$ are 1$-$type.
(3) $f(u) = u$, $h(u) = \frac{1}{2}\left(u\sqrt{u^2-1} - \log(u+\sqrt{u^2-1})\right) + c$ and $\phi = H$. Therefore, M and its deformed surface $\overline{M}$ are infinite type.
(4) $f(u) = a$, $h(u) = cu$ and $\phi = \sin(u+v)$ where c is constant. Hence, M (circular cylinder) is null 2$-$type and the deformed surface $\overline{M}$ is infinite type.

The above four cases are translated to the Figures [1 - 4].

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