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UNIQUENESS OF A MEROMORPHIC FUNCTIONS THAT SHARE ONE SMALL FUNCTION AND ITS DERIVATIVE.

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ABSTRACT. In this paper we consider the problem of uniqueness of meromorphic functions that share one small function and its derivatives, and obtain two theorems which improve the result of Qingcai Zhang [11].

1. Introduction

Let f be a non-constant meromorphic function defined in the whole complex plane $\mathbb{C}$. It is assumed that the reader is familiar with the following notations of Nevanlinna theory such as $T(r,f), m(r,f), N(r,f), S(r,f)$ and so on, that can be found, for instance in [1,2].

Let f and g be two non-constant meromorphic functions, $a \in \mathbb{C} \cup \{\infty\}$, we say that f and g share the value a CM (counting multiplicity) if $f-a$ and $g-a$ have the same zeroes with the same multiplicities and they share the value a IM (ignoring multiplicities) if we do not consider the multiplicities. When $a=\infty$ the zeroes of $f-a$ means the poles of f(see [7]).

Let k be a non-negative integer or infinity. For any $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a,f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a,f) = E_k(a,g)$, we say that f, g share the value a with weight k.(see[3],[5]).

We write f and g share (a,k) to mean that f and g share the value a with weight k. Clearly, if f and g share (a,k), then f and g share (a,p) for all integers p with $0 \leq p \leq k$. Also, we note that f, g share a value a IM or CM if and only if they share $(a,0)$ or (a,∞) respectively.

A function $a(z)$ is said to be a small function of f if $a(z)$ is a meromorphic function satisfying $T(r,a) = S(r,f)$, i.e,$T(r,a) = o(T(r,f))$ as $r \to +\infty$ possibly outside of set of finite linear measure. Similarly, we define that f and g share a small function a IM or CM or with weight k by $f-a$ and $g-a$ sharing the value 0 IM or CM or with weight k respectively.

For any constant a, we denote by $N_{k)}(r, \frac{1}{f-a})$ the counting function for zeros of $f-a$ with multiplicity no more than k, and by $\overline{N}_{k)}(r, \frac{1}{f-a})$ the corresponding one

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for which multiplicity is not counted. Let $N_{(k}(r,\frac{1}{f-a})$ be the counting function for zeros of $f-a$ with multiplicity at least k and $\overline{N}_{(k}(r,\frac{1}{f-a})$ be the corresponding one for which multiplicity is not counted. Set
$N_k(r,\frac{1}{f-a})=\overline{N}(r,\frac{1}{f-a})+\overline{N}_{(2}(r,\frac{1}{f-a})+...+\overline{N}_{(k}(r,\frac{1}{f-a})$.
We define

$$\Theta(a,f)=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,\frac{1}{f-a})}{T(r,f)}\quad,\quad\delta(a,f)=1-\limsup_{r\longrightarrow\infty}\frac{N(r,\frac{1}{f-a})}{T(r,f)},$$

We further define

$$\delta_k(a,f)=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}_k(r,\frac{1}{f-a})}{T(r,f)}$$

Clearly,

$$0\leq\delta(a,f)\leq\delta_k(a,f)\leq\delta_{k-1}(a,f)...\leq\delta_2(a,f)\leq\delta_1(a,f)=\Theta(a,f).$$

In additional, we shall also use the following notations:

Let f and g be two non-constant meromorphic functions such that f and g share 1 IM. We denote by $\overline{N}_L(r,\frac{1}{f-1})$ the counting function for 1-point of both f and g about which f has larger multiplicity than g, with multiplicity being not counted, and denote by $N_{11}(r,\frac{1}{f-1})$ the counting function for common simple 1-point of both f and g, and denote by $N_{(22}(r,\frac{1}{f-1})$ the counting function of those same multiplicity 1-point of both f and g and the multiplicity is ≥ 2. In the same way, we can define $\overline{N}_L(r,\frac{1}{g-1}),N_{11}(r,\frac{1}{g-1})$, and $N_{(22}(r,\frac{1}{g-1})$. Especially, if f and g share 1 CM, then $\overline{N}_L(r,\frac{1}{g-1})=0$.

R.Bruck [4] first considered the uniqueness problems of an entire function sharing one value with its derivative and proved the following result.

Theorem A. Let f be a entire function which is not constant. If f and f' share the value 1 CM and if $N(r,\frac{1}{f'})=S(r,f)$, then $\frac{f'-1}{f-1}\equiv c$ for some nonzero constant $c\in\mathbb{C}\setminus\{0\}$.

Bruck [4] further posed the following conjecture.

Conjecture 1.1. Let f be an entire function, which is not constant, $\rho_1(f)$ be the first iterated order of f. If $\rho_1(f)<+\infty$ and $\rho_1(f)$ is not a positive integer, and if f and f' share one value a CM, then $\frac{f'-a}{f-a}\equiv c$ for some nonzero constant $c\in\mathbb{C}\setminus\{0\}$.

Yang [8] proved that the conjecture is true if f is an entire function of finite order. Zhang[10] extended Theorem A to meromorphic functions. Yu[9] recently considered the problem of an entire or meromorphic function sharing one small function with its derivative and proved the following two theorems.

Theorem B([9]). Let f be a non-constant entire function and $a\equiv a(z)$ be a meromorphic function such that $a\not\equiv 0,\infty$ and $T(r,a)=o(T(r,f))$ as $r\to+\infty$. If $f-a$ and $f^{(k)}-a$ share the value 0 CM and $\delta(0,f)>\frac{3}{4}$, then $f\equiv f^{(k)}$.

Theorem C([9]). Let f be a non-constant, non-entire meromorphic function and $a\equiv a(z)$ be a meromorphic function such that $a\not\equiv 0,\infty$ and $T(r,a)=o(T(r,f))$ as $r\to+\infty$. If
(i) f and a have no common poles,
(ii) $f-a$ and $f^{(k)}-a$ share the value 0 CM,
(iii) $4\delta(0,f)+2\Theta(\infty,f)>19+2k$,
then $f\equiv f^{(k)}$ where k is a positive integer.
In the same paper, Yu[9] further posed the following open questions.

(i) Can a CM shared be replaced by an IM shared value ?
(ii) Can the condition $\delta(0, f) > \frac{3}{4}$ of Theorem B be further relaxed ?
(iii) Can the condition (iii) of Theorem C be further relaxed ?
(iv) Can in general the condition (i) of Theorem C be dropped ?

Lahiri[5] improved the results of Zhang[10] with weighted shared value obtained the following two theorems.

Theorem D([5]). Let f be a non-constant meromorphic function and k be a positive integer. If f and $f^{(k)}$ share (1,2) and

$$2\overline{N}(r, f) + N_2(r, \frac{1}{f^{(k)}}) + N_2(r, \frac{1}{f}) < (\lambda + o(1))T(r, f^{(k)})$$

for $r \in I$, where $0 < \lambda < 1$ and I is a set of infinite linear measure, then $\frac{f^{(k)}-1}{f-1} \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.

Theorem E([5]). Let f be a non-constant meromorphic function and k be a positive integer. If f and $f^{(k)}$ share (1,1) and

$$2\overline{N}(r, f) + N_2(r, \frac{1}{f^{(k)}}) + 2\overline{N}(r, \frac{1}{f}) < (\lambda + o(1))T(r, f^{(k)})$$

for $r \in I$, where $0 < \lambda < 1$ and I is a set of infinite linear measure, then $\frac{f^{(k)}-1}{f-1} \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.

In the same paper Lahiri[5] also obtained the following result which is an improvement of Theorem C.

Theorem F([5]). Let f be a non-constant meromorphic function and k be a positive integer. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic function such that $T(r, a) = S(r, f)$. If

(i) a has no zero(pole) which is also a zero(pole) of f or $f^{(k)}$ with the same multiplicity.
(ii) $f - a$ and $f^{(k)} - a$ share (0,2) CM,
(iii) $2\delta_{2+k}(0, f) + (4 + k)\Theta(\infty, f) > 5 + k$,

then $f \equiv f^{(k)}$.

In 2005, Zhang[11] improved the above results and proved the following theorems.

Theorem G([11]). Let f be a non-constant meromorphic function and $k(\geq 1), l(\geq 0)$ be integers. Also, let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic function such that $T(r, a) = S(r, f)$. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. If $l \geq 2$ and

$$2\overline{N}(r, f) + N_2(r, \frac{1}{f^{(k)}}) + N_2(r, \frac{1}{(f/a)'}) < (\lambda + o(1))T(r, f^{(k)}), \quad (1)$$

or $l = 1$ and

$$2\overline{N}(r, f) + N_2(r, \frac{1}{f^{(k)}}) + 2\overline{N}(r, \frac{1}{(f/a)'}) < (\lambda + o(1))T(r, f^{(k)}), \quad (2)$$

or $l = 0$, i.e, $f - a$ and $f^{(k)} - a$ share the value 0 IM and

$$4\overline{N}(r, f) + 3N_2(r, \frac{1}{f^{(k)}}) + 2\overline{N}(r, \frac{1}{(f/a)'}) < (\lambda + o(1))T(r, f^{(k)}) \quad (3)$$

for $r \in I$, where $0 < \lambda < 1$ and I is a set of infinite linear measure, then $\frac{f^{(k)}-a}{f-a} \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.

Theorem H([11]). Let f be a non-constant meromorphic function and $k(\geq 1), l(\geq$

0) be integers. Also let $a \equiv a(z)(\not\equiv 0,\infty)$ be a meromorphic function such that $T(r,a) = S(r,f)$. Suppose that $f - a$ and $f^{(k)} - a$ share $(0,l)$. If $l \geq 2$ and

$$(3+k)\Theta(\infty,f) + 2\delta_{2+k}(0,f) > k+4, \tag{4}$$

or $l = 1$ and

$$(4+k)\Theta(\infty,f) + 3\delta_{2+k}(0,f) > k+6, \tag{5}$$

or $l = 0$ ie $f - a$ and $f^{(k)} - a$ share the value 0 IM and

$$(6+2k)\Theta(\infty,f) + 5\delta_{2+k}(0,f) > 2k+10, \tag{6}$$

then $f \equiv f^{(k)}$.

In this paper we pay our attention to the uniqueness of more generalised form of a function namely f^n and $(f^{(k)})^m$sharing a small function for two arbitrary positive integer n and m.

Theorem 1.1. Let f be a non-constant meromorphic function and $k(\geq 1), n(\geq 1), m(\geq 2), l(\geq 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0,\infty)$ be a meromorphic function such that $T(r,a) = S(r,f)$. Suppose that $f^n - a$ and $(f^{(k)})^m - a$ share $(0,l)$.
If $l \geq 2$ and

$$\frac{2}{m}\overline{N}(r,f) + \frac{2}{m}\overline{N}(r,\frac{1}{f^{(k)}}) + N_2(r,\frac{1}{(f/a)'}) < (\lambda + o(1))T(r,f^{(k)}) \tag{7}$$

or $l = 1$ and

$$\frac{2}{m}\overline{N}(r,f) + \frac{2}{m}\overline{N}(r,\frac{1}{f^{(k)}}) + 2N(r,\frac{1}{(f/a)'}) < (\lambda + o(1))T(r,f^{(k)}) \tag{8}$$

or $l = 0$ ie $f - a$ and $(f^{(k)})^m - a$ share the value 0 IM and

$$\frac{4}{m}\overline{N}(r,f) + \frac{6}{m}\overline{N}(r,\frac{1}{f^{(k)}}) + 2\overline{N}(r,\frac{1}{(f/a)'}) < (\lambda + o(1))T(r,f^{(k)}) \tag{9}$$

for $r \in I$, where $0 < \lambda < 1$ and I is a set of infinite linear measure, then $\frac{(f^{(k)})^m - a}{f^n - a} \equiv c$ for some constant $c \in \mathbb{C} \setminus \{0\}$.

Theorem 1.2. Let f be a non-constant meromorphic function and $k(\geq 1), n(\geq 1), m(\geq 2), l(\geq 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0,\infty)$ be a meromorphic function such that $T(r,a) = S(r,f)$. Suppose that $f^n - a$ and $(f^{(k)})^m - a$ share $(0,l)$.
If $l \geq 2$ and

$$(3+2k)\Theta(\infty,f) + 2\Theta(0,f) + 2\delta_{1+k}(0,f) > 2k+7-n \tag{10}$$

or $l = 1$ and

$$(4+2k)\Theta(\infty,f) + 4\Theta(0,f) + 2\delta_{1+k}(0,f) > 2k+10-n \tag{11}$$

or $l = 0$ ie $f - a$ and $(f^{(k)})^m - a$ share the value 0 IM and

$$(6+4k)\Theta(\infty,f) + 6\Theta(0,f) + \delta_{1+k}(0,f) > 16+4k-n, \tag{12}$$

then $f^n \equiv (f^{(k)})^m$.

From Theorem 1.2 we have the following corollary.

Corollary 1.3. Let f be a non-constant entire function and $a \equiv a(z)(\neq 0,\infty)$ be a meromorphic function such that $T(r,a) = S(r,f)$. If $f^n - a$ and $(f^{(k)})^m - a$ share the value 0 CM and $\delta(0,f) > 1 - \frac{n}{2}$, or if $f^n - a$ and $(f^{(k)})^m - a$ share the value 0 IM and $\delta(0,f) > 1 - \frac{n}{4}$, then $f^n \equiv (f^{(k)})^m$.

2. Main Lemmas

Lemma 2.1[5]. Let f be a non-constant meromorphic function, k be a positive integer,then

$$N_p(r, \frac{1}{f^{(k)}}) \leq N_{p+k}(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f)$$

Lemma 2.2[7]. Let f be a non-constant meromorphic function, n be a positive integer. $P(f) = a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f$ where a_i is a meromorphic function such that $T(r, a_i) = S(r, f)(i = 1, 2, ...n)$ Then $T(r, P(f)) = nT(r, f) + S(r, f)$.

3. Proof of Theorem 1.1

Let $F = \frac{f^n}{a}$, $G = \frac{(f^{(k)})^m}{a}$, then $F - 1 = \frac{f^n - a}{a}$, $G - 1 = \frac{(f^{(k)})^m - a}{a}$. Since $f^n - a$ and $(f^{(k)})^m - a$ share $(0, l)$, F and G share $(1, l)$ except the zeros and poles of $a(z)$. Define

$$H = (\frac{F''}{F'} - \frac{2F'}{F-1}) - (\frac{G''}{G'} - \frac{2G'}{G-1}), \tag{13}$$

we have the following two cases to investigate
Case 1. $H \equiv 0$. Integration yields

$$\frac{1}{F-1} \equiv C\frac{1}{G-1} + D, \tag{14}$$

where C and D are constants and $C \neq 0$. If there exists a pole z_0 of f with multiplicity p which is not the pole and zero of $a(z)$, then z_0 is the pole of F with multiplicity p and the pole of G with multiplicity $p+k$. This contradicts with (14). So

$$\overline{N}(r, f) \leq \overline{N}(r, a) + \overline{N}(r, \frac{1}{a}) = S(r, f), \tag{15}$$

$$\overline{N}(r, F) = S(r, f) \quad \overline{N}(r, G) = S(r, f)$$

(14) also shows F and G share the value 1 CM. Next we prove $D = 0$. We first assume that $D \neq 0$, then

$$\frac{1}{F-1} \equiv \frac{D(G - 1 + \frac{C}{D})}{G-1} \tag{16}$$

So,

$$\overline{N}(r, \frac{1}{G - 1 + \frac{C}{D}}) = \overline{N}(r, F) = S(r, f) \tag{17}$$

If $\frac{C}{D} \neq 1$, by the second fundamental theorem and (15),(17) and $S(r, G) = S(r, f)$, we have

$$T(r, G) \leq \overline{N}(r, G) + \overline{N}(r, \frac{1}{G}) + \overline{N}(r, \frac{1}{G - 1 + \frac{C}{D}}) + S(r, G)$$

$$\leq \overline{N}(r, \frac{1}{G}) + S(r, f) \leq T(r, G) + S(r, f)$$

$$So, \qquad T(r, G) = \overline{N}(r, \frac{1}{G}) + S(r, f), \tag{18}$$

i.e, $$T(r, (f^{(k)})^m) = \overline{N}(r, \frac{1}{(f^{(k)})^m}) + S(r, f)$$

$mT(r,(f^{(k)})) = \overline{N}(r, \frac{1}{f^{(k)}}) + S(r,f).$

this contradicts with conditions (1),(2) and (3) of this theorem.
If $\frac{C}{D} = 1$, from (16) we know

$$\frac{1}{F-1} \equiv C\frac{G}{G-1}$$

then

$$(F-1-\frac{1}{C})G = -\frac{1}{C}.$$

Noticing that $F = \frac{f^n}{a}, \quad G = \frac{(f^{(k)})^m}{a}, \quad wehave$

$$\frac{1}{f^n(f^n-(1+\frac{1}{C})a)} \equiv \frac{-C}{a^2}.\frac{(f^{(k)})^m}{f^n} \tag{19}$$

By Lemma 2.2 and (15) and (19), then

$$2T(r,f^n) = T(r, f^n(f^n-(1+\frac{1}{C})a)) + S(r,f) \tag{20}$$

$$\begin{aligned}
2nT(r,f) &= T(r, \frac{1}{f^n(f^n-(1+\frac{1}{C})a)}) + S(r,f) \\
&= T(r, \frac{(f^{(k)})^m}{f^n}) + S(r,f) \\
&\le N(r, \frac{1}{f^n}) + m\overline{N}(r, f^{(k)}) + S(r,f) \\
&\le nN(r, \frac{1}{f}) + S(r,f) \\
&\le nT(r,f) + S(r,f)
\end{aligned}$$

So, $nT(r,f) = S(r,f)$, which is impossible. Hence D=0, and $\frac{G-1}{F-1} \equiv C$, ie, $\frac{(f^{(k)})^m - a}{f^n - a} \equiv C$. This is just the conclusion of this theorem.
Case 2.$H \not\equiv 0$, From (13) it is easy to see that $m(r,H) = S(r,f)$.
Subcase 2.1. $l \ge 1$. From (13) we have

$$\begin{aligned}
N(r,H) \le \overline{N}(r,F) + \overline{N}_{(l+1}(r, \frac{1}{F-1}) + \overline{N}_{(2}(r, \frac{1}{F}) + \overline{N}_{(2}(r, \frac{1}{G}) \\
+ \overline{N}_0(r, \frac{1}{G'}) + \overline{N}(r,a) + \overline{N}(r, \frac{1}{a}).
\end{aligned} \tag{21}$$

where $N_0(r, \frac{1}{F'})$ denotes the counting function of the zeros of F' which are not the zeros of F and $F-1$, and $\overline{N}_0(r, \frac{1}{F'})$ denotes its reduced form. In the same way, we can define $N_0(r, \frac{1}{G'})$ and $\overline{N}_0(r, \frac{1}{G'})$, Let z_0 be a simple zero of $F-1$ but $a(z_0) \ne 0, \infty$, then z_0 is also the simple zero of $G-1$. By calculating z_0 is the zero of H, So

$$N_{1)}(r, \frac{1}{F-1}) \le N(r, \frac{1}{H}) + N(r,a) + N(r, \frac{1}{a}) \le N(r,H) + S(r,f) \tag{22}$$

Noticing that $N_{1)}(r, \frac{1}{G}) = N_{1)}(r, \frac{1}{F}) + S(r, f)$
we have

$$\begin{aligned}\overline{N}(r, \frac{1}{G-1}) &= N_{1)}(r, \frac{1}{F-1}) + \overline{N}_{(2}(r, \frac{1}{F-1})\\ &\leq \overline{N}(r, F) + \overline{N}_{(l+1}(r, \frac{1}{F-1}) + \overline{N}_{(2}(r, \frac{1}{F-1})\\ &\quad + \overline{N}_{(2}(r, \frac{1}{F}) + \overline{N}_{(2}(r, \frac{1}{G}) + \overline{N}_0(r, \frac{1}{F'}) + \overline{N}_0(r, \frac{1}{G}) + S(r, f)\end{aligned} \tag{23}$$

By the second fundamental theorem and (23) and noticing

$$\overline{N}(r, F) = \overline{N}(r, G) + S(r, f),$$

then

$$\begin{aligned}T(r, G) &\leq \overline{N}(r, G) + \overline{N}(r, \frac{1}{G}) + \overline{N}(r, \frac{1}{G-1}) - N_0(r, \frac{1}{G'}) + S(r, G)\\ &\leq 2\overline{N}(r, F) + \overline{N}(r, \frac{1}{G}) + \overline{N}_{(2}(r, \frac{1}{G}) + \overline{N}_{(2}(r, \frac{1}{F})\\ &\quad + \overline{N}_{(l+1}(r, \frac{1}{F-1}) + \overline{N}_{(2}(r, \frac{1}{F-1}) + \overline{N}_0(r, \frac{1}{F'}) + S(r, f).\end{aligned} \tag{24}$$

While $l \geq 2$,

$$\overline{N}_{(2}(r, \frac{1}{F}) + \overline{N}_{(l+1}(r, \frac{1}{F-1}) + \overline{N}_{(2}(r, \frac{1}{F-1}) + \overline{N}_0(r, \frac{1}{F'}) \leq N_2(r, \frac{1}{F'}), \tag{25}$$

So

$$T(r, G) \leq 2\overline{N}(r, F) + N_2(r, \frac{1}{G}) + N_2(r, \frac{1}{F'}) + S(r, f)$$

i.e,

$$mT(r, f^{(k)}) \leq 2\overline{N}(r, f) + N_2(r, \frac{1}{(f^{(k)})^m}) + N_2(r, \frac{1}{(\frac{f^n}{a})'}) + S(r, f)$$

$$T(r, f^{(k)}) \leq \frac{2}{m}\overline{N}(r, f) + \frac{2}{m}\overline{N}(r, \frac{1}{f^{(k)}}) + N_2(r, \frac{1}{(\frac{f^n}{a})'}) + S(r, f)$$

this contradicts with (1).
While $l = 1$, (25) turns into

$$\overline{N}_{(2}(r, \frac{1}{F}) + \overline{N}_{(l+1}(r, \frac{1}{F-1}) + \overline{N}_{(2}(r, \frac{1}{F-1}) + \overline{N}_0(r, \frac{1}{F'}) \leq 2\overline{N}(r, \frac{1}{F})$$

Similarly as above , we have

$$T(r, f^{(k)}) \leq \frac{2}{m}\overline{N}(r, f) + \frac{2}{m}\overline{N}(r, \frac{1}{f^{(k)}}) + 2N(r, \frac{1}{(\frac{f^n}{a})'}) + S(r, f)$$

This contradicts with (2).
Subcase 2.2. $l = 0$. In this case, F and G share 1 IM except the zeros and poles of $a(z)$. Let z_0 be the zero of $F-1$ with multiplicity p and the zero of $G-1$ with multiplicity q.
We denote by $N_E^{1)}(r, \frac{1}{F})$ the counting function of the zeros of $F-1$ where $p-q=1$; by $N_E^{2)}(r, \frac{1}{F})$ the counting function of the zeros of $F-1$ where $p = q \geq 2$; by $\overline{N}_L(r, \frac{1}{F})$ the counting function of the zeros of $F-1$ where $p > q \geq 1$, each point

in these counting functions is counted only once. In the same way, we can define $N_E^{1)}(r,\frac{1}{G})$,$N_E^{(2}(r,\frac{1}{G})$ and $\overline{N}_L(r,\frac{1}{G})$. It is easy to see that

$$N_E^{1)}(r,\frac{1}{F-1}) = N_E^{1)}(r,\frac{1}{G-1}) + S(r,f),$$

$$\overline{N}_E^{2)}(r,\frac{1}{F-1}) = \overline{N}_E^{(2}(r,\frac{1}{G-1}) + S(r,f),$$

$$\begin{aligned}\overline{N}(r,\frac{1}{F-1}) &= \overline{N}(r,\frac{1}{G-1}) + S(r,f)\\ &= N_E^{1)}(r,\frac{1}{F-1}) + N_E^{(2}(r,\frac{1}{F-1}) + \overline{N}_L(r,\frac{1}{F-1})\\ &+ \overline{N}_L(r,\frac{1}{G-1}) + S(r,f)\end{aligned} \tag{26}$$

From (13) we have now

$$\begin{aligned}N(r,H) \le\ & \overline{N}(r,F) + \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) + \overline{N}_L(r,\frac{1}{F-1})\\ &+ \overline{N}_L(r,\frac{1}{G-1}) + \overline{N}_0(r,\frac{1}{F'}) + \overline{N}_0(r,\frac{1}{G'}) + S(r,f).\end{aligned} \tag{27}$$

In this case, (22) is replaced by

$$N_E^{1)}(r,\frac{1}{F-1}) \le N(r,H) + S(r,f). \tag{28}$$

From (26),(27) and (28), we have

$$\begin{aligned}\overline{N}(r,\frac{1}{G-1}) \le\ & \overline{N}(r,F) + \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) + \overline{N}_E^{(2}(r,\frac{1}{F-1})\\ &+ 2\overline{N}_L(r,\frac{1}{F-1}) + 2\overline{N}_L(r,\frac{1}{G-1}) + \overline{N}_0(r,\frac{1}{F'})\\ &+ \overline{N}_0(r,\frac{1}{G'}) + S(r,f)\\ \le\ & \overline{N}(r,F) + 2\overline{N}(r,\frac{1}{F'}) + 2\overline{N}_L(r,\frac{1}{G-1})\\ &+ \overline{N}_{(2}(r,\frac{1}{G}) + \overline{N}_0(r,\frac{1}{G'}) + S(r,f)\end{aligned}$$

By the second fundamental theorem, then

$$\begin{aligned}T(r,G) &\le \overline{N}(r,G) + \overline{N}(r,\frac{1}{G}) + \overline{N}(r,\frac{1}{G-1}) - N_0(r,\frac{1}{G'}) + S(r,G)\\ &\le 2\overline{N}(r,G) + 2\overline{N}(r,\frac{1}{F'}) + \overline{N}(r,\frac{1}{G}) + 2\overline{N}(r,\frac{1}{G'}) + S(r,f)\end{aligned}$$

From Lemma 2.1 for $p = 1, k = 1$ we know

$$\overline{N}(r,\frac{1}{G'}) \le N_2(r,\frac{1}{G}) + \overline{N}(r,G) + S(r,G),$$

So,

$$T(r,G) \le 4\overline{N}(r,F) + 3N_2(r,\frac{1}{G}) + 2\overline{N}(r,\frac{1}{F'}) + S(r,f)$$

i.e,

$$mT(r,f^{(k)}) \leq 4\overline{N}(r,f) + 3N_2(r,\frac{1}{(f^{(k)})^m}) + 2\overline{N}(r,\frac{1}{(\frac{f^n}{a})'}) + S(r,f).$$

$$T(r,f^{(k)}) \leq \frac{4}{m}\overline{N}(r,f) + \frac{6}{m}\overline{N}(r,\frac{1}{f^{(k)}}) + 2\overline{N}(r,\frac{1}{(\frac{f^n}{a})'}) + S(r,f)$$

This contradicts with (3). The proof is complete.

4. Proof of Theorem 1.2

The proof is similar to that of Theorem 1.1. We define F and G and (13) as above, and we also distinguish two cases to discuss.

Case 3. $H \equiv 0$. We also have (14). From (15) we know that $\Theta(\infty, f) = 1$, and from (4),(5) and (6), We further know $\delta_{1+k}(0,f) > 1 - \frac{n}{2}$. Assume that $D \neq 0$, then

$$\frac{-D(F-1-\frac{1}{D})}{F-1} \equiv C\frac{1}{G-1},$$

so

$$\overline{N}(r,\frac{1}{F-1-\frac{1}{D}}) = \overline{N}(r,G) = S(r,f).$$

If $D \neq -1$, using the second fundamental theorem for F, similarly as (18)

we have $T(r,F) = \overline{N}(r,\frac{1}{F}) + S(r,f)$,

i.e., $T(r,f^n) = \overline{N}(r,\frac{1}{f^n}) + S(r,f)$,

$nT(r,f) = \overline{N}(r,\frac{1}{f}) + S(r,f)$

Hence $\Theta(0,f) = 0$, this contradicts with $\Theta(0,f) \geq \delta_{1+k}(0,f) > 1 - \frac{n}{2}$.

If $D = -1$, then $\overline{N}(r,\frac{1}{F}) = S(r,f)$, i.e., $\overline{N}(r,\frac{1}{f}) = S(r,f)$, and

$$\frac{F}{F-1} \equiv C\frac{1}{G-1}.$$

Then $F(G-1-C) \equiv -C$

and thus,

$$(f^{(k)})^m((f^{(k)})^m - (1+C)a) \equiv -C\frac{a^2(f^{(k)})^m}{f^n}. \tag{29}$$

As same as (20), by Lemma 2.2 and (15) and $\overline{N}(r,\frac{1}{f}) = S(r,f)$. from (29) we have

$$\begin{aligned}2T(r,(f^{(k)})^m) &= T(r,\frac{(f^{(k)})^m}{f}) + S(r,f)\\ &= N(r,\frac{(f^{(k)})^m}{f}) + S(r,f)\\ &\leq mk\overline{N}(r,f) + m\overline{N}(r,\frac{1}{f}) + S(r,f)\\ &= S(r,f)\end{aligned}$$

So, $T(r,(f^{(k)})^m)=S(r,f)$ and $T(r,\frac{(f^{(k)})^m}{f})=S(r,f)$.
Hence

$$\begin{aligned} T(r,f^n) &\le T(r,\frac{f^n}{(f^{(k)})^m})+T(r,(f^{(k)})^m)+O(1) \\ &= T(r,\frac{(f^{(k)})^m}{f^n})+mT(r,f^{(k)})+O(1) \\ &= S(r,f), \end{aligned}$$

this is impossible. Therefore $D=0$, and from (14) then

$$G-1\equiv\frac{1}{C}(F-1)$$

If $C\neq 1$, then $G=\frac{1}{C}(F-1+C)$,
and $N(r,\frac{1}{G})=N(r,\frac{1}{F-1+C})$
By the second fundamental theorem and (15) we have

$$\begin{aligned} T(r,F) &\le \overline{N}(r,F)+\overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{F-1+C})+S(r,G) \\ &\le \overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{G})+S(r,f) \end{aligned}$$

By Lemma 2.1 for $p=1$ and (15), we have

$$\begin{aligned} nT(r,f) &\le \overline{N}(r,\frac{1}{f^n})+\overline{N}(r,\frac{1}{(f^{(k)})^m})+S(r,G) \\ &\le \overline{N}(r,\frac{1}{f})+\overline{N}(r,\frac{1}{f^{(k)}})+S(r,f) \\ &\le 2N_{1+k}(r,\frac{1}{f})+S(r,f) \end{aligned}$$

Hence $\delta_{1+k}(0,f)\le 1-\frac{n}{2}$. This is a contradiction with $\delta_{1+k}(0,f)\le 1-\frac{n}{2}$. So $C=1$ and $F\equiv G$, i.e., $f^n=(f^{(k)})^m$. This is just the conclusion of this theorem.
Case 4. $H\not\equiv 0$
Subcase 4.1 $l\ge 1$ As similar as Subcase 2.1, From (21) and (22) we have

$$\begin{aligned} \overline{N}(r,\frac{1}{F-1})+\overline{N}(r,\frac{1}{G-1}) &= N_{1)}(r,\frac{1}{F-1})+\overline{N}_{(2}(r,\frac{1}{F-1})+\overline{N}(r,\frac{1}{G-1}) \\ &\le \overline{N}(r,F)+\overline{N}_{(2}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{G})+\overline{N}_{(l+1}(r,\frac{1}{G-1}) \\ &+\overline{N}_{(2}(r,\frac{1}{G-1})+\overline{N}(r,\frac{1}{G-1})+\overline{N}_0(r,\frac{1}{F'}) \\ &+\overline{N}_0(r,\frac{1}{G'})+S(r,f) \end{aligned}$$

While $l\ge 2$,

$$\overline{N}_{(l+1}(r,\frac{1}{G-1})+\overline{N}_{(2}(r,\frac{1}{G-1})+\overline{N}(r,\frac{1}{G-1})\le N(r,\frac{1}{G-1})\le T(r,G)+O(1),$$

So,

$$\begin{aligned} \overline{N}(r,\frac{1}{F-1})+\overline{N}(r,\frac{1}{G-1}) &\le \overline{N}(r,F)+\overline{N}_{(2}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{G}) \\ &+\overline{N}_0(r,\frac{1}{F'})+\overline{N}_0(r,\frac{1}{G'})+T(r,G)+S(r,f). \end{aligned}$$

By the second fundamental theorem, we have

$$T(r,F)+T(r,G)\leq \overline{N}(r,F)+\overline{N}(r,G)+\overline{N}(r,\frac{1}{F})+\overline{N}(r,\frac{1}{G})+\overline{N}(r,\frac{1}{F-1})$$
$$+\overline{N}(r,\frac{1}{G-1})-N_0(r,\frac{1}{F'})-N_0(r,\frac{1}{G'})+S(r,F)+S(r,G)$$
$$\leq 3\overline{N}(r,F)+N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+T(r,G)+S(r,f),$$

$$So,\ \ T(r,F)\leq 3\overline{N}(r,F)+N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+S(r,f),$$
$$i.e,\ \ nT(r,f)\leq 3\overline{N}(r,f)+N_2(r,\frac{1}{f})+N_2(r,\frac{1}{(f^{(k)})^m})+S(r,f)$$
$$nT(r,f)\leq 3\overline{N}(r,f)+N_2(r,\frac{1}{f})+2N(r,\frac{1}{f^{(k)}})+S(r,f)$$

By Lemma 2.1 for $p=2$ we have

$$nT(r,f)\leq (3+2k)\overline{N}(r,f)+2\overline{N}(r,\frac{1}{f})+2N_{1+k}(r,\frac{1}{f})+S(r,f)$$

So, $\qquad (3+2k)\Theta(\infty,f)+2\Theta(0,f)+2\delta_{1+k}(0,f)\leq 7+2k-n.$
This contradicts with (4).
While $l=1$,

$$\overline{N}_{(l+1}(r,\frac{1}{G-1})+\overline{N}(r,\frac{1}{G-1})\leq N(r,\frac{1}{G-1})\leq T(r,G)+O(1),$$

so by Lemma 2.1 for $p=1,k=1$, we have

$$\overline{N}(r,\frac{1}{F-1})+\overline{N}(r,\frac{1}{G-1})\leq \overline{N}(r,F)+\overline{N}_{(2}(r,\frac{1}{F})+\overline{N}_{(2}(r,\frac{1}{G})+\overline{N}_{(2}(r,\frac{1}{F-1})+\overline{N}_0(r,\frac{1}{F'})$$
$$+\overline{N}_0(r,\frac{1}{G'})+T(r,G)+S(r,f).$$
$$\leq \overline{N}(r,F)+\overline{N}_{(2}(r,\frac{1}{G})+\overline{N}(r,\frac{1}{F'})+\overline{N}_0(r,\frac{1}{G'})+T(r,G)+S(r,f)$$
$$\leq 2\overline{N}(r,F)+\overline{N}_{(2}(r,\frac{1}{G})+N_2(r,\frac{1}{F})+\overline{N}_0(r,\frac{1}{G'})+T(r,G)+S(r,f)$$

As same as above, by the second fundamental theorem we have

$$T(r,F)+T(r,G)\leq 4\overline{N}(r,F)+2N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+T(r,G)+S(r,f),$$

so

$$T(r,F)\leq 4\overline{N}(r,F)+2N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+S(r,f),$$

i.e.,

$$nT(r,f)\leq 4\overline{N}(r,f)+2N_2(r,\frac{1}{f^n})+N_2(r,\frac{1}{(f^{(k)})^m})+S(r,f),$$

$$nT(r,f)\leq 4\overline{N}(r,f)+4\overline{N}(r,\frac{1}{f})+2\overline{N}(r,\frac{1}{f^{(k)}})+S(r,f)$$
$$\leq 4\overline{N}(r,f)+4\overline{N}(r,\frac{1}{f})+2\{N_{1+k}(r,\frac{1}{f})+k\overline{N}(r,f)\}+S(r,f)$$

By Lemma 2.1 for p=2 we have

$$nT(r,f) \leq (4+2k)\overline{N}(r,f) + 2N_{1+k}(r,\frac{1}{f}) + 4\overline{N}(r,\frac{1}{f}) + S(r,f)$$

So,

$$(4+2k)\Theta(\infty,f) + 4\Theta(0,f) + 2\delta_{1+k}(0,f) \leq 10 + 2k - n$$

This contradicts with (5).
Subcase 4.2. $l = 0$. From (26),(27) and (28) and Lemma 2.1 for $p = 1, k = 1$, noticing

$$N_E^{(2}(r,\frac{1}{G-1}) + \overline{N}_L(r,\frac{1}{G-1}) + \overline{N}(r,\frac{1}{G-1}) \leq N(r,\frac{1}{G-1}) \leq T(r,G) + S(r,f)$$

then

$$\begin{aligned}\overline{N}(r,\frac{1}{F-1}) + \overline{N}(r,\frac{1}{G-1}) &= N_E^{1)}(r,\frac{1}{F-1}) + N_E^{(2}(r,\frac{1}{F-1}) + \overline{N}_L(r,\frac{1}{F-1}) + \overline{N}_L(r,\frac{1}{G-1})\\ &\quad + \overline{N}(r,\frac{1}{G-1})\\ &\leq \overline{N}(r,F) + \overline{N}_{(2}(r,\frac{1}{F}) + \overline{N}_{(2}(r,\frac{1}{G}) + 2\overline{N}_L(r,\frac{1}{F-1}) + \overline{N}_L(r,\frac{1}{G-1})\\ &\quad + \overline{N}_E^{(2}(r,\frac{1}{G-1}) + \overline{N}_L(r,\frac{1}{G-1}) + \overline{N}(r,\frac{1}{G-1}) + \overline{N}_0(r,\frac{1}{F'}) + \overline{N}_0(r,\frac{1}{G'})\\ &\quad + S(r,f)\\ &\leq \overline{N}(r,F) + 2\overline{N}(r,\frac{1}{F'}) + \overline{N}(r,\frac{1}{G'}) + T(r,G) + S(r,f)\\ &\leq 4\overline{N}(r,F) + 2N_2(r,\frac{1}{F}) + N_2(r,\frac{1}{G}) + T(r,G) + S(r,f)\end{aligned}$$

As same as above, by the second fundamental theorem,we can obtain

$$T(r,F) + T(r,G) \leq 6\overline{N}(r,F) + 3N_2(r,\frac{1}{F}) + 2N_2(r,\frac{1}{G}) + T(r,G) + S(r,f)$$

So

$$\begin{aligned}T(r,F) &\leq 6\overline{N}(r,F) + 3N_2(r,\frac{1}{F}) + 2N_2(r,\frac{1}{G}) + S(r,f),\\ nT(r,f) &\leq 6\overline{N}(r,f) + 6\overline{N}(r,\frac{1}{f^n}) + 2N_2(r,\frac{1}{(f^{(k)})^m}) + S(r,f)\\ nT(r,f) &\leq 6\overline{N}(r,f) + 6\overline{N}(r,\frac{1}{f}) + 4\overline{N}(r,\frac{1}{f^{(k)}}) + S(r,f)\end{aligned}$$

By Lemma 2.1 for $p = 2$ we have

$$nT(r,f) \leq (6+4k)\overline{N}(r,f) + 6\overline{N}(r,\frac{1}{f}) + 4N_{1+k}(r,\frac{1}{f}) + S(r,f)$$

$$(6+4k)\Theta(\infty,f) + 6\Theta(0,f) + 4\delta_{1+k}(0,f) \leq 16 + 4k - n$$

this contradicts with (6). Now the proof has been completed.

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