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RESULTS IN PRIME RINGS WITH TWO SIDED α-DERIVATION

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Résumé. In this paper, we study the commutativity of prime rings satisfying certain identities involving two sided α-derivations on rings. Furthermore, we give examples to show that the restrictions imposed on the hypothesis of various theorems are not superfluous.

1. Introduction

Throughout $\mathcal{R}$ will represent an associative ring with center $Z(\mathcal{R})$. For all x, y in $\mathcal{R}$, as usual the commutator $xy - yx$ will be denoted by $[x, y]$. Recall that $\mathcal{R}$ is prime if $a\mathcal{R}b = \{0\}$ implies $a = 0$ or $b = 0$. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all pairs $x, y \in \mathcal{R}$. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is called a (α, β)-derivation if there exist functions $\alpha, \beta : \mathcal{R} \to \mathcal{R}$ such that $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in \mathcal{R}$, furthermore, an additive mapping $d : \mathcal{R} \to \mathcal{R}$ is called a two-sided α-derivation if d is an $(\alpha, 1)$-derivation as well as $(1, \alpha)$-derivation. Moreover, if d commutes with α, then d is called a semiderivation. Clearly every semiderivation is a two-sided α-derivation, but the converse is not true (for example see Example 2 of this paper, it is clear that d is a two-sided α-derivation but not a semiderivation because $d\alpha \neq \alpha d$). In case $\alpha = id_{\mathcal{R}}$, a two-sided α-derivation is just a derivation but the converse is not true. A mapping $f : \mathcal{R} \to \mathcal{R}$ is said to be commuting if $[f(x), x] = 0$ for all $x, y \in \mathcal{R}$. The study of such mappings was initiated by a paper of E. Posner. In the last few years, there has been a considerable interest in the structure of a ring, proof of this is that, a lot of researchers working in this domain. In [1], [4] and [5] the authors study several well known results in prime rings with derivation. A continuous approach in this direction is still on. In this connection, our aim in the present paper is to generalize the theorem of Posner in the case of an $(\alpha, 1)$-derivation or $(1, \alpha)$-derivation on a prime ring and to investigate some properties satisfying certain differential identities. Furthermore, at end of paper, we try to construct some examples which shows that the restrictions imposed on the hypotheses of these results were not superfluous.

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2. Some preliminaries

We facilitate our discussion with the following lemmas which are essential for developing the proof of our results of the present paper.

Lemma 2.1. [2, Lemma 1.2] Let $\mathcal{R}$ be a prime ring. If $z \in Z(\mathcal{R}) - \{0\}$ and $xz \in Z(\mathcal{R})$, then $x \in Z(\mathcal{R})$.

Lemma 2.2. *Let $\mathcal{R}$ be a prime ring. If d is a nonzero two sided α-derivation associated with an onto map α, then $d(Z(\mathcal{R})) \subseteq Z(\mathcal{R})$.*

Proof. Let $z \in Z(\mathcal{R})$. By definition of d, we have

$$d(x)z + \alpha(x)d(z) = d(z)\alpha(x) + zd(x) \quad \text{for all} \;\; x \in \mathcal{R}.$$

Since $z \in Z(\mathcal{R})$, then, the above equation becomes $\alpha(x)d(z) = d(z)\alpha(x)$ for all $x \in \mathcal{R}$, using the fact α is onto, we arrive at $xd(z) = d(z)x$ for all $x \in \mathcal{R}$ which implies that $d(Z(\mathcal{R})) \subseteq Z(\mathcal{R})$. □

Lemma 2.3. [3, Theorem 2.9] Let $\mathcal{R}$ be a prime ring. If $[x, y] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is commutative.

Lemma 2.4. [3, Theorem 2.10] Let $\mathcal{R}$ be a 2-torsion free prime ring. If $x \circ y \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is commutative.

Lemma 2.5. [7, Theorem 3] Let $\mathcal{R}$ be a prime ring. If d is a nonzero two sided α-derivation on $\mathcal{R}$ such that $d([x, y]) = [x, y]$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is commutative or $d = -\alpha + Id_R$.

Lemma 2.6. [7, Theorem 4] Let R be a prime ring. If d is a nonzero two sided α-derivation on $\mathcal{R}$ such that $d(x \circ y) = x \circ y$ for all $x, y \in \mathcal{R}$, then $d = -\alpha + Id_{\mathcal{R}}$.

Lemma 2.7. *Let R be a prime ring and d be an $(\alpha, 1)$-derivation or $(1, \alpha)$-derivation associated with a map α. If α is onto, then α is a ring homomorphism.*

Proof. First, since $d(x(y + z)) = d(xy) + d(xz)$, it can be shown that α must be additive. Next, since $d((xy)z) = d(x(yz))$, it is easy to see that α is a ring homomorphism. □

3. Results for $(\alpha, 1)$-derivations and $(1, \alpha)$-derivations

In [6] Posner proved that a prime ring $\mathcal{R}$ must be commutative if it possesses a nonzero centralizing derivation. In the following theorem, we extend this result for $(\alpha, 1)$-derivation and $(1, \alpha)$-derivations in rings.

Theorem 3.1. *Let $\mathcal{R}$ be a prime ring. If d is a nonzero commuting $(\alpha, 1)$-derivation or $(1, \alpha)$-derivation associated with an onto map α, then $\mathcal{R}$ is commutative.*

Proof. (i) Suppose that d is an $(\alpha, 1)$-derivation and $[d(x), x] = 0$ for all $x \in \mathcal{R}$. Linearizing the last expression, we get $[d(x) + d(y), x + y] = 0$ for all $x, y \in \mathcal{R}$, by given hypothesis we have

$$d(x)y + d(y)x = yd(x) + xd(y) \quad \text{for all} \;\; x, y \in \mathcal{R}. \tag{1}$$

Replacing y by xy in (1) and using it again, we arrive at

$$d(x)xy + \Big(d(x)\alpha(y) + xd(y)\Big)x = xyd(x) + x\Big(d(x)\alpha(y) + xd(y)\Big) \quad \text{for all} \;\; x, y \in \mathcal{R}.$$

Therefore

$$d(x)xy + d(x)\alpha(y)x + xd(y)x = xyd(x) + xd(x)\alpha(y) + x^2d(y) \quad \text{for all } \ x, y \in \mathcal{R}.$$

In view of the latter relation, we obtain

$$d(x)xy + d(x)\alpha(y)x + xd(y)x = x\bigg(yd(x) + xd(y)\bigg) + xd(x)\alpha(y) \quad \text{for all } \ x, y \in \mathcal{R}. \tag{2}$$

By application of (1), (2) implies that

$$d(x)xy + d(x)\alpha(y)x + xd(y)x = xd(x)y + xd(y)x + xd(x)\alpha(y) \quad \text{for all } \ x, y \in \mathcal{R}.$$

Hence

$$d(x)\alpha(y)x = xd(x)\alpha(y) \quad \text{for all } \ x, y \in \mathcal{R}. \tag{3}$$

Using the fact that α is onto and substituting yz for y in (3), we find that

$$d(x)[x, y] = 0 \quad \text{for all } \ x, y \in \mathcal{R}. \tag{4}$$

Replacing y by ty in (4), we get

$$d(x)\mathcal{R}[x, y] = \{0\} \quad \text{for all } \ x, y \in \mathcal{R}. \tag{5}$$

Since $\mathcal{R}$ is prime, then $x \in Z(\mathcal{R})$ or $d(x) = 0$ for all $x \in \mathcal{R}$. Therefore, $\mathcal{R}$ is the union of its additive subgroups $Z(\mathcal{R})$ and $H = \{x \in \mathcal{R} \mid d(x) = 0\}$. But a group cannot be the union of two of its proper subgroups. Hence, either $R = Z(\mathcal{R})$ or $\mathcal{R} = H$. Using the fact that $d \neq 0$, we conclude that $R = Z(\mathcal{R})$ proving that $\mathcal{R}$ is commutative.
(ii) For an $(1, \alpha)$-derivation d, using the similar manner as in (i), we find the required result. □

The following example shows that the hypothesis "d is commuting" in above result is necessary.

Example 1. Let $\mathcal{R} = M_2(\mathbb{Z}_2)$, then it is clear that $\mathcal{R}$ is a non-commutative prime ring. We define the mapping d by $d(X) = AX + XA$ where $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It is easy to see that d is an $(1, \alpha)$-derivation and $(\alpha, 1)$-derivation associated with $\alpha = id_{\mathcal{R}}$, but d is not commuting in $\mathcal{R}$.

Corollary 3.1. *Let $\mathcal{R}$ be a prime ring. If $\mathcal{R}$ admits a nonzero $(\alpha, 1)$-derivation or $(1, \alpha)$-derivation d associated with an onto map α satisfying any one of the following conditions :*

(i) $d([x, y]) = [d(x), y]$ *for all* $x, y \in \mathcal{R}$.

(ii) $[d(x), y] = [x, y]$ *for all* $x, y \in \mathcal{R}$,

then $\mathcal{R}$ is commutative.

Theorem 3.2. *Let $\mathcal{R}$ be a prime ring. If $\mathcal{R}$ admits a nonzero $(\alpha, 1)$-derivation or $(1, \alpha)$-derivation d associated with an onto map α for which $[d(x), y] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

Proof. (a) Suppose that d is an $(\alpha, 1)$-derivation and

$$[d(x), y] \in Z(\mathcal{R}) \quad \text{for all } \ x, y \in \mathcal{R}.$$

This implies that

$$[[d(x), y], x] = 0 \quad \text{for all } \ x, y \in \mathcal{R}. \tag{6}$$

Replacing y by yx in (6), we obtain

$$[y[d(x), x], x] = 0 \quad \text{for all } \ x, y \in \mathcal{R}. \tag{7}$$

Since $[d(x), x] \in Z(\mathcal{R})$, then $[d(x), x]R[y, x] = \{0\}$ for all $x, y \in \mathcal{R}$.
In particular, for $y = d(x)$, we get

$$[d(x), x]R[d(x), x] = \{0\} \quad \text{for all } \ x \in \mathcal{R}. \tag{8}$$

Since $\mathcal{R}$ is prime, then $[d(x), x] = 0$ for all $x \in \mathcal{R}$. Hence Theorem 3.1 assures that $\mathcal{R}$ is commutative.
(b) For an $(1, \alpha)$-derivation d, using the similar manner as in (a), we find the required result. ☐

4. Results for two sided α-derivations

In [1], M. Asraf and N. Rehman showed that a prime ring $\mathcal{R}$ must be commutative if it admits a derivation d satisfying either of the properties $d(xy) + xy \in Z(\mathcal{R})$ or $d(xy) - xy \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. The aim of following results is to prove these theorems for a two sided α-derivation.

Theorem 4.1. *Let $\mathcal{R}$ be a prime ring. If $\mathcal{R}$ admits a two sided α-derivation d associated with an onto map α. Then, the following assertions are equivalent*

- (i) $d(xz) - xz \in Z(\mathcal{R})$ *for all* $x, z \in \mathcal{R}$,
- (ii) $d(xz) + xz \in Z(\mathcal{R})$ *for all* $x, z \in \mathcal{R}$,
- (iii) $\mathcal{R}$ *is commutative.*

Proof. It is easy to see that $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (i)$.
$(i) \Rightarrow (iii)$ If $d = 0$, then $xz \in Z(\mathcal{R})$ for all $x, z \in \mathcal{R}$. In particular, $[xz, y] = 0$ and hence $x[z, y] + [x, y]z = 0$ for all $x, z \in \mathcal{R}$. Replacing x by xt in the previous relation, we get $xt[z, y] = 0$ for all $x, y, z, t \in \mathcal{R}$. In particular, $[z, y]\mathcal{R}[z, y] = \{0\}$ for all $y, z \in \mathcal{R}$. Since $\mathcal{R}$ is prime, then $[z, y] = 0$ for all $y, z \in \mathcal{R}$, hence $\mathcal{R}$ is commutative.
If $d \neq 0$. Suppose that $Z(\mathcal{R}) = \{0\}$, then $d(xz) = xz$ for all $x, z \in \mathcal{R}$, then

$$d(x)z + \alpha(x)d(z) = xz \quad \text{for all } \ x, z \in \mathcal{R}. \tag{9}$$

Replacing x by xy in (9), then $d(xy)z + \alpha(xy)d(z) = xyz$ for all $x, y, z \in \mathcal{R}$. Since $d(xy) = xy$ and α is onto, then $xyd(z) = 0$ for all $x, y, z \in \mathcal{R}$. In particular, $d(z)\mathcal{R}d(z) = \{0\}$ for all $z \in \mathcal{R}$. Since $\mathcal{R}$ is prime, then $d = 0$; a contradiction. Hence $Z(\mathcal{R}) \neq \{0\}$.
Assume that $d(xz) - xz \in Z(\mathcal{R})$ for all $x, z \in \mathcal{R}$, then

$$\alpha(x)d(z) + (d(x) - x)z \in Z(\mathcal{R}) \quad \text{for all } \ x, z \in \mathcal{R}. \tag{10}$$

Replacing x by xy in (10), then $\alpha(xy)d(z) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$, $z \in Z(\mathcal{R})$. By Lemma 2.1, we arrive at $d(Z(\mathcal{R})) = \{0\}$ or $\alpha(xy) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$.
Suppose that $d(Z(\mathcal{R})) = \{0\}$. By (10), we have $(d(x) - x)z \in Z(R)$ for all $x \in R$, $z \in Z(R)$. As $Z(R) \neq \{0\}$, Lemma 2.1, yields $d(x) - x \in Z(R)$ for all $x \in R$.

Consequently, $[d(x), x] = 0$, by Theorem 3.1, $\mathcal{R}$ is commutative.
If $\alpha(xy) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$, using the fact that α is onto, we arrive at $xy \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$ and according to the above, we conclude that $\mathcal{R}$ is commutative.
$(ii) \Rightarrow (iii)$ Now assume that $d(xz) + xz \in Z(\mathcal{R})$ for all $x, z \in \mathcal{R}$. Using similar arguments as used in the proof of $(i) \Rightarrow (iii)$, we can prove that $\mathcal{R}$ is commutative □

Theorem 4.2. *Let $\mathcal{R}$ be a prime ring. If $\mathcal{R}$ admits a two sided α-derivation associated with an onto map α. Then, the following assertions are equivalent*

(i) $d(x)d(z) - xz \in Z(\mathcal{R}$ *for all* $x, z \in \mathcal{R}$,

(ii) $d(x)d(z) + xz \in Z(\mathcal{R}$ *for all* $x, z \in \mathcal{R}$,

(iii) $\mathcal{R}$ *is commutative.*

Proof. It is obvious that $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$.
$(i) \Rightarrow (iii)$ If $d = 0$, then $xz \in Z(\mathcal{R})$ for all $x, z \in \mathcal{R}$. Using the same arguments as used in the beginning of the proof of Theorem 4.1, we get the required result. Hence, onward we shall assume that $d \neq 0$. Suppose that $Z(\mathcal{R}) = \{0\}$, then $d(x)d(z) - xz = 0$ for all $x, z \in \mathcal{R}$. Replacing z by yz, then $d(x)\alpha(y)d(z) + (d(x)d(y) - xy)z = 0$ for all $x, y, z \in \mathcal{R}$. Since $d(x)d(y) - xy = 0$ and α is onto, then $d(x)yd(z) = 0$ for all $x, y, z \in \mathcal{R}$. By primeness of $\mathcal{R}$, we conclude that $d = 0$; a contradiction. Hence $Z(\mathcal{R}) \neq \{0\}$.
Now assume that

$$d(x)d(z) - xz \in Z(\mathcal{R}) \quad \text{for all } x, z \in \mathcal{R}. \tag{11}$$

Replacing z by yz in (11), then

$$d(x)\alpha(y)d(z) + (d(x)d(y) - xy)z \in Z(\mathcal{R}) \quad \text{for all } x, y, z \in \mathcal{R}.$$

In particular,

$$d(x)\alpha(y)d(z) + (d(x)d(y) - xy)z \in Z(\mathcal{R}) \quad \text{for all } x, z \in Z(\mathcal{R}), y \in \mathcal{R}.$$

Using (11) again and $z \in Z(\mathcal{R})$, we get $d(x)\alpha(y)d(z) \in Z(\mathcal{R})$ for all $x, z \in Z(\mathcal{R})$, $y \in \mathcal{R}$, by Lemma 2.1, we find that $[y, t] = 0$ or $d(x)d(z) = 0$ for all $x, z \in Z(\mathcal{R})$, $y, t \in \mathcal{R}$. Hence we conclude that $\mathcal{R}$ is commutative or $d(Z(\mathcal{R}))\mathcal{R}d(Z(\mathcal{R})) = \{0\}$. Using the primeness of $\mathcal{R}$ again, we infer that $\mathcal{R}$ is a commutative or $d(Z(\mathcal{R})) = \{0\}$. Assume that $d(Z(\mathcal{R})) = \{0\}$, by (11), we obtain $xz \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$, $z \in Z(\mathcal{R})$, and Lemma (2.1) assures that $\mathcal{R}$ is commutative.
$(ii) \Rightarrow (iii)$ Suppose that $d(x)d(z) + xz \in Z(\mathcal{R})$ for all $x, z \in \mathcal{R}$, arguing in the similar demonstration as used in the proof of $(i) \Rightarrow (iii)$, we can conclude that $\mathcal{R}$ is commutative. □

Daif and Bell [4] proved that if a semiprime ring $\mathcal{R}$ admits a derivation d such that either $d([x, y]) + [x, y] = 0$ or $d([x, y]) - [x, y] = 0$, for all x, y, in a non-zero ideal I of $\mathcal{R}$, then $\mathcal{R}$ is necessarily commutative. Hongan [5] generalized the above result, considering $\mathcal{R}$ satisfying the conditions $d([x, y]) + [x, y] \in Z(\mathcal{R})$ and $d([x, y]) - [x, y] \in Z(\mathcal{R})$, for all $x, y \in I$. Motivated by the above observations, we explore the commutativity of a prime ring admitting a two sided α-derivation d satisfying any one of the following conditions :

(i) $d([x,y]) - [x,y] \in Z(\mathcal{R})$, **(ii)** $d([x,y]) + [x,y] \in Z(\mathcal{R})$,

(iii) $d(x \circ y) - (x \circ y) \in Z(\mathcal{R})$, and **(iv)** $d(x \circ y) + (x \circ y) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$.

Theorem 4.3. *Let $\mathcal{R}$ be a prime ring. If $\mathcal{R}$ admits a two sided α-derivation d associated with an onto map α such that $(\alpha - id_{\mathcal{R}})(Z(\mathcal{R})) \neq \{0\}$, then the following conditions are equivalents :*

(i) $d([x,y]) - [x,y] \in Z(\mathcal{R})$ *for all* $x, z \in \mathcal{R}$,

(ii) $d([x,y]) + [x,y] \in Z(\mathcal{R})$ *for all* $x, z \in \mathcal{R}$,

(iii) $\mathcal{R}$ *is commutative.*

Proof. It is clear that $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$.
$(i) \Rightarrow (iii)$ If $d = 0$, then $[x,y] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. By Lemma 2.3, we conclude that $\mathcal{R}$ is commutative. Onward we shall assume that $d \neq 0$. Suppose that $Z(\mathcal{R}) = \{0\}$, then

$$d([x,y]) = [x,y] \quad \text{for all} \;\; x, y \in \mathcal{R}. \tag{12}$$

According to Lemma 2.5, we arrive at $\mathcal{R}$ is commutative or $d = -\alpha + id_{\mathcal{R}}$.
If $d = -\alpha + id_{\mathcal{R}}$, by (12), we obtain $\alpha([x,y]) = 0$ for all $x, y \in \mathcal{R}$, and since α is an onto homomorphism of $\mathcal{R}$, we conclude that $[x,y] = 0$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is commutative.
Assume that $Z(\mathcal{R}) \neq \{0\}$ and

$$d([x,y]) - [x,y] \in Z(\mathcal{R}) \quad \text{for all} \;\; x, y \in \mathcal{R}. \tag{13}$$

Replacing y by yz such that $z \in Z(\mathcal{R})$, thus $\alpha([x,y])d(z) + (d([x,y]) - [x,y])z \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$, $z \in Z(\mathcal{R})$. Since $d([x,y]) - [x,y] \in Z(\mathcal{R})$ and $z \in Z(\mathcal{R})$, the preceding relation implies that

$$\alpha([x,y])d(z) \in Z(\mathcal{R}) \quad \text{for all} \;\; x, y \in \mathcal{R}, z \in Z(\mathcal{R}).$$

This means that

$$[x,y]d(z) \in Z(\mathcal{R}) \quad \text{for all} \;\; x, y \in \mathcal{R}, z \in Z(\mathcal{R}). \tag{14}$$

By Lemma 2.1 and Lemma 2.2, we arrive at $[x,y] \in Z(\mathcal{R})$ or $d(Z(\mathcal{R})) = \{0\}$ for all $x, y \in \mathcal{R}$. According to Lemma 2.3, we conclude that $\mathcal{R}$ is commutative or $d(Z(\mathcal{R})) = \{0\}$.
Suppose that $d(Z(\mathcal{R})) = \{0\}$. Replacing y by yz such that $z \in Z(\mathcal{R})$ in (13), we get

$$d(z)[x,y] + \alpha(z)d([x,y]) - z([x,y]) \in Z(\mathcal{R}) \quad \text{for all} \;\; x, y \in \mathcal{R}, z \in Z(\mathcal{R}) \tag{15}$$

which implies that

$$\alpha(z)\Big(d([x,y]) - [x,y]\Big) + \Big(\alpha(z) - z\Big)[x,y] \in Z(\mathcal{R}) \quad \text{for all} \;\; x, y \in \mathcal{R}, z \in Z(\mathcal{R}). \tag{16}$$

Since α is an onto homomorphism of $\mathcal{R}$, then $\alpha(Z(\mathcal{R})) \subset Z(\mathcal{R})$. In this cases (16) becomes

$$\Big(\alpha(z) - z\Big)[x,y] \in Z(\mathcal{R}) \quad \text{for all} \;\; x, y \in \mathcal{R}, z \in Z(\mathcal{R}). \tag{17}$$

Since $(\alpha(z) - z) \in Z(\mathcal{R})$ for all $z \in Z(\mathcal{R})$, then by Lemma 2.1, we arrive at

$$\Big(\alpha(z) - z\Big) = 0 \text{ or } [x,y] \in Z(\mathcal{R}) \quad \text{for all} \;\; x, y \in \mathcal{R}, z \in Z(\mathcal{R}). \tag{18}$$

Since $(\alpha - id_{\mathcal{R}})(Z(\mathcal{R})) \neq \{0\}$, then the first case is indisputable, in this case, the second case implies that $\mathcal{R}$ is commutative by Lemma 2.3.
$(ii) \Rightarrow (iii)$ Using similar arguments as used in the proof of $(i) \Rightarrow (iii)$, we can prove that $\mathcal{R}$ is commutative. □
The following example demonstrate that the condition "3-primeness of N" in various Theorems is crucial.

Example 2. Let $\mathcal{R} = \left\{ \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix} \mid \alpha, \beta, \gamma \in \mathcal{R} \right\}$ where $\mathcal{R}$ is the field of reels integers, it is obvious that R is not prime ring. Next, we define the maps $d, \alpha : \mathcal{R} \to \mathcal{R}$ by $d \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\alpha \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\alpha & -\beta \\ 0 & 0 & -\gamma \\ 0 & 0 & 0 \end{pmatrix}$. It is easy to see that d is a two sided α-derivation associated with an onto homomorphism α such that :

(i) $d(AB) - AB \in Z(\mathcal{R})$ **(ii)** $d(AB) + AB \in Z(\mathcal{R})$
(iii) $d(A)d(B) - AB \in Z(\mathcal{R})$ **(iv)** $d(A)d(B) + AB \in Z(\mathcal{R})$
(v) $d([A, B]) - [A, B] \in Z(\mathcal{R})$ **(vi)** $d([A, B]) + [A, B] \in Z(\mathcal{R})$ for all $A, B \in \mathcal{R}$.

However, $\mathcal{R}$ is not a commutative.

Theorem 4.4. *Let $\mathcal{R}$ be a 2-torsion free prime ring. If $\mathcal{R}$ admits a two sided α-derivation d associated with an onto map α such that $(\alpha - id_{\mathcal{R}})(Z(\mathcal{R})) \neq \{0\}$, then the following conditions are equivalents :*

(i) $d(x \circ y) - x \circ y \in Z(\mathcal{R})$ *for all* $x, y \in \mathcal{R}$,
(ii) $d(x \circ y) + x \circ y \in Z(\mathcal{R})$ *for all* $x, y \in \mathcal{R}$,
(iii) $\mathcal{R}$ *is commutative.*

Proof. It is obvious that $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$.
$(i) \Rightarrow (iii)$ Suppose that

$$d(x \circ y) - x \circ y \in Z(R) \quad \text{for all } x, y \in \mathcal{R}. \tag{19}$$

If $d = 0$, then $x \circ y \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$. By Lemma 2.4, we conclude that $\mathcal{R}$ is commutative.
Suppose that $d \neq 0$ and $Z(\mathcal{R}) = \{0\}$, then

$$d(x \circ y) = x \circ y \quad \text{for all } x, y \in \mathcal{R}. \tag{20}$$

According to Lemma 2.6, we get $d = -\alpha + id_{\mathcal{R}}$. In this cases, using (20), we arrive at $\alpha(x \circ y) = 0$ for all $x, y \in \mathcal{R}$, and the fact that α is onto homomorphism assures

$$x \circ y = 0 \quad \text{for all } x, y \in \mathcal{R}. \tag{21}$$

Taking x instead of y in (21) and using the 2-torsion freeness of $\mathcal{R}$, we obtain $x^2 = 0$ for all $x \in \mathcal{R}$, in this cases, putting y instead of yx in (21) we get $x\mathcal{R}x = \{0\}$ for all $x \in \mathcal{R}$ and the primeness of $\mathcal{R}$ forces $x = 0$ for all $x \in \mathcal{R}$; a contradiction. Hence $Z(\mathcal{R}) \neq \{0\}$.
Assuming that

$$d(x \circ y) - (x \circ y) \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}. \tag{22}$$

Replacing y by yz where $z \in Z(\mathcal{R})$ in the latter relation and using the same again we arrive at, $\alpha(x \circ y)d(z) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$, $z \in Z(\mathcal{R})$, which implies that $(x \circ y)d(z) \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$, $z \in Z(\mathcal{R})$. By application of Lemma 2.1 and Lemma 2.2, we arrive at $x \circ y \in Z(\mathcal{R})$ or $d(Z(\mathcal{R})) = \{0\}$ for all $x, y \in \mathcal{R}$. In view of Lemma 2.4, $\mathcal{R}$ is commutative or $d(Z(\mathcal{R})) = \{0\}$.
Suppose that $d(Z(\mathcal{R})) = \{0\}$. Replacing y by yz where $z \in Z(\mathcal{R})$ in (22), we get

$$d(z)(x \circ y) + \alpha(z)d(x \circ y) - z(x \circ y) \in Z(N) \quad \text{for all } x, y \in \mathcal{R}, z \in Z(\mathcal{R}) \qquad (23)$$

which implies that

$$\alpha(z)\bigg(d(x \circ y) - x \circ y\bigg) + \bigg(\alpha(z) - z\bigg)(x \circ y) \in Z(N) \quad \text{for all } x, y \in \mathcal{R}, z \in Z(\mathcal{R}). \qquad (24)$$

Since $\alpha(Z(\mathcal{R})) \subset Z(\mathcal{R})$, because α is an onto homomorphism of $\mathcal{R}$, then (24) becomes

$$\bigg(\alpha(z) - z\bigg)(x \circ y) \in Z(R) \quad \text{for all } x, y \in \mathcal{R}, z \in Z(\mathcal{R}). \qquad (25)$$

Since $(\alpha(z) - z) \in Z(\mathcal{R})$ for all $z \in Z(\mathcal{R})$, then by Lemma 2.1, we arrive at

$$\bigg(\alpha(z) - z\bigg) = 0 \text{ or } x \circ y \in Z(\mathcal{R}) \quad \text{for all } x, y \in \mathcal{R}, z \in Z(\mathcal{R}). \qquad (26)$$

Since $(\alpha - id_{\mathcal{R}})(Z(\mathcal{R})) \neq \{0\}$, then the first case is indisputable, in this case, the second case implies that $\mathcal{R}$ is commutative by Lemma 2.4.
$(ii) \Rightarrow (iii)$ Using similar arguments as used in the proof of $(i) \Rightarrow (iii)$, we can prove that $\mathcal{R}$ is commutative. □

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