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VARIATIONS OF HYPERBOLIC SEMIGROUPS

S.PANAYAPPAN

Abstract. In this article, hyperbolic semigroup is characterised in terms of the spectrum of its cogenerator. Further, (λ,μ) hyperbolic semigroups and essentially hyperbolic semigroups are studied.

1. Introduction

Hyperbolic operator semigroups arise naturally in the theory of dynamical systems. The notion of hyperbolicity of operator semigroups is vital in differentiable dynamics and in the theory of differential equations.
Over the years, the notion of hyperbolicity was broadened (non-uniform hyperbolicity) [1] and relaxed (partial hyperbolicity) [6] to encompass a much larger class of systems and has become a paradigm for complex dynamical evolution systems.
In this article, we define and study (λ,μ) hyperbolic semigroups and essentially hyperbolic semigroups.

2. Preliminaries

We begin with some basic definitions.
Definition 1:
A C_0-semigroup $T(.)$ on a Banach space X is called bounded if $sup_{t\geq 0}\|T(t)\| < \infty$.
We remark that if A be the generator of a bounded semigroup of $T(.)$ then $\sigma(A) \subset \{z : Rez \leq 0\}$ but not conversely.
Definition 2:
A C_0-semigroup $T(.)$ is called uniformly exponentially stable, if there exist $M \geq 1$ and $\epsilon > 0$ such that
$\|T(t)\| \leq Me^{-\epsilon t}$ for all $t \geq 0$.
A C_0-semigroup $\left(e^{tA}\right)_{t\geq 0}$ on finite - dimensional Banach space is uniformly exponentially stable if and only if $Re\lambda < 0$ for every $\lambda \in \sigma(A)$. But, on an infinite dimensional space,this result is no longer valid.
The following elementary description of uniformly exponentially stable C_0-semigroups is repeatedly used in this article.

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Theorem 3:
For C_0-semigroup $T(.)$ on a Banach space X the following assertions are equivalent:
(i) $T(.)$ is uniformly exponentially stable.
(ii) $lim_{t\to\infty}\|T(t)\| = 0$
(iii) $\|T(t_0)\| < 1$ for some $t_0 > 0$
(iv) $r(T_0) < 1$ for some $t_0 > 0$
(v) $r(T(t)) < 1$ for all $t > 0$
This theorem shows that to check uniform exponential stability, it suffices to determine the spectral radius of $T(t_0)$ for some t_0.
For bounded operators, the spectral mapping theorem $\sigma(T(t)) = e^{t\sigma(A)}$ for every $t \geq 0$ holds. We have the following terminology.
Definition 4:
A C_0-semigroup $T(.)$ on a Banach space X with generator A satisfies the spectral mapping theorem if $\sigma(T(t))/\{0\} = e^{t\sigma(A)}$ for every $t \geq 0$.
Spectral inclusion theorem:
For the generator (A,D(A)) of a C_0-semigroup $T(t)_{t\geq 0}$ on a Banach space X, we have the inclusions $\sigma(T(t)) \supset e^{t\sigma(A)}$ for $t \geq 0$.

3. **Hyperbolic decompositions**

In hyperbolic semigroup theory, we try to decompose the Banach space into the direct sum of two closed subspaces such that the semigroup becomes "forward" exponentially stable on one subspace and "backward" exponentially stable on other subspace.
Definition 5:
A C_0-semigroup $T(t)$ of bounded operators on a Banach space X is called hyperbolic if X can be written as a direct sum $X = X_s \oplus X_u$ of two $(T(t))_{t\geq 0}$ - invariant closed subspaces X_s, X_u called stable and unstable subspaces and such that for some $\epsilon > 0$ and $M > 0$ and one has
$\|T(t)x\| \leq Me^{-\epsilon t}\|x\|$ for $x \in X_s$, $t > 0$
$\|T(t)x\| \geq \frac{1}{M}e^{\epsilon t}\|x\|$ for $x \in X_u$, $t > 0$
It is known that $T(t)_{t\geq 0}$ is hyperbolic if and only if $\sigma(T(t)) \cap \Gamma = \phi$ for all or equivalently for some $t > 0$ where Γ is the unit circle [4].
More precisely, we have
Theorem 6:
For a C_0-semigroup $(T(t))_{t\geq 0}$ the following assertions are equivalent.
(a) $(T(t))_{t\geq 0}$ is hyperbolic.
(b) $\sigma(T(t)) \cap \{e^{-\epsilon t} \leq |z| \leq e^{\epsilon t}\} = \phi$ for one /all $t > 0$.
Proof:
$(a) \Rightarrow (b)$
Let the restricted semigroups of $(T(t))_{t\geq 0}$ on X_s and X_u be $(T_s(t))_{t\geq 0}$ and $(T_u(t))_{t\geq 0}$ respectively.Then $\sigma(T(t)) = \sigma(T_s(t)) \cup \sigma(T_u(t))$
By assumption $\|T_s(t)x\| \leq Me^{-\epsilon t}\|x\|$ for $t > 0$ then $r(T_s(t)) \leq e^{-\epsilon t}$ for $t > 0$ and so
$\sigma(T_s(t)) \cap a\Gamma = \phi$ for $a \geq e^{-\epsilon t}$ ————(1)
By the same argument, from $\|T_u(t)x\| \geq \frac{1}{M}e^{\epsilon t}\|x\|$, $t \geq 0$ we obtain
$r(T_u(t))^{-1} \leq e^{-\epsilon t}$
Since $\sigma(T_u(t)) = \left\{\lambda^{-1} : \lambda \in \sigma(T_u(t)^{-1})\right\}$. We conclude that $|\lambda| \geq e^{\epsilon t}$ for each $\lambda \in \sigma(T_u(t))$.

So $\sigma(T_u(t)) \cap a\Gamma = \phi$ for $a \leq e^{\epsilon t}$ ————(2)
Combining (1) and (2)
$\sigma(T(t)) \cap a\Gamma = \phi$ for $a \in [e^{-\epsilon t}, e^{\epsilon t}]$ and so
$\sigma(T(t)) \cap \{e^{-\epsilon t} \leq |z| \leq e^{\epsilon t}\} = \phi$
Conversly, we fix $s > 0$ such that
$\sigma(T(s)) \cap \{e^{-\epsilon s} \leq |z| \leq e^{\epsilon s}\} = \phi$
The operators $P_s = \frac{1}{2\pi i}\int_{|z|\leq e^{-\epsilon s}} \frac{dz}{z-T(s)}$ and $P_u = I - P_s$ define projection maps onto the corresponding subspaces X_s, X_u and
$\sigma(T(s)/X_s) = \sigma(T(s)) \cap \{|z| \leq e^{-\epsilon t}\}$
$\sigma(T(s)/X_u) = \sigma(T(s)) \cap \{|z| \geq e^{\epsilon t}\}$
Then the spectral radius of $(T(s)/_{X_s})$ is given by $r(T_s(s)) \leq e^{-\epsilon s}$. Then by theorem 3, for $x \in X_s$, $t \geq 0$, there exists $M > 0$ such that
$\|T(t)x\| \leq Me^{-\epsilon t}\|x\|$.
Now the restriction $T_u(s)$ of $T(s)$ in X_u has the spectrum
$\sigma(T_u(s)) = \{\lambda \in \sigma(T(s))/|\lambda| \geq e^{\epsilon s}\}$
and so is invertible on X_u.This then shows that $(T_u(t))$ is invertible for $0 \leq t < s$ and for $t > s$.
We choose $n \in N$ so that $ns > t$.
Then $(T_u(s))^n = T_u(ns) = T_u(t)T_u(ns - t)$ and so $T_u(t)$ is invertible for $t > s$.
Again by theorem 3,
$\|T(t)x\| \geq \frac{1}{M}e^{\epsilon t}\|x\|$ for $x \in X_u$, $t \geq 0$.
Thus $(T(t))_{t\geq 0}$ is hyperbolic.

Definition 7:
Let A generate a C_0 -semigroup $T(.)$ on a Banach space X satisfying $1 \in P(A)$, the resolvent set of A.The operator V defined by
$V = (A + I)(A - I)^{-1} \in L(X)$ is called the co generator of $T(.)$.
We observe that $A = I + 2(V - I)^{-1}$ holds and note that the co-generator determines the generator and therefore the semi-group uniquely.
The following theorem characterises a semigroup in terms of the spectrum of its co-generator.

Theorem 8:
Let A be the generator and V be the co-generator of a C_0 -semigroup $(T(t))_{t\geq 0}$. Assume further $\sigma(T(t)) \subset \Gamma e^{t\sigma(A)} = \left\{ze^{t\lambda} : \lambda \in \sigma(A), |z| = 1\right\}$. Then $(T(t))_{t\geq 0}$ is hyperbolic if and only if $\sigma(V) \cap \Gamma = \phi$

Proof:
By Engel, Nagel [4],[Theorem V.1.17] $(T(t))_{t\geq 0}$ is hyperbolic if and only if
$\sigma(A) \cap iR = \phi$ whenever $\sigma(T(t)) \subset \Gamma e^{t\sigma(A)}$ holds. Using the identity
$V = (A - I + 2I)(A - I)^{-1} = I - 2R(1, A)$
and the spectral mapping theorem for the resolvent we get
$\sigma(V)/\{1\} = \left\{\frac{\lambda+1}{\lambda-1} : \lambda \in \sigma(A)\right\}$.
Now $ir \notin \sigma(A)$ is equivalent to $\frac{ir+1}{ir-1} \notin \sigma(V)/\{1\}$ for every real r. This then means $\sigma(V)/\{1\} \cap \Gamma = \phi$, as desired.

4. (λ, μ) Hyperbolic semigroup

The rescaled semigroup $(e^{\mu t}T(\alpha t))_{t\geq 0}$ for some fixed $\mu \in C$ and $\alpha > 0$ has generator $B = \alpha A + \mu I$ with domain $D(A) = D(B)$. Further we have $\sigma(B) = \alpha\sigma(A) + \mu$

If $(T(t))_{t\geq 0}$ is a hyperbolic semigroup, it can be observed that the rescaled semigroup operator $e^{\alpha t}T(t)$ has no spectrum in the annulus $\left(e^{(\alpha-\epsilon)t} \leq |z| \leq e^{(\alpha+\epsilon)t}\right)$ for every $\alpha \in R$.

Definition 9:

For $0 < \lambda < \mu$. We call $(T(t))_{t\geq 0}$ is (λ, μ) hyperbolic whenever the rescaled operator $a^t T(t)$ is hyperbolic for every $a \in [\lambda, \mu]$.

If $0 < \lambda < 1$ and $\mu > 1$ then (λ, μ) hyperbolic semigroup becomes a hyperbolic semigroup.

Theorem 10:

A co-semigroup $(T(t))_{t\geq 0}$ is (λ, μ) hyperbolic if and only if
$\sigma(T(t)) \cap \{\lambda \leq |z| \leq \mu\} = \phi$ for one/all $t > 0$.

Proof:

$(T(t))_{t\geq 0}$ is (λ, μ) hyperbolic if and only if $a^t T(t)$ is hyperbolic for every $a \in (\lambda, \mu)$ if and only if
$\sigma(T(t)) \cap (a^t e^{-\epsilon t} \leq |z| \leq a^t e^{\epsilon t}) = \phi$
for every $a \in [\lambda, \mu]$. Since $(T(t))_{t\geq 0}$ is hyperbolic if and only if $T(1)$ is hyperbolic,we have $\sigma(T(t)) \cap a\Gamma = \phi$ for every $a \in (\lambda, \mu)$. So, $\sigma(T(t))$ cannot have any points in common with $\{\lambda \leq |z| \leq \mu\}$ as desired.

Theorem 11:

Let A be the generator of a co-semigroup $(T(t))_{t\geq 0}$. Assume that $\sigma(T(t)) \subset a\Gamma e^{t\sigma(A)} = \left\{aze^{t\lambda}/\lambda \in \sigma(A), |z| = 1, a \in (\lambda, \mu)\right\}$ Then the following assertions are equivalent

(a) $(T(t))_{t\geq 0}$ is (λ, μ) hyperbolic.

(b) $\sigma(T(t)) \cap (\lambda < |z| < \mu) = \phi$ for one /all $t > 0$

(c) $\sigma(A) \cap (loga + iR) = \phi$

Proof:

$(a) \Leftrightarrow (b)$: follows from Theorem 10

$(b) \Rightarrow (c)$:

By assumption, $\sigma(T(t)) \cap a\Gamma = \phi$ for every $a \in [\lambda, \mu]$ and for all $t > 0$.

Using spectral mapping theorem, we then have $e^{t\sigma(A)} \cap a\Gamma = \phi$ for every $a \in [\lambda, \mu]$ for all $t > 0$. Let $\lambda \in \sigma(A)$ so that $e^{t\lambda} \notin a\Gamma$ for all $a \in [\lambda, \mu]$ and for all $t > 0$ which implies $\lambda t \neq loga + i\theta$ for all $t > 0$ and hence $\lambda \notin loga + iR$ as desired.

$(c) \Rightarrow (b)$:

Observe that , if $\lambda \in \sigma(A)$ then $\lambda \notin loga + iR$ for $\lambda \leq a \leq \mu$.
Then $\sigma(T(t)) \subset a\Gamma e^{t\sigma(A)} = \left\{aze^{t\lambda}/\lambda \notin loga + iR, |z| = 1, \right\}$
for all $t > 0$ and $\lambda \leq a \leq \mu$ and which shows $\sigma(T(t)) \cap a\Gamma = \phi$ for $\lambda \leq a \leq \mu$.

5. Essentially hyperbolic semigroups

The spectrum $\sigma(T)$ of a general bounded operator T splits into disjoint union of discrete and essential parts. Following Browder [2], we call $\lambda \in \sigma(T)$, a point of discrete spectrum, $\sigma_{disc}(T)$ if

(a) λ is an isolated point in $\sigma(T)$ and

(b) The corresponding spectral subspace,determined by the image of the Riesz projection
$P = \frac{1}{2\pi i}\int_{|z-\lambda|<\delta} \frac{dz}{z-T}$ is finite dimensional.

The essential spectrum, denoted by $\sigma_{ess}(T)$ is defined as $\sigma(T)/\sigma_{disc}(T)$.

Definition 12:

A semigroup $(T(t))_{t\geq 0}$ on a Banach space is called essentially hyperbolic if

$\sigma_{ess}(T(t)) \cap \Gamma = \phi$ for all $t > 0$.

Theorem 13:

A semigroup $(T(t))_{t \geq 0}$ on a Banach space is essentially hyperbolic if and only if there exist constants $M, \epsilon > 0$ and X can be written as a direct sum $X = X_s \oplus X_c \oplus X_u$ of invariant closed subspaces X_s, X_c, X_u such that

(a) $\|T(t)x\| \leq M e^{-\epsilon t} \|x\|$ for all $x \in X_s$, $t > 0$

(b) $\|T(t)x\| \geq M^{-1} e^{\epsilon t} \|x\|$ for all $x \in X_u$, $t > 0$ and

(c) X_c is finite dimensional.

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S.Panayappan

Post Graduate and Research Department of Mathematics,Government Arts College(Autonomous),Coimbatore-641 018,Tamil Nadu, India.

E-mail address: panayappan@gmail.com