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HIGHER ORDER DUALITY IN NON-DIFFERENTIABLE MULTIPLICATIVE PROGRAMMING WITH GENERALIZED CONVEX FUNCTIONS OVER CONES

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ABSTRACT. In this paper, we consider a new class of generalized higher order (F, α, ρ, d)-convex and (F, α, ρ, d)-pseudo convex functions with examples. Mond-Weir type higher order duality is proposed for the non-differentiable multiobjective problem involving cone constraints, where every component of the objective function contains square root term of positive semidefinite quadratic form. For this problem, duality results are established for efficient solution under suitable higher order (F, α, ρ, d)-convexity conditions.

1. Introduction

In recent years, several extension and generalization have been considered for classical convexity. Hanson and Mond [14] introduced the concept of F-convexity. The concept of generalized (F, ρ) convexity is introduced by Preda [15]. Based on the various generalized convex functions, Liang et al.[8] introduced a unified formulation of generalized convex function, called (F, α, ρ, d)-convex function. Ahmad and Husain [2] generalized (F, α, ρ, d)-convex functions to second order (F, α, ρ, d)-convex functions and discussed duality theorems for second order Mond-Weir type multiobjective dual model under (F, α, ρ, d)-convexity/pseudo convexity assumption. Ahmad et al.[1] introduced a new generalized higher order (F, α, ρ, d)-type 1 function and proposed a general Mond-Weir type higher order dual and established the duality results under higher order (F, α, ρ, d)-type 1 function .

Higher order duality in non-linear programming has been studied in last few years by many researchers. One practical advantage of second order and higher order duality is that it provides tighter bounds for the value of the objective function of the primal problem when approximations are used because there are more parameters involved. Mangasarian [9] first formulated a class of second order and higher order for non-linear programming problem involving twice differentiable functions. Higher order duality has been studied by many researchers like Chen [3], Mond and

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Zhang [14], Yang et al. [19, 20], Zhang [21], Kim and Lee [7], Mishra and Rueda [10, 11].

Mishra and Rueda [11] considered higher order duality for non-differentiable mathematical programming problem. They formulated a number of higher order duals to a non-differentiable programming problem and established duality under the higher order generalized invexity conditions introduced in Mishra and Rueda [10]. Zhang [21] introduced higher-order $((F, \rho)$-convexity and established higher-order duality for multi-objective programming problems, there by extending the results of Gulati and Islam [4], Mangasarian [9], Preda [15], Mishra and Rueda [11] and Mond and Weir [12].

In this paper, the concept of higher order (F, α, ρ, d)-convexity and higher order (F, α, ρ, d)-pseudoconvexity are introduced with examples. A pair of Mond-Weir type higher order dual programs is considered for the non-differentiable multiobjective problems involving cone constraints, where every component of objective functions contain a square root term of positive semidefinite quadratic form and established the duality results under suitable higher order (F, α, ρ, d)-convexity conditions.

2. Preliminaries and Definitions

The following conventions for vectors in R^n will be followed throughout this paper: $x < y \Leftrightarrow x_i < y_i$, for $i = 1, 2, ..., n$, $x \leq y \Leftrightarrow x_i \leq y_i, i = 1, 2, ..., n, x \neq y$,

Definition 2.1 A set C is called a cone, if for each $x \in C$ and $\lambda \in R, \lambda \geq 0$, we have $\lambda x \in C$. Moreover, if C is convex, then it is a convex cone.

Definition 2.2 The positive polar cone C^* of a cone C is defined by

$$C^* = \{z : x^T z \geq 0, \forall x \in C\}$$

Let $C_1 \subset R^n$ and $C_2 \subset R^m$ be closed convex cones with nonempty interior having positive polars C_1^* and C_2^* respectively. Let $S_1 \subseteq R^n$ and $S_2 \subseteq R^m$ be open and $C_1 \subseteq S_1$ and $C_2 \subseteq S_2$.
We now introduced higher order (F, α, ρ, d) convex function and pseudo convex function.

Definition 2.3 A function $F : S_1 \times S_1 \times R^n \to R$ is sublinear in its third component if for all $(x, u) \in S_1 \times S_1$,
(i) $F(x, u; a_1 + a_2) \leq F(x, u; a_1) + F(x, u; a_2), \forall a_1, a_2 \in R^n$ and
(ii)$F(x, u; \alpha a) = \alpha F(x, u; a), \alpha \geq 0, \forall a \in R^n$.
For simplicity, we denote $F(x, u; a) = F_{x,u}(a)$

Definition 2.4 A twice differentiable function $f_i : S_1 \to R$ is called higher order (F, α, ρ, d)-convex function at $u \in S_1$ with respect to a differentiable function $h_i : S_1 \times S_1 \to R$ if for all $x \in S_1$ there exist real valued function $\alpha : S_1 \times S_1 \to R_+\backslash\{0\}, d : S_1 \times S_1 \to R$ and $\rho \in R$ such that
$f_i(x) - f_i(u) - h_i(u, p) + p^T \nabla_p h_i(u, p) \geq F(x, u; \alpha(x, u)(\nabla f_i(u) + \nabla_p h_i(u, p)) + \rho d^2(x, u)$.

Definition 2.5 A twice differentiable function $f_i : S_1 \to R$ is called higher order (F, α, ρ, d)-pseudo convex function at $u \in S_1$ with respect to a differentiable function $h_i : S_1 \times S_1 \to R$ if for all $x \in S_1$ there exist real valued function α :

$S_1 \times S_1 \to R_+ \backslash \{0\}, d : S_1 \times S_1 \to R$ and $\rho \in R$ such that
$F(x, u; \alpha(x, u)(\nabla f_i(u) + \nabla_p h_i(u, p)) + \rho d^2(x, u) \geq 0$
$\Rightarrow f_i(x) - f_i(u) - h_i(u, p) + p^T \nabla_p h_i(u, p) \geq 0.$

Example 2.1 Let $S \subseteq R, S = \{x : x \geq 1, f : S \to R, F : S \times S \times R \to R$, $h : S \times R \to R$ and $d : S \times S \to R$ defined as follows;

$$f(x) = x + \frac{2}{x+1}, F(x, u; a) = |a|(x-u)^2, h(u, p) = \frac{p}{u+1}, d(x, u) = x - u.$$

And let $u = 1, \rho = -1, \alpha(x, u) = \frac{3}{4}$. Then for all $(x, p) \in S \times R$,

$$f(x) - f(u) = \frac{x^2-x}{x+1} \geq F(x, u; \alpha(x, u)[\nabla_x f(u) + \nabla_p(h(u, p))])$$

$$+h(u, p) - p^T \nabla_p h(u, p) + \rho d^2(x, u) = -\frac{1}{4}(x-1)^2.$$

This implies that f is higher order (F, α, ρ, d)-convex function at $u = 1$ with respect to $h(u, p)$.

But when we take $x = 2, p = 3$ and $x = 6, p = 3$ respectively, we have

$$f(2) - f(1) = \frac{2}{3} < F(x, u; (\nabla_x f(u) + \nabla_p h(u, p))) + h(u, p) - p^T \nabla_p h(u, p) = \frac{3}{4}$$

and $f(6) - f(1) + \frac{1}{2} p^T \nabla_{uu} f(u) p = \frac{1488}{343} = 4.39$

$$< F(x, u; (\nabla_x f(u) + \nabla_{xx} f(u) p)) + \rho d^2(x, u) = \frac{475}{16} = 29.6.$$

Hence f is neither a higher order F-convex function as in [3] nor a second order (F, α, ρ, d)-convex function as in [2]. So higher order (F, α, ρ, d)-convex function is more generalized than higher order F-convex function as in [3] nor a second order (F, α, ρ, d)-convex function as in [2].

Example 2.2 Let $S \subseteq R, S = \{x : x \geq 1\}, f : S \to R, F : S \times S \times R \to R$, $h : S \times R \to R$ and $d : S \times S \to R$ defined as follows;

$$f(x) = \frac{1+x^2}{x+1}, F(x, u; a) = |a|(x-u)^2, h(u, p) = \frac{p}{u+1}, d(x, u) = x - u.$$

And let $u = 1, \rho = \frac{1}{2}, \alpha(x, u) = \frac{2}{3}$.

Then for all $(x, p) \in S \times R$,

$$F(x, u; \alpha(x, u)[\nabla_x f(u) + \nabla_p(h(u, p))]) + h(u, p) - p^T \nabla_p h(u, p) + \rho d^2(x, u)$$

$$= \frac{1}{2}(x-1)^2 \geq 0$$

$$\Rightarrow f(x) - f(u) = \frac{x^2-x}{x+1} \geq 0, \forall x \in S.$$

This implies that f is higher order (F, α, ρ, d)-pseudo convex function at $u = 1$ with respect to $h(u, p)$.

But when we take $x = 3$ we have

$f(3) - f(1) = \frac{3}{2} < F(x, u; \alpha(x, u)(\nabla_x f(u) + \nabla_p h(u, p))) + h(u, p) - p^T \nabla_p h(u, p)$ $+\rho d^2(x, u) = 2.$
Hence f not higher order (F, α, ρ, d)-convex function. So (F, α, ρ, d)-pseudo convex function is more generalized than higher order (F, α, ρ, d) convex function.

We now onsider the following nonsmooth multiobjective programming problem:

(NMPP) Minimize $f(x) + (x^T Bx)^{\frac{1}{2}} = (f_1(x) + (x^T B_1 x)^{\frac{1}{2}}, ..., f_k(x) + (x^T B_k x)^{\frac{1}{2}})$

Subject to

$$-g(x) \in C_2^*, x \in C_1; \tag{1}$$

where,
(i) $f : R^n \to R^k$ and $g : R^n \to R^m$ are continuously differentiable functions,
(ii) $B_i, i = 1, 2, .., k$; are $n \times n$ positive semi definite symmetric matrix,
(iii) C_1 and C_2 are are closed convex cone with nonempty interior in R^n and R^m respectively, and
(iv) C_1^* and C_2^* are positive polar cones of C_1 and C_2 respectively.

Let $X_0 = \{x \in C_1 : -g(x) \in C_2^*\}$ be the set of all feasible solutions of (NMPP).

Since the objectives in multiobjective programming problems generally conflict with one another, an optimal solution is chosen from the set of efficient or weak efficient solution in following sense.

Definition 2.6 [17] A point $x^0 \in X_0$ is an efficient (pareto optimal) solution of (NMPP) if there does not exist any other $x \in X_0$ such that
$f_i(x) + (x^T B_i x)^{\frac{1}{2}} \leq f_i(x^0) + (x^{0T} B_i x^0)^{\frac{1}{2}}, \forall i = 1, 2, ..., k$ and
$f_j(x) + (x^T B_j x)^{\frac{1}{2}} < f_j(x^0) + (x^{0T} B_j x^0)^{\frac{1}{2}}$, for at least one index $j \in \{1, 2, ..., k\}$.

Lemma 2.1 Let $x, w \in R^n$ and $B \in R^n \times R^n$ be a positive semi definite matrix, then $x^T Bw \leq (x^T Bx)^{\frac{1}{2}} (w^T Bw)^{\frac{1}{2}}$. Equality holds if for some $\lambda \geq 0, Bx = \lambda Bw$.

3. Mond-Weir type higher order duality

In this section, we have introduced the Mond-Weir type multiobjective higher order dual of (MP) and established weak and strong duality theorems under generalized higher order (F, α, ρ, d)-convex functions. We consider the following Mond-Weir type higher order dual for (NMPP):

(MWHD): Maximize $f(u) + u^T Bw + \lambda^T h(u, p)e - p^T \nabla_p (\lambda^T h(u, p))e$

Subject to

$$\lambda^T [\nabla f(u) + Bw + \nabla_p h(u, p)] - y^T [\nabla g(u) + \nabla_p K(u, p)] = 0, \tag{2}$$

$$g(u) + K(u, p) - p^T \nabla_p K(u, p) \in C_2^*, \tag{3}$$

$$w_i^T B_i w_i \leq 1, i = 1, 2, ..., k, \tag{4}$$

$$y \in C_2, \tag{5}$$

$$\lambda > 0, \lambda^T e = 1, \tag{6}$$

where (i) $f : S_1 \to R^k$ and $g : S_1 : R^m$ are continuously differentiable functions,
(ii) C_1 and C_2 are closed convex cones in R^n and R^m with nonempty interior respectively,
(iii) C_1^* andC_2^* are polar cones of C_1 and C_2 respectively,
(iv) $e = (1, 1, ..., 1)^T$ is a vector in R^k,
(v) $w_i \in R^n$,
(vi) $B_i, i = 1, 2, ..., k$, are positive semidefinite symmetric matrix of order $n \times n$ and
(vii) $h : S \times S \to R^k$ and $K : S \times S \to R^m$ are differentiable functions and $p \in R^n$.

Theorem 3.1 Let x and (u, y, λ, w, p) be the feasible solution for (NMPP) and (MWHD) respectively. Assume that
$\lambda^T[f(.) + (.)^T Bw]$ is higher order (F, α, ρ, d)-convex at u with respect to $\lambda^T h(u, p)$ and $-y^T g(.)$ is higher order (F, α, σ, d)-convex at u with respect to $-y^T K(u, p)$ along with $\rho + \sigma \geq 0$.
Then $f(x) + (x^T Bx)^{\frac{1}{2}} > f(u) + u^T Bw + \lambda^T h(u, p)e - p^T \nabla_p(\lambda^T h(u, p))$.

Proof: Suppose that contradiction holds.
That is $f(x) + (x^T Bx)^{\frac{1}{2}} \leq f(u) + u^T Bw + (\lambda^T h(u, p))e - p^T \nabla_p(\lambda^T h(u, p))e$.

Since $\lambda > 0$, we obtain

$$\lambda^T[f(x) + (x^T Bx)^{\frac{1}{2}}] < \lambda^T[f(u) + u^T Bw] + \lambda^T h(u, p) - p^T \nabla_p(\lambda^T h(u, p)). \tag{7}$$

From (3) and (5), we have

$$y^T[g(u) + K(u, p) - p^T \nabla_p K(u, p)] \leq 0. \tag{8}$$

From (7) and (8), we have

$$\begin{aligned}\lambda^T[f(x) + (x^T Bx)^{\frac{1}{2}}] < \lambda^T[f(u) + u^T Bw] + \lambda^T h(u, p) - p^T \nabla_p(\lambda^T h(u, p))\\ - y^T[g(u) + K(u, p) - p^T \nabla_p K(u, p)].\end{aligned} \tag{9}$$

Now by Schwartz inequality and (4), we obtain $x^T Bw \leq (x^T Bx)^{\frac{1}{2}}$.

So, (9) becomes

$$\begin{aligned}\lambda^T[f(x) + (x^T Bw] < \lambda^T[f(u) + u^T Bw] + \lambda^T h(u, p) - p^T \nabla_p(\lambda^T h(u, p))\\ - y^T[g(u) + K(u, p) - p^T \nabla_p K(u, p)].\end{aligned} \tag{10}$$

Since $\lambda^T[f(.) + (.)^T Bw]$ is higher order (F, α, ρ, d)-convex at u with respect to $\lambda^T h(u, p)$ and $-y^T g(.)$ is higher order (F, α, σ, d)-convex at u with respect to $-y^T K(u, p)$, we have
$\lambda^T[f(x) + x^T Bw] - \lambda^T[f(u) + u^T Bw] - \lambda^T h(u, p) + p^T \nabla_p(\lambda^T h(u, p))$

$$\geq F(x, u; \alpha(x, u)[\lambda^T(\nabla f(u) + Bw) + \nabla_p(\lambda^T h(u, p))]) + \rho d^2(x, u) \tag{11}$$

and $-y^T g(x) + y^T g(u) + y^T K(u, p) - p^T \nabla_p(y^T K(u, p))$

$$\geq F(x, u; \alpha(x, u)[-y^T(\nabla g(u)) - \nabla_p(y^T K(u, p))]) + \sigma d^2(x, u). \tag{12}$$

Now from (1), (2) and (5), we obtained

$$-y^T g(x) + y^T g(u) + y^T K(u, p) - p^T \nabla_p(y^T K(u, p)) \leq 0. \tag{13}$$

So using (13) in (12), we obtained

$$F(x,u;\alpha(x,u)[-y^T(\nabla g(u)) - \nabla_p(y^T K(u,p))]) + \sigma d^2(x,u) \leq 0. \tag{14}$$

Since $\alpha(x,y) > 0$ and F is sublinear, from (2), we obtained,

$F(x,u;\alpha(x,u)[\lambda^T(\nabla f(u) + Bw) + \nabla_p(\lambda^T h(u,p))])$

$+ F(x,u;\alpha(x,u)[-y^T(\nabla g(u))\nabla_p(y^T K(u,p))])$

$\geq F(x,u;\alpha(x,u)[\lambda^T(\nabla f(u)+Bw)+\nabla_p(\lambda^T h(u,p)) - y^T(\nabla g(u)+\nabla_p K(u,p))]) = 0$

$\Rightarrow F(x,u;\alpha(x,u)[\lambda^T(\nabla f(u) + Bw) + \nabla_p(\lambda^T h(u,p))])$

$\geq -F(x,u;\alpha(x,u)[-y^T(\nabla g(u))\nabla_p(y^T K(u,p))])$

$\Rightarrow F(x,u;\alpha(x,u)[\lambda^T(\nabla f(u) + Bw) + \nabla_p(\lambda^T h(u,p))]) - \sigma d^2(x,u)$

$\geq -F(x,u;\alpha(x,u)[-y^T(\nabla g(u))\nabla_p(y^T K(u,p))]) - \sigma d^2(x,u) \geq 0$ $(using(3.13))$

$\Rightarrow F(x,u;\alpha(x,u)[\lambda^T(\nabla f(u) + Bw) + \nabla_p(\lambda^T h(u,p))])$

$+\rho d^2(x,u) - \rho d^2(x,u) - \sigma d^2(x,u) \geq 0.$

$\Rightarrow F(x,u;\alpha(x,u)[\lambda^T(\nabla f(u)+Bw)+\nabla_p(\lambda^T h(u,p))])+\rho d^2(x,u) \geq 0.$ (as $\rho+\sigma \geq 0$)
So (11) implies

$\lambda^T[f(x) + x^T Bw] \geq \lambda^T[f(u) + u^T Bw] + \lambda^T h(u,p) + p^T\nabla_p(\lambda^T h(u,p)).$

This is a contradiction to (7). Hence we proved.

Theorem 3.2 Let x and (u,y,λ,w,p) be the feasible solution for (NMPP) and (MWHD) respectively. Assume that
$\lambda^T[f(.) + (.)^T Bw] - y^T g(.)$ is higher order (F,α,ρ,d)- convex at u with respect to $\lambda^T h(u,p) - y^T K(u,p)$ along with $\rho \geq 0$.
Then $f(x) + (x^T Bx)^{\frac{1}{2}} > f(u) + u^T Bw + \lambda^T h(u,p)e - p^T\nabla_p(\lambda^T h(u,p))$.
Proof: Suppose that contradiction holds.
That is $f(x) + (x^T Bx)^{\frac{1}{2}} \leq f(u) + u^T Bw + (\lambda^T h(u,p))e - p^T\nabla_p(\lambda^T h(u,p))e$.

Since $\lambda > 0$, we obtain

$$\lambda^T[f(x) + (x^T Bx)^{\frac{1}{2}}] < \lambda^T[f(u) + u^T Bw] + \lambda^T h(u,p) - p^T\nabla_p(\lambda^T h(u,p)). \tag{15}$$

From (3) and (5), we have

$$y^T[g(u) + K(u,p) - p^T\nabla_p K(u,p)] \leq 0. \tag{16}$$

From (15) and (16), we have

$$\begin{aligned}\lambda^T[f(x) + (x^T Bx)^{\frac{1}{2}}] &< \lambda^T[f(u) + u^T Bw] + \lambda^T h(u,p) - p^T\nabla_p(\lambda^T h(u,p)) \\ &\quad - y^T[g(u) + K(u,p) - p^T\nabla_p K(u,p)].\end{aligned} \tag{17}$$

Now by Schwartz inequality and (4), we obtain $x^TBw \le (x^TBx)^{\frac{1}{2}}$.

So (17) becomes

$$\lambda^T[f(x) + (x^TBw] < \lambda^T[f(u) + u^TBw] + \lambda^T h(u,p) - p^T\nabla_p(\lambda^T h(u,p)) - y^T[g(u) + K(u,p) - p^T\nabla_p K(u,p)]. \quad (18)$$

Using $-y^Tg(x) \le 0$ in (18), we obtain

$$\lambda^T[f(x) + (x^TBw] - y^Tg(x) < \lambda^T[f(u) + u^TBw] + \lambda^T h(u,p) - p^T\nabla_p(\lambda^T h(u,p))$$

$$-y^T[g(u) + K(u,p) - p^T\nabla_p K(u,p)]$$

$$\Rightarrow \lambda^T[f(x) + (x^TBw] - y^Tg(x) - \lambda^T[f(u) + u^TBw] + \lambda^T h(u,p) - p^T\nabla_p(\lambda^T h(u,p)) - y^T[g(u) + K(u,p) - p^T\nabla_p K(u,p)] < 0. \quad (19)$$

Since $\lambda^T[f(.)+(.)^TBw]-y^Tg(.)$ is higher order (F,α,ρ,d)- convex at u with respect to $\lambda^T h(u,p) - y^T K(u,p)$, we have

$$\lambda^T[f(x)+(x^TBw]-y^Tg(x)-\lambda^T[f(u)+u^TBw]+y^Tg(u)-\lambda^T h(u,p)+p^T\nabla_p(\lambda^T h(u,p)) + y^T[K(u,p) - p^T\nabla_p K(u,p)]$$
$$\ge F_{x,u}(\alpha(x,u)\{\lambda^T(\nabla f(u)+Bw)+\nabla_p(\lambda^T h(u,p))-y^T\nabla g(u)-\nabla_p K(u,p)\})+\rho d^2(x,u). \quad (20)$$

From (2), (19) and (20) along with $F(x,u;0) = 0$, we get $\rho < 0$, which contradicts the assumption that $\rho \ge 0$. Hence we proved.

Theorem 3.3 Let x and (u,y,λ,w,p) be the feasible solution for (NMPP) and (MWHD) respectively. Assume that
$\lambda^T[f(.) + (.)^TBw] - y^Tg(.)$ is higher order (F,α,ρ,d)- pseudo convex at u with respect to $\lambda^T h(u,p) - y^T K(u,p)$ along with $\rho \ge 0$.
Then $f(x) + (x^TBx)^{\frac{1}{2}} > f(u) + u^TBw + \lambda^T h(u,p)e - p^T\nabla_p(\lambda^T h(u,p))$

Proof: Suppose that contradiction holds.
That is $f(x) + (x^TBx)^{\frac{1}{2}} \le f(u) + u^TBw + (\lambda^T h(u,p))e - p^T\nabla_p(\lambda^T h(u,p))e$
Since $\lambda > 0$, we obtain

$$\lambda^T[f(x) + (x^TBx)^{\frac{1}{2}}] < \lambda^T[f(u) + u^TBw] + \lambda^T h(u,p) - p^T\nabla_p(\lambda^T h(u,p)). \quad (21)$$

From (3) and (5), we have

$$y^T[g(u) + K(u,p) - p^T\nabla_p K(u,p)] \le 0. \quad (22)$$

From (21) and (22), we have

$$\lambda^T[f(x) + (x^TBx)^{\frac{1}{2}}] < \lambda^T[f(u) + u^TBw] + \lambda^T h(u,p) - p^T\nabla_p(\lambda^T h(u,p)) - y^T[g(u) + K(u,p) - p^T\nabla_p K(u,p)]. \quad (23)$$

Now by Schwartz inequality and (4), we obtain $x^TBw \le (x^TBx)^{\frac{1}{2}}$.
So (23) becomes

$$\lambda^T[f(x) + (x^TBw] < \lambda^T[f(u) + u^TBw] + \lambda^T h(u,p) - p^T\nabla_p(\lambda^T h(u,p)) - y^T[g(u) + K(u,p) - p^T\nabla_p K(u,p)]. \quad (24)$$

From (2), we obtain

$$F_{x,u}(\alpha(x,u)\{\lambda^T(\nabla f(u)+Bw)+\nabla_p(\lambda^T h(u,p))-y^T\nabla g(u)-\nabla_p K(u,p)\})=0.$$

So by assumption of theorem 3.3, we get

$$\lambda^T[f(x)+(x^TBw]-y^Tg(x)-\lambda^T[f(u)+u^TBw]+y^Tg(u)-\lambda^Th(u,p)+p^T\nabla_p(\lambda^Th(u,p))$$

$$+y^T[K(u,p)-p^T\nabla_pK(u,p)]\geq 0$$

$$\Rightarrow \lambda^T[f(x)+(x^TBw]-y^Tg(x)\geq\lambda^T[f(u)+u^TBw]+y^Tg(u)-\lambda^Th(u,p)+p^T\nabla_p(\lambda^Th(u,p))$$
$$+\,y^T[K(u,p)-p^T\nabla_pK(u,p)]. \tag{25}$$

Since $-g(x)\in C_2^*$, for $y\in C_2$, we get $-y^Tg(x)\leq 0$.
So, (25) implies
$\lambda^T[f(x)+(x^TBw]-y^Tg(x)\geq\lambda^T[f(u)+u^TBw]+\lambda^Th(u,p)-p^T\nabla_p(\lambda^Th(u,p))$
$-\,y^T[g(u)+K(u,p)-p^T\nabla_pK(u,p)]$.
This is a contradiction to (24). Hence we proved.

To prove the strong duality theorem, we shall make use the following lemma established by Suneja et al. [16]. It gives Fritz John type necessary optimality conditions for a weakly efficient solution of (NMPP).

Lemma 3.1 If that $\bar{x}\in X_0$ is an efficient solution of (NMPP), then there exist $\bar{\lambda}\in R^k_+,\bar{w}_i\in R^n, i=1,2,...,k$ and $\bar{y}\in C_2$ with $(\bar{\lambda},\bar{y})\neq 0$ such that

$$(x-\bar{x})^T[\bar{\lambda}^T(\nabla f(\bar{x})+B\bar{w})-\bar{y}^T\nabla g(\bar{x})]\geq 0,\forall x\in C_1, \tag{26}$$
$$\bar{y}g(\bar{x})=0, \tag{27}$$
$$(\bar{x}B_i\bar{x})^{\frac{1}{2}}=\bar{x}^TB_i\bar{w}_i, i=1,2,...,k, \tag{28}$$
$$\bar{w}_iB_i\bar{w}_i\leq 1, i=1,2,...,k \tag{29}$$

Theorem 3.4 (**Strong Duality**) Let $\bar{x}$ be an efficient solution of (NMPP), and

$$h(\bar{x},0)=0,K(\bar{x},0)=0,\nabla_ph(\bar{x},0)=\nabla f(\bar{x}),\nabla_pK(\bar{x},0)=\nabla g(\bar{x}). \tag{30}$$

Then there exist $\bar{\lambda}\in R^k_+,\bar{w}_i\in R^n,i=1,2,...,k$ and $\bar{y}\in C_2$ with $(\bar{\lambda},\bar{y})\neq 0$ such that $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p}=0)$ is a feasible solution of (MWHD) and the corresponding value of objective functions are equal.. Further, if the conditions of weak duality Theorems 3.1 or Theorems 3.2 or Theorems 3.3 are satisfied then $(\bar{x},\bar{y},\bar{\lambda},\bar{w},\bar{p}=0)$ is an efficient solution of (MWHD).

Proof: Since $\bar{x}$ be an efficient solution of (NMPP), then by lemma 3.1, there exist $\bar{\lambda}\in R^k_+,\bar{w}_i\in R^n,i=1,2,...,k$ and $\bar{y}\in C_2$ with $(\bar{\lambda},\bar{y})\neq 0$ such that

$$(x-\bar{x})^T[\bar{\lambda}^T(\nabla f(\bar{x})+B\bar{w})-\bar{y}^T\nabla g(\bar{x})]\geq 0,\forall x\in C_1, \tag{31}$$
$$\bar{y}g(\bar{x})=0, \tag{32}$$
$$(\bar{x}B_i\bar{x})^{\frac{1}{2}}=\bar{x}^TB_i\bar{w}_i, i=1,2,...,k, \tag{33}$$
$$\bar{w}_iB_i\bar{w}_i\leq 1, i=1,2,...,k. \tag{34}$$

Since $x\in C_1,\bar{x}\in C_1$ and C_1 is a closed convex cone, we have $x+\bar{x}\in C_1$.
So replacing x by $x+\bar{x}$ in (31), we get

$x^T[\bar{\lambda}^T(\nabla f(\bar{x}) + B\bar{w} - \bar{y}\nabla g(\bar{x})] \geq 0, \forall x \in C_1$

i.e.

$$x^T[\bar{\lambda}^T(\nabla f(\bar{x}) + B\bar{w} - \bar{y}\nabla g(\bar{x})] \geq 0, \forall x \in C_1. \tag{35}$$

So, by relation (30), (32), (33), (34) along with (35) imply that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p} = 0)$ is feasible for (MWHD) and the corresponding value of objective functions are equal. The proof of the remaining part follows from the weak duality Theorem 3.1, Theorem 3.2 and Theorem 3.3.

4. Special Cases

(1) If $C_1 = R^n_+, R^m_+, k = 1$ then we get higher order dual programs studied by Mishra and Rueda [11].

(2) If $C_1 = R^n_+, R^m_+, s(x \mid D) = (x^T Bx)^T$, where $D = \{Bw \mid w^T Bw \leq 1\}$ then we get higher order dual programs studied by Yang et al. [18].

(3) If $C_1 = R^n_+, R^m_+, B_i = 0, i = 1, 2, ..., k; h(u, p) = p^T \nabla f(u) + \frac{1}{2} p^T \nabla^2 f(u) p$ and $K(u, p) = p^T \nabla g(u)$, then our higher order dual programs become first order dual program in Wolfe [18].

(4) $C_1 = R^n_+, R^m_+, B_i = 0, i = 1, 2, ..., k; h(u, p) = p^T \nabla f(u) + \frac{1}{2} p^T \nabla^2 f(u) p$ then we obtain second order dual programs studied by [9].

5. Conclusion

In this paper, we consider a new class of generalized higher order (F, α, ρ, d)-convex and (F, α, ρ, d)-pseudo convex functions with examples. Mond-Weir type higher order duality is proposed for the non-differentiable multiobjective problem involving cone constraints, where every component of the objective function contains square root term of positive semidefinite quadratic form. For this problem, duality results are established for efficient solution under suitable higher order (F, α, ρ, d)-convexity conditions. Duality and sufficient optimality conditions with generalized (F, α, ρ, d)-convexity will be studied for non-differentiable variational and control problems, which will orient the future research of authors.

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