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COMMON FIXED POINT RESULTS FOR INFINITE FAMILIES IN PARTIALLY ORDERED b-METRIC SPACES AND APPLICATIONS

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Abstract. In this paper, we prove common fixed point results of sequence of self mappings in b-metric spaces. The results presented in this paper generalize some recent results announced by many authors. We demonstrate these facts by some examples. Finally, an application to existence problem for an integral equation is presented.

1. Introduction

During the last few decades, there have appeared a lot of papers on common fixed points of metric spaces, b-metric spaces, G-metric spaces and partial metric spaces with different methods(see for example [2, 4, 5, 6, 7, 15, 16, 20]. The family of contraction mappings was introduced and studied by Ćirić [10] and Tasković [21]. Also in the process, the study of existence of common fixed point for finite and infinite family of self-mapping has been carried out by many authors. For example, one may refer[1, 3, 9, 13, 14, 22, 23, 24]. In [11, 12], Czerwik introduced the notion of a b-metric space, which is a generalization of the usual metric space, and generalized the Banach contraction principle in the context of complete b-metric spaces. Consistent with [12], the following definition and results will be needed in the sequel.

Definition 1.1. [12] *Let X be a (nonempty) set and $s \geq 1$ be given a real number. A function $d : X \times X \longrightarrow \mathbb{R}^+$ is said to be a b-metric space if and only if for all $x, y, z \in X$, the following conditions are satisfied:*
(i) $d(x, y) = 0$ *if and only if* $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leq s[d(x, z) + d(z, y)]$.

Then the triplet (X, d, s) is called a $b-$metric space with the parameter s. Clearly, a (standard)metric space is also a $b-$metric space, but the converse is not always true.

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Example 1.1. *Let $X = [0,1]$ and $d : X \times X \longrightarrow \mathbb{R}^+$ be defined by $b(x,y) = |x-y|^2$ for all $x, y \in X$. Clearly, $(X, d, s = 2)$ is a $b-$metric space that is not a metric space.*

Also, the following example of a $b-$metric space is given in [8].

Example 1.2. *Let $p \in (0,1)$. Then the space $L^p([0,1])$ of all real functions $f : [0,1] \longrightarrow \mathbb{R}$ such that $\int_0^1 |f(x)|^p dx < \infty$ endowed with the functional $d : L^p([0,1]) \times L^p([0,1]) \longrightarrow \mathbb{R}$ given by*

$$d(f,g) = (\int_0^1 |f(x) - g(x)|^p dx)^{\frac{1}{p}},$$

for all $f, g \in L^p([0,1])$ is a $b-$metric space with $s = 2^{\frac{1}{p}}$.

Definition 1.2. *Let X be a nonempty set and let $\{T_n\}$ be a family of self mappings on X. A point $x_0 \in X$ is called a common fixed point for this family iff $T_n(x_0) = x_0$, for each $n \in N$.*

2. MAIN RESULTS

In this section, we will present common fixed point theorems for contractive mappings in the setting of b-metric spaces. Furthermore, we will give examples to support our main results. The first result in this paper is the following fixed point theorem.

Throughout the paper, let Ψ be the family of all functions $\psi, \varphi : [0,\infty) \longrightarrow [0,\infty)$ satisfying the following conditions:

(a) $\varphi(t) < \psi(t)$ for each $t > 0$, $\varphi(0) = \psi(0) = 0$;
(b) φ and ψ are continuous functions;
(c) ψ is increasing.

We denote by Θ the set of all functions $\theta : [0,\infty)^4 \longrightarrow [0,\infty)$ satisfying the following conditions:

(a) θ is continuous,
(b) $\theta(a,b,c,d) = 0$ if and only if $abcd = 0$.

Example 2.1. *The following functions belong to Θ:*

- $\theta(t_1, t_2, t_3, t_4) = k \min\{t_1, t_2, t_3, t_4\} + t_1 \times t_2 \times t_3 \times t_4$, $k > 0$,
- $\theta(t_1, t_2, t_3, t_4) = ln(1 + t_1 t_2 t_3 t_4)$.

Definition 2.1. *Let X be a nonempty set. Then $(X, d, \leq)$ is called a partially ordered $b-$metric space if d is a $b-$metric on a partially ordered set $(X, \leq)$. The space $(X, d, \leq)$ is called regular if the following condition hold:*
if a non-decreasing $x_n \longrightarrow x$, then $x_n \leq x$ for all n.

Theorem 2.2. *Suppose that $(X, d, \leq)$ is a partially ordered complete b-metric space and $\{T_n\}$ be a nondecreasing sequence of self maps on X. If there exists a continuous function $\alpha : X \times X \longrightarrow [0,1)$ such that for all $x, y \in X$*

$$\alpha(T_i x, T_j y) \leq a_{i,j} \alpha(x,y)$$

and

$$\psi(s^3 d(T_i x, T_j y)) \leq \alpha(x,y)\varphi(M_{i,j}(x,y)) + \theta(d(x, T_i x), d(y, T_j y), d(x, T_j y), d(y, T_i x)), \quad (1)$$

for all $x, y \in X$ with $x \leq y$, where $(\psi, \varphi) \in \Psi$, $\theta \in \Theta$ and

$$M_{i,j}(x,y) = \max\{d(x,y), d(x,T_ix), d(y,T_jy), \frac{d(x,T_jy)+d(y,T_ix)}{2s}\},$$

and $0 \leq a_{i,j}$ $(i,j = 1,2,\cdots)$, satisfy

i) *for each n, $A_n = \prod_{i=1}^{n} a_{i,i+1} < 1$,*

ii) *for each j, $\overline{\lim_{i\longrightarrow\infty}} a_{i,j} < 1$.*

Suppose that

(i) *T is continuous, or*

(ii) *X is regular.*

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then all T_n's have a common fixed point in X.

Proof. If $x_0 = Tx_0$, then we have the result. Suppose that $x_0 < Tx_0$. Then we construct a sequence $\{x_n\}$ in X such that

$$x_n = T_n(x_{n-1}) \quad \forall n \in \mathbb{N}. \tag{2}$$

Since T is a nondecreasing mapping, we obtain by induction that

$$x_0 \leq T_1(x_0) = x_1 \leq T_2(x_1) = x_2 \leq \cdots \leq T_n(x_{n-1}) = x_n \leq T_{n+1}(x_n) = x_{n+1} \leq \cdots. \tag{3}$$

If there exists some $k \in \mathbb{N}$ such that $x_{k+1} = x_k$, then from (2), $x_{k+1} = T_{k+1}(x_k) = x_k$, that is, x_k is a common fixed point of T_k and the proof is finished. So, we suppose that $x_{n+1} \neq x_n$, for all $n \in \mathbb{N}$. Since $x_n < x_{n+1}$, for all $n \in \mathbb{N}$, we set $x = x_n$ and $y = x_{n+1}$ in (1), we have

$$\begin{aligned}
\psi(d(x_n,x_{n+1})) =& \psi(d(T_n(x_{n-1}),T_{n+1}(x_n))) \\
\leq& \psi(s^3 d(T_n(x_{n-1}),T_{n+1}(x_n))) \\
\leq& \alpha(x_{n-1},x_n)\varphi(M_{n,n+1}(x_{n-1},x_n)) \\
&+ \theta(d(x_{n-1},T_n(x_{n-1})), d(x_n,T_{n+1}(x_n)), d(x_{n-1},T_{n+1}(x_n)), d(x_n,T_n(x_{n-1}))) \\
=& \alpha(x_{n-1},x_n)\varphi(M_{n,n+1}(x_{n-1},x_n)) \\
&+ \theta(d(x_{n-1},x_n), d(x_n,x_{n+1}), d(x_{n-1},x_{n+1}), d(x_n,x_n)).
\end{aligned}$$

Since

$$\frac{d(x_{n-1},x_{n+1})}{2s} \leq \frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2s} \leq \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\},$$

then we get,

$$\begin{aligned}
M_{n,n+1}(x_{n-1},x_n) =& \max\{d(x_{n-1},x_n), d(x_{n-1},T_n(x_{n-1})), d(x_n,T_{n+1}(x_n)), \\
&\frac{d(x_{n-1},T_{n+1}(x_n))+d(x_n,T_n(x_{n-1}))}{2s}\} \\
=& \max\{d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}), \\
&\frac{d(x_{n-1},x_{n+1})+d(x_n,x_n)}{2s}\} \\
=& \max\{d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{1}{2s}d(x_{n-1},x_{n+1})\} \\
=& \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\},
\end{aligned}$$

that is,

$$\begin{aligned}\psi(d(x_n,x_{n+1})) &\le \alpha(x_{n-1},x_n)\varphi(\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\})\\ &= \alpha(T_{n-1}(x_{n-2}),T_n(x_{n-1}))\varphi(\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\})\\ &\le a_{n-1,n}\,\alpha(x_{n-2},x_{n-1})\varphi(\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\})\\ &\vdots \\ &\le \prod_{i=1}^{n-1} a_{i,i+1}\,\alpha(x_0,x_1)\varphi(\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\})\\ &= A_{n-1}\,\alpha(x_0,x_1)\varphi(\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\})\end{aligned} \tag{4}$$

Consider the following cases:

Case 1. If $M_{n,n+1}(x_{n-1},x_n)=d(x_n,x_{n+1})$, then by (4), we have

$$\begin{aligned}\psi(d(x_n,x_{n+1})\}) &\le A_{n-1}\ \alpha(x_0,x_1)\varphi(d(x_n,x_{n+1}))\\ &< A_{n-1}\ \alpha(x_0,x_1)\psi(d(x_n,x_{n+1}))\\ &< \psi(d(x_n,x_{n+1})),\end{aligned}$$

which is a contradiction.

Case 2. If $M_{n,n+1}(x_{n-1},x_n)=d(x_{n-1},x_n)$, then by (4), we have

$$\begin{aligned}\psi(d(x_n,x_{n+1})\}) &\le A_{n-1}\ \alpha(x_0,x_1)\varphi(d(x_{n-1},x_n))\\ &< A_{n-1}\ \alpha(x_0,x_1)\psi(d(x_{n-1},x_n))\\ &< \psi(d(x_{n-1},x_n)).\end{aligned} \tag{5}$$

Using the properties of the function ψ, we get

$$B_n=d(x_n,x_{n+1})\le d(x_{n-1},x_n)=B_{n-1}.$$

Then the sequence $\{B_n\}$ is non-increasing and bounded below, therefor there exists $B\ge 0$ such that,

$$\lim_{n\longrightarrow\infty} B_n = B. \tag{6}$$

We show that $B=0$. Suppose, on the contrary, that $B>0$. Taking the limit as $n\longrightarrow\infty$ in (5) and ψ and φ are continuous, we get

$$\psi(B)=\varphi(B),$$

and so $B=0$, a contradiction. Thus

$$\lim_{n\longrightarrow\infty} d(x_n,x_{n+1})=0. \tag{7}$$

Now, we claim that

$$\lim_{n,m\longrightarrow\infty} d(x_n,x_m)=0. \tag{8}$$

Assume on the contrary that there exists $0<\epsilon<1$ and subsequences $\{x_{m(k)}\}$, $\{x_{n(k)}\}$ of $\{x_n\}$ with $m(k)>n(k)\ge k$ such that

$$d(x_{n(k)},x_{m(k)})\ge\epsilon. \tag{9}$$

Additionally, corresponding to $n(k)$, we may choose $m(k)$ such that it is the smallest integer satisfying (9) and $m(k)>n(k)\ge k$. Thus,

$$d(x_{n(k)},x_{m(k)-1})<\epsilon. \tag{10}$$

Using the triangle inequality in $b-$metric space and (9) and (10), we obtain that

$$\begin{aligned}\epsilon \leq d(x_{m(k)}, x_{n(k)}) &\leq s\ d(x_{m(k)}, x_{m(k)-1}) + s\ d(x_{m(k)-1}, x_{n(k)})\\ &< s\ d(x_{m(k)}, x_{m(k)-1}) + s\epsilon.\end{aligned}$$

Taking the upper limit as $k \longrightarrow \infty$ and using (7), we obtain

$$\epsilon \leq \limsup_{k\longrightarrow\infty} d(x_{n(k)}, x_{m(k)}) \leq s\epsilon. \tag{11}$$

Also

$$\begin{aligned}\epsilon &\leq d(x_{n(k)}, x_{m(k)}) \leq s\ d(x_{n(k)}, x_{m(k)+1}) + s\ d(x_{m(k)+1}, x_{m(k)})\\ &\leq s^2\ d(x_{n(k)}, x_{m(k)}) + s^2\ d(x_{m(k)}, x_{m(k)+1}) + s\ d(x_{m(k)+1}, x_{m(k)})\\ &\leq s^2\ d(x_{n(k)}, x_{m(k)}) + (s^2+s)\ d(x_{m(k)}, x_{m(k)+1}).\end{aligned}$$

So from (7) and (11), we have

$$\frac{\epsilon}{s} \leq \limsup_{k\longrightarrow\infty} d(x_{n(k)}, x_{m(k)+1}) \leq s^2\epsilon. \tag{12}$$

Also

$$\begin{aligned}\epsilon \leq d(x_{m(k)}, x_{n(k)}) &\leq s\ d(x_{m(k)}, x_{n(k)+1}) + s\ d(x_{n(k)+1}, x_{n(k)})\\ &\leq s^2\ d(x_{m(k)}, x_{n(k)}) + s^2\ d(x_{n(k)}, x_{n(k)+1}) + s\ d(gx_{n(k)+1}, gx_{n(k)})\\ &\leq s^2\ d(x_{m(k)}, x_{n(k)}) + (s^2+s)\ d(x_{n(k)}, x_{n(k)+1}).\end{aligned}$$

So from (7) and (11), we have

$$\frac{\epsilon}{s} \leq \limsup_{k\longrightarrow\infty} d(x_{m(k)}, x_{n(k)+1}) \leq s^2\epsilon. \tag{13}$$

Also

$$d(x_{n(k)+1}, x_{m(k)}) \leq s\ d(x_{n(k)+1}, x_{m(k)+1}) + s\ d(x_{m(k)+1}, x_{m(k)}),$$

so from (7) and (12), we have

$$\frac{\epsilon}{s^2} \leq \limsup_{k\longrightarrow\infty} d(x_{n(k)+1}, x_{m(k)+1}). \tag{14}$$

Linking (7),(11),(12) together with (13), we get

$$\begin{aligned}\frac{\epsilon}{s^2} &= \min\{\epsilon, 0, 0, \frac{\frac{\epsilon}{s}+\frac{\epsilon}{s}}{2s}\}\\ &\leq \max\{\limsup_{k\longrightarrow\infty} d(x_{n(k)}, x_{m(k)}), \limsup_{k\longrightarrow\infty} d(x_{n(k)}, x_{n(k)+1}), \limsup_{k\longrightarrow\infty} d(x_{m(k)}, x_{m(k)+1}),\\ &\quad \limsup_{k\longrightarrow\infty} \frac{d(x_{m(k)}, x_{n(k)+1}) + d(x_{n(k)}, x_{m(k)+1})}{2s}\\ &\leq \max\{s\epsilon, 0, 0, \frac{s^2\epsilon + s^2\epsilon}{2s}\} = s\epsilon.\end{aligned}$$

So,

$$\frac{\epsilon}{s^2} \leq \limsup_{k\longrightarrow\infty} M_{n(k)+1, m(k)+1}(x_{n(k)}, x_{m(k)}) \leq \epsilon s. \tag{15}$$

Similarly, we have

$$\frac{\epsilon}{s^2} \leq \liminf_{k\longrightarrow\infty} M_{n(k)+1, m(k)+1}(x_{n(k)}, x_{m(k)}) \leq \epsilon s. \tag{16}$$

Also, we have

$$\lim_{k\longrightarrow\infty}\theta(d(x_{n(k)},T_{n(k)+1}(x_{n(k)})),d(x_{m(k)},T_{m(k)+1}(x_{m(k)})),d(x_{n(k)},T_{m(k)+1}(x_{m(k)})),d(x_{m(k)},T_{n(k)+1}(x_{n(k)})))$$
$$=\lim_{k\longrightarrow\infty}\theta(d(x_{n(k)},x_{n(k)+1}),d(x_{m(k)},x_{m(k)+1}),d(x_{n(k)},x_{m(k)+1}),d(x_{m(k)},x_{n(k)+1}))=0.$$

Since $m(k)>n(k)$ from (3), we have

$$x_{n(k)}\leq x_{m(k)}.$$

Thus,

$$\begin{aligned}\psi(s^3 d(x_{n(k)+1},x_{m(k)+1})) &= \psi(s^3\ d(T_{n(k)+1}(x_{n(k)}),T_{m(k)+1}(x_{m(k)})))\\ &\leq \alpha(x_{n(k)},x_{m(k)})\varphi(M_{n(k)+1,m(k)+1}(x_{n(k)},x_{m(k)}))\\ &= \alpha(T_{n(k)}(x_{n(k)-1}),T_{m(k)}(x_{m(k)-1}))\varphi(M_{n(k)+1,m(k)+1}(x_{n(k)},x_{m(k)}))\\ &\leq a_{n(k),m(k)}\ \alpha(x_{n(k)-1},x_{m(k)-1})\varphi(M_{n(k)+1,m(k)+1}(x_{n(k)},x_{m(k)}))\\ &\vdots\\ &\leq a_{n(k),m(k)}\times a_{n(k)-1,m(k)-1}\times\cdots a_{0,m(k)-n(k)+1}\ \alpha(x_0,x_{m(k)-n(k)})\\ &\quad\ \varphi(M_{n(k)+1,m(k)+1}(x_{n(k)},x_{m(k)})),\end{aligned}$$

Taking the upper limit as $k\longrightarrow\infty$, and using (14), (15) and (16), we get

$$\begin{aligned}\psi(s\,\epsilon) &\leq \psi(s^3\limsup_{k\longrightarrow\infty} d(x_{n(k)+1},x_{m(k)+1}))\\ &\leq \limsup_{k\longrightarrow\infty}[a_{n(k),m(k)}\times a_{n(k)-1,m(k)-1}\times\cdots a_{0,m(k)-n(k)+1}\ \alpha(x_0,x_{m(k)-n(k)})]\\ &\quad\ \varphi(\limsup_{k\longrightarrow\infty} M_{n(k)+1,m(k)+1}(x_{n(k)},x_{m(k)}))\\ &< \limsup_{k\longrightarrow\infty}[a_{n(k),m(k)}\times a_{n(k)-1,m(k)-1}\times\cdots a_{0,m(k)-n(k)+1}\ \alpha(x_0,x_{m(k)-n(k)})]\\ &\quad\ \psi(\limsup_{k\longrightarrow\infty} M_{n(k)+1,m(k)+1}(x_{n(k)},x_{m(k)}))\\ &< \psi(s\,\epsilon),\end{aligned}$$

this is contradiction. Therefore, (8) holds and we have

$$\lim_{n,m\longrightarrow\infty} d(x_n,x_m)=0.$$

Since X is a complete b-metric space, there exist $x\in X$ such that

$$\lim_{n\longrightarrow\infty} x_{n+1}=x. \tag{17}$$

Case1. Suppose that the assumption (i) holds.
Letting $n\longrightarrow\infty$ in (18) and from the continuity of T, we get

$$x=\lim_{n\longrightarrow\infty} x_{n+1}=\lim_{n\longrightarrow\infty} T(x_n)=T(\lim_{n\longrightarrow\infty} x_n)=T(x).$$

This implies that x is a fixed point of T.

Case2. Suppose that the assumption (ii) holds.
Since $\{x_n\}$ is a non-decreasing sequence and $x_n\longrightarrow x$, hence we have $x_n\leq x$ for all n, then by the triangle inequality in $b-$metric space and (1) and for any positive integer m, we get

$$d(x,T_m x)\leq sd(x,x_n)+sd(x_n,T_m x)=sd(x,x_n)+sd(T_n x_{n-1},T_m x).$$

Taking $\overline{\lim}$ as $n \longrightarrow \infty$, we get

$$d(x, T_m x) \leq \overline{\lim_{n\to\infty}} s d(T_n x_{n-1}, T_m x).$$

Because

$$\begin{aligned} M_{n,m}(x_{n-1}, x) &= \max\{d(x_{n-1}, x), d(x_{n-1}, T_n x_{n-1}), d(x, T_m x), \frac{d(x_{n-1}, T_m x) + d(x, T_n x_{n-1})}{2s}\} \\ &= \max\{d(x_{n-1}, x), d(x_{n-1}, x_n), d(x, T_m x), \frac{d(x_{n-1}, T_m x) + d(x, x_n)}{2s}\}, \end{aligned}$$

and

$$\begin{aligned} &\theta(d(x_{n-1}, T_n(x_{n-1}), d(x, T_m x), d(x_{n-1}, T_m x), d(x, T_n(x_{n-1}))) \\ &= \theta(d(x_{n-1}, x_n), d(x, T_m x), d(x_{n-1}, T_m x), d(x, x_n)), \end{aligned}$$

which,

$$\overline{\lim_{n\longrightarrow\infty}} M_{n,m}(x_{n-1}, x) = d(x, T_m x),$$

$$\overline{\lim_{n\longrightarrow\infty}} \theta(d(x_{n-1}, x_n), d(x, T_m x), d(x_{n-1}, T_m x), d(x, x_n)) = 0$$

So,

$$\begin{aligned} \psi(d(x, T_m x)) &\leq \overline{\lim_{n\to\infty}} \psi(s d(T_n x_{n-1}, T_m x)) \leq \overline{\lim_{n\to\infty}} \psi(s^3 d(T_n x_{n-1}, T_m x)) \\ &\leq \overline{\lim_{n\to\infty}} \alpha(x_{n-1}, x) \varphi(M_{n,m}(x_{n-1}, x)) < \alpha(x, x) \psi(d(x, T_m x)) \\ &< \psi(d(x, T_m x)), \end{aligned}$$

this is contradiction. It follows that $d(x, T_m x) = 0$ gives x as a common fixed point of $\{T_m\}$.

$\square$

Example 2.3. *Let $X = [0,1]$ with the usual order $\leq$. Define $d(x,y) = |x-y|^2$. Then d is a $b-$metric with $s = 2$. Also define $\psi(t) = t$, $\varphi(t) = \frac{1}{2}t$, $\alpha(x,y) = \frac{xy}{4}$,*

$$T_n(x) = \begin{cases} 0, & 0 \leq x < 1, \\ \frac{1}{n+24}, & x = 1, \end{cases}$$

and $\theta \in \Theta$ is arbitrary function. Let $a_{i,j} = \frac{1}{3} + \frac{1}{|i-j|+2}$, then for each j, $\overline{\lim_{i\longrightarrow\infty}} a_{i,j} < 1$ and $A_n = (\frac{2}{3})^n$. Obviously,

$$\alpha(?T_i x, T_j y) \leq a_{i,j}\, \alpha(x,y).$$

for all $x \in [0,1)$, $y \in [0,1)$ or $x \in [0,1)$, $y = 1$ or $y \in [0,1)$, $x = 1$. Also, if $x = y = 1$, we have

$$\alpha(?T_i x, T_j y) = \frac{\frac{1}{i+24} . \frac{1}{j+24}}{4} \leq \frac{\frac{1}{25} . \frac{1}{25}}{4} \leq \frac{\frac{1}{3}}{4} \leq \frac{\frac{1}{3} + \frac{1}{|i-j|+2}}{4} = a_{i,j} \frac{1}{4} = a_{i,j} \frac{xy}{4} = a_{i,j} \alpha(x,y).$$

Now we prove that for each comparable $x, y \in X$,

$$\psi(s^3 d(T_i x, T_j y)) \leq \alpha(x,y)\, \varphi(M_{i,j}(x,y)) + \theta(d(x, T_i x), d(y, T_j y), d(x, T_j y), d(y, T_i x)), \quad (18)$$

Obviously, if $x \in [0,1)$, $y \in [0,1)$, then the condition (18) holds. If $x \in [0,1)$, $y = 1$, then

$$\begin{aligned}\psi(s^3 d(T_ix, T_jy)) =& 8|T_ix - T_jy|^2 = 8|\frac{1}{j+24}|^2 \leq \frac{8}{625} \leq \frac{1}{4}\frac{1}{2}(\frac{24}{25})^2\\ \leq& \frac{1}{4}\frac{1}{2}(1-\frac{1}{j+24})^2 = \frac{xy}{4}\frac{1}{2}d(y, T_jy)\\ \leq& \alpha(x,y)\frac{1}{2}M_{i,j}(x,y) = \alpha(x,y)\varphi(M_{i,j}(x,y))\\ \leq& \alpha(x,y)\varphi(M_{i,j}(x,y)) + \theta(d(x,T_ix), d(y,T_jy), d(x,T_jy), d(y,T_ix)).\end{aligned}$$

We assume that (18) holds for $y \in [0,1)$, $x = 1$. Now, if $x = y = 1$, $i < j$, then

$$\begin{aligned}\psi(s^3 d(T_ix, T_jy)) =& 8|T_ix - T_jy|^2 = 8|\frac{1}{i+24} - \frac{1}{j+24}|^2 \leq \frac{8}{(i+24)^2} \leq \frac{8}{25^2} \leq \frac{1}{8}(\frac{24}{25})^2\\ \leq& \frac{xy}{4}\frac{1}{2}(1-\frac{1}{i+24})^2 = \alpha(x,y)\frac{1}{2}d(x,T_ix) \leq \alpha(x,y)\frac{1}{2}M_{i,j}(x,y)\\ =& \alpha(x,y)\varphi(M_{i,j}(x,y))\\ \leq& \alpha(x,y)\varphi(M_{i,j}(x,y)) + \theta(d(x,T_ix), d(y,T_jy), d(x,T_jy), d(y,T_ix)).\end{aligned}$$

Moreover, $\{T_n\}$ are nondecreasing mappings with respect to the usual order $\leq$. So all the conditions of Theorem 2.2 are satisfied and note that $x = 0$ is the only fixed point for all T_n.

The following result is the immediate consequence of Theorem 2.2.

Theorem 2.4. *Suppose that $(X, d, \leq)$ is a partially ordered complete b-metric space and $\{T_n\}$ be a nondecreasing sequence of self maps on X. If there exists a $\lambda \in [0,1)$ such that for all $x, y \in X$ with $x \leq y$,*

$$\psi(s^3 d(T_ix, T_jy)) \leq \lambda\,\varphi(M_{i,j}(x,y)) + \theta(d(x,T_ix), d(y,T_jy), d(x,T_jy), d(y,T_ix)), \tag{19}$$

where $(\psi, \varphi) \in \Psi$, $\theta \in \Theta$ and

$$M_{i,j}(x,y) = \max\{d(x,y), d(x,T_ix), d(y,T_jy), \frac{d(x,T_jy) + d(y,T_ix)}{2s}\}.$$

Suppose that

(i) *T is continuous, or*

(ii) *X is regular.*

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then all T_n's have a common fixed point in X.

We can prove this result by applying Theorem 2.2 with $\alpha(x,y) = \lambda$.

Example 2.5. *Let $X = [0,1]$ with the usual order $\leq$. Define $d(x,y) = |x-y|^2$. Then d is a $b-$metric with $s = 2$. Also define $\psi(t) = t$, $\varphi(t) = \frac{1}{2}t$, $\lambda = \frac{1}{4}$,*

$$T_n(x) = \begin{cases} 1, & 0 < x \leq 1,\\ \frac{15}{16} + \frac{1}{n+15}, & x = 0,\end{cases}$$

and $\theta \in \Theta$ is arbitrary function. Now we prove that for each comparable $x, y \in X$,

$$\psi(s^3 d(T_ix, T_jy)) \leq \lambda\,\varphi(M_{i,j}(x,y)) + \theta(d(x,T_ix), d(y,T_jy), d(x,T_jy), d(y,T_ix)),$$

There are three possible cases:

(1) $x \in (0,1]$, $y \in (0,1]$. *Then*

$$\begin{aligned}\psi(s^3 d(T_i x, T_j y)) =& 8|T_i x - T_j y|^2 = 0 \leq \frac{1}{8}\frac{|x-1|^2 + |y-1|^2}{4} = \frac{1}{8}\frac{d(x,T_j y) + d(y, T_i x)}{2s} \\ \leq& \frac{1}{8} M_{i,j}(x,y) = \lambda\varphi(M_{i,j}(x,y)) \\ \leq& \lambda\varphi(M_{i,j}(x,y)) + \theta(d(x,T_i x), d(y,T_j y), d(x,T_j y), d(y,T_i x)).\end{aligned}$$

(2) $x \in (0,1]$, $y = 0$. *Then*

$$\begin{aligned}\psi(s^3 d(T_i x, T_j y)) =& 8|T_i x - T_j y|^2 = 8|\frac{1}{16} - \frac{1}{j+15}|^2 \leq \frac{1}{32} \leq \frac{1}{4}\frac{1}{2}(\frac{15}{16})^2 \\ \leq& \frac{1}{4}\frac{1}{2}(\frac{15}{16} + \frac{1}{j+15})^2 = \frac{1}{4}\frac{1}{2} d(y, T_j y) \\ \leq& \frac{1}{4}\frac{1}{2} M_{i,j}(x,y) = \lambda\varphi(M_{i,j}(x,y)) \\ \leq& \lambda\varphi(M_{i,j}(x,y)) + \theta(d(x,T_i x), d(y,T_j y), d(x,T_j y), d(y,T_i x)).\end{aligned}$$

(3) $x = y = 0$, $i < j$. *Then*

$$\begin{aligned}\psi(s^3 d(T_i x, T_j y)) =& 8|T_i x - T_j y|^2 = 8|\frac{1}{i+15} - \frac{1}{j+15}|^2 \leq \frac{8}{(i+15)^2} \leq \frac{8}{16^2} \\ \leq& \frac{1}{8}(\frac{15}{16})^2 \leq \frac{1}{8}(\frac{15}{16} + \frac{1}{i+15})^2 = \frac{1}{8} d(x, T_i x) \leq \frac{1}{4}\frac{1}{2} M_{i,j}(x,y) = \lambda\varphi(M_{i,j}(x,y)) \\ \leq& \lambda\varphi(M_{i,j}(x,y)) + \theta(d(x,T_i x), d(y,T_j y), d(x,T_j y), d(y,T_i x)).\end{aligned}$$

Moreover, $\{T_n\}$ are nondecreasing mappings with respect to the usual order $\leq$. So all the conditions of Theorem 2.4 are satisfied and note that $x = 1$ is the only fixed point for all T_n.

Corollary 2.6. *Suppose that $(X, d, \leq)$ is a partially ordered complete b-metric space and $\{T_n\}$ be a nondecreasing sequence of self maps on X. If there exists a $k \in [0,1)$ such that for all $x, y \in X$ with $x \leq y$,*

$$\psi(s^3 d(T_i x, T_j y)) \leq k\,\psi(M_{i,j}(x,y)) + \theta(d(x,T_i x), d(y,T_j y), d(x,T_j y), d(y,T_i x)),$$

where φ, θ and $M_{i,j}(x,y)$ satisfy in conditions theorem 2.4 and suppose that

(i) *T is continuous, or*
(ii) *X is regular.*

If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then all T_n's have a common fixed point in X.

Proof. We can prove this result by applying Theorem 2.4 with $\varphi(t) = k\ \psi(t)$, $0 \leq k < 1$. □

Remark 2.7. *Letting $\psi(t) = t$ and $\varphi(t) = k\,t$ with $0 \leq k < 1$ in Theorem 2.4, we retrieve immediately the Banach contraction principle and we obtain a generalized version of KhanNi, Ra.*

Remark 2.8. *Since a b-metric space is a metric space when $s = 1$, so our results can be viewed as the generalization and the extension of several comparable results.*

3. Application to integral equations

Here, in this section, we wish to study the existence of a solution to a nonlinear quadratic integral equation, as an application to the our common fixed point theorem. Consider the integral equation

$$x(t) = h(t) + \gamma \int_0^1 k(t,s) f_n(s, x(s)) ds, \ t \in I = [0,1], \gamma \geq 0, \ n \in \mathbb{N}. \tag{20}$$

Let Φ *denote the class of those non-decreasing functions* $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ *which* $\varphi(t) < t$, $t > 0$, $\varphi(0) = 0$ *and* $(\varphi(t))^2 \leq \varphi(t^2)$.
For example, $\varphi_1(t) = kt$, *where* $0 \leq k < 1$ *and* $\varphi_2(t) = \frac{t}{t+1}$ *are in* Φ.
We will analyze Eq. (20) under the following assumptions:

(a_1) $f_n : I \times \mathbb{R} \longrightarrow \mathbb{R}$ *is continuous monotone non-decreasing in* x, $f_n(t,x) \geq 0$ *and there exists constant* $0 \leq L_{i,j} < 1$ *and* $\varphi \in \Phi$ *such that for all* $x, y \in \mathbb{R}$ *and* $x \geq y$

$$|f_i(t,x) - f_j(t,y)| \leq L_{i,j} \varphi(x - y).$$

(a_2) $h : I \longrightarrow \mathbb{R}$ *is a continuous function.*

(a_3) $k : I \times I \longrightarrow \mathbb{R}$ *is continuous in* $t \in I$ *for every* $s \in I$ *and measurable in* $s \in I$ *for all* $t \in I$ *such that*

$$\int_0^1 k(t,s) ds \leq K$$

and $k(t,s) \geq 0$.

(a_4) *There exists* $\alpha \in C(I)$ *such that*

$$\alpha(t) \leq h(t) + \gamma \int_0^1 k(t,s) f_n(s, \alpha(s)) ds$$

(a_5) $8L_{i,j}^2 \gamma^2 K^2 = \lambda < 1$.

We consider the space $X = C(I)$ *of continuous functions defined on* $I = [0,1]$ *with the standard metric given by*

$$\rho(x,y) = \sup_{t \in I} |x(t) - y(t)|, \ for \ x, y \in C(I).$$

This space can also be equipped with a partial order given by

$$x, y \in C(I), \ x \leq y \Longleftrightarrow x(t) \leq y(t) \ for \ any \ t \in I.$$

Now, we define

$$d(x,y) = (\rho(x,y))^2 = (\sup_{t \in I} |x(t) - y(t)|)^2 = \sup_{t \in I} |x(t) - y(t)|^2, \ for \ x, y \in C(I).$$

It is easy to see that (X, d) *is a complete* $b-$*metric space. For any* $x, y \in X$ *and each* $t \in I$, $max\{x(t), y(t)\}$ *and* $min\{x(t), y(t)\}$ *belong to* X *and are upper and lower bounds of* x, y, *respectively. Therefore, for every* $x, y \in X$, *one can take* $max\{x, y\}, min\{x, y\} \in X$ *which are comparable to* x, y. *Now, we formulate the main result of this section.*

Theorem 3.1. *Under assumptions* $(a_1) - (a_5)$, *Eq. (20) has a solution in* $C(I)$.

Proof. *We consider the operator* $T_n : X \longrightarrow X$ *defined by*

$$T_n(x)(t) = h(t) + \gamma \int_0^1 k(t,s) f_n(s, x(s)) ds, \ for \ t \in I, \ n \in \mathbb{N}.$$

By virtue of our assumptions, T_n is well defined (this means that if $x \in X$ then $T_n(x) \in X$). For $x \leq y$, and $t \in I$ we have

$$T_n(x)(t) - T_n(y)(t) = h(t) + \gamma \int_0^1 k(t,s) f_n(s, x(s)) ds - h(t) - \gamma \int_0^1 k(t,s) f_n(s, y(s)) ds$$
$$= \gamma \int_0^1 k(t,s)[f_n(s, x(s)) - f_n(s, y(s))] ds \leq 0.$$

Therefore, T_n has the monotone nondecreasing property. Also, for $x \leq y$, we have

$$\begin{aligned}|T_i(x)(t) - Tj(y)(t)| &= |h(t) + \gamma \int_0^1 k(t,s) f_i(s, x(s)) ds - h(t) - \gamma \int_0^1 k(t,s) f_j(s, y(s)) ds| \\ &\leq \gamma \int_0^1 k(t,s) |f_i(s, x(s)) - f_j(s, y(s))| ds \\ &\leq \gamma \int_0^1 k(t,s) L_{i,j} \varphi(y(s) - x(s)) ds.\end{aligned}$$

Since the function φ is non-decreasing and $x \leq y$, we have

$$\varphi(y(s) - x(s)) \leq \varphi(\sup_{t \in I} |x(s) - y(s)|) = \varphi(\rho(x,y)),$$

hence

$$|T_i(x)(t) - T_j(y)(t)| \leq \gamma \int_0^1 k(t,s) L_{i,j} \varphi(\rho(x,y) ds \leq \lambda K L_{i,j} \varphi(\rho(x,y).$$

Then, we can obtain

$$\begin{aligned}\psi(s^3 d(T_i(x), T_j(y))) =& 8d(T_i(x), T_j(y)) = 8 \sup_{t \in I} |T_i(x)(t) - T_j(y)(t)|^2 \\ \leq& 8\{\gamma K L_{i,j} \varphi(\rho(x,y))\}^2 = 8\gamma^2 K^2 L_{i,j}^2 \varphi(\rho(x,y))^2 \\ \leq& \lambda \varphi(\rho(x,y)^2) = \lambda \varphi(d(x,y)) \\ \leq& \lambda\, \varphi(M_{i,j}(x,y)) + \theta(d(x, T_i x), d(y, T_j y), d(x, T_j y), d(y, T_i x)).\end{aligned}$$

This proves that the operator T_n satisfies the contractive condition (19) appearing in Theorem 2.4. Also, let α be the function appearing in assumption (a_4); then, by (a_4), we get $\alpha \leq T_n(\alpha)$. So, the Eq. (20) has a solution and the proof is complete.

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