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NEW REFINEMENT ALGORITHM FOR A POSTERIORI ERROR ESTIMATE IN GOAL-ORIENTED PROBLEMS

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Abstract. In this work, a modified algorithm is presented for the mesh refinement involved in evaluating a posteriori error estimate for goal-oriented problems using recovery techniques. As the results show in some problems, the effectivity index exhibits some peaks before tending to 1. To overcome these peaks, we propose an improvement to the local refinement algorithm.

1. Introduction

In this section we explain some of the terms and derive some of the mathematical formulas related to the work in this article.

1.1. **Adaptive finite element.** Adaptive methods based on a posteriori error estimates are now widely used in the scientific computation to achieve better accuracy with minimum degrees of freedom. Adaptive finite element methods (FEM) typically consist of successive loops of the sequence (Solve – Estimate – Mark – Refine), until a stopping criterion is satisfied, as in [1].

1.2. **Recovery techniques.** Error estimators can be categorized under two classes. The first one is the residual type estimators, as in [2], and the second one is the recovery type estimators, as in [3]. Many of the popular techniques are based on the residual type estimators, as in [4]-[11].

Finite element recovery techniques are post-processing methods that reconstruct numerical approximations from finite element solutions to obtain the improved solutions. The practical usage of the recovery technique is not only to improve the quality of the approximation, but also to construct recovery type a posteriori error estimators in adaptive computations [12].

Recently, gradient recovery has gained a lot of interest from scientists and engineers. Several gradient recovery techniques are proposed based on weighted averaging [13], [14], local or global projections [15]-[17], smoothing techniques [18], [19] and least-square type methods [20]-[22]. Gradient recovery technique has been widely used in engineering practice due to its good properties. These properties

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include robustness as an a posteriori error estimator, superconvergence of the recovered derivatives, and efficiency in implementation; see, e.g., [23]-[30].

1.3. **Polynomial preserving recovery technique.** In [21], the authors designed a new systematic gradient recovery technique that is applicable to FEMs of all orders in 2D and 3D and at the same time inherits the good properties of the superconvergent patch recovery (SPR) technique. This is the polynomial preserving recovery (PPR) technique. The authors introduced [26], analyzed and showed that the PPR technique is as good as or better than the SPR.

To recover the gradient using the PPR technique at a mesh node p, a patch κ_p of elements is selected. Then, a polynomial that best fits the finite element solution is constructed, in least-squares sense, at the mesh nodes in κ_p. The recovered gradient is defined to be the gradient of the fitting polynomial. Nodes on domain boundary $\partial\Omega$ are handled in the same way, although they need extra care in constructing their patches

The PPR-recovered gradient has a superconvergence property i.e. the convergence rate at some exceptional points of the domain exceeds the global known optimal rates. As it is known, if the recovered gradient is superconvergent to the exact gradient, then the a posteriori error estimator based on this recovered gradient is exact in asymptotic sense [23].

1.4. **Goal-oriented problem.** In finite element analysis it is frequently the case that the analyst is more interested in certain output data of the finite element approximation than in the global energy norm. In order to find an estimate for the error in the output data related to a specific quantity, or to find at least an effective mesh to accurately solve for this quantity, error estimators for the energy norm are not useful. Hence, more recently so-called goal-oriented error estimates were developed, which estimate the error in quantities of interest [31].

Goal-oriented methods visualized as a generalization to classical a posteriori error estimation methods. The mission of goal-oriented adaptive algorithms is to estimate the discretization error in such quantities of interest and use these estimates to adapt the mesh in order to control their accuracy.

In [32], a new approach for evaluating a posteriori error estimate for goal-oriented problems is presented. This approach is based on replacing the gradient in the goal-oriented error estimate by the recovered gradient obtained by PPR. Also, a local refinement algorithm that properly implements the proposed technique is suggested.

1.5. **Model problem.** Suppose that we are given the elliptic boundary-value problem

$$-\Delta u = f \ \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega, \tag{1}$$

and a linear functional G such that $G(u)$ is a quantity of interest. In order to approximate $G(u)$, one may compute $G(u_h)$, where u_h is the linear finite element approximation to u over a conforming mesh τ of Ω.

We are interested in estimating the goal-oriented error e_h

$$e_h = G\left(u - u_h\right) = \langle u - u_h\ ,\ G\rangle\,, \tag{2}$$

where $\langle .\ ,\ .\rangle$ denotes the inner product in $L_2(\Omega)$. Numerous contributions have been made to have an upper bound to e_h in both the mathematical and engineering literature as in [5]-[10]. In fact, combining the solution of the dual problem

$$-\Delta z = f \quad \text{in } \Omega, \quad z = 0 \text{ on } \partial\Omega, \tag{3}$$

with Galerkin orthogonality yields the representation formula

$$\begin{aligned} e_h &= \langle -\Delta z \ , \ u - u_h \rangle \\ &= \langle \nabla z \ , \ \nabla (u - u_h) \rangle \\ &= \langle \nabla (u - u_h) \ , \ \nabla (z - z_h) \rangle , \end{aligned} \tag{4}$$

where z_h is the linear finite element approximation of z [9]. The approach in [32] replaced the exact gradients in (4) by the recovered gradients obtained by the PPR technique. So, the estimated error η_h is calculated by the inner product

$$\eta_h = \langle R_h u_h - \nabla u_h \ , \ R_h z_h - \nabla z_h \rangle , \tag{5}$$

where $R_h u_h$ and $R_h z_h$ are the PPR recovered gradients of u_h and z_h, respectively.

Then the proposed approach is used to obtain an estimate for the goal-oriented error to one-dimensional problems. Numerical examples are presented to show the resulting estimator provide tight bound with the effectivity index λ_h defined by

$$\lambda_h = |\eta_h / e_h| \ , \tag{6}$$

tending to 1. But, sometimes the results show that, the effectivity index exhibits some peaks before tending to 1. To overcome these peaks, an improvement to the local refinement algorithm is introduced in the next section.

It is customary to choose the mollifiers G_ϵ of the form

$$G_\epsilon(x - x_0) = C \ . \ exp\left(\frac{\epsilon^2}{(x - x_0)^2 - \epsilon^2}\right), \tag{7}$$

if $|x - x_0| < \epsilon$ and $G_\epsilon(x - x_0) = 0$ elsewhere. The constant C, depends on ϵ and x_0, is selected to satisfy

$$\int_{x_0-\epsilon}^{x_0+\epsilon} G_\epsilon(x - x_0)dx = 1, \tag{8}$$

a numerical integration of the last integral provides that $C \approx 2.2523\ \epsilon^{-1}$ [4].

2. Modified local refinement algorithm

After having computed the local error estimate, we resort to marking the elements that need to be refined. Different approaches for marking strategies can be found in [1]. Let $\tau_{h,0}$ be the initial mesh. Now we start the procedure to compute a sequence of meshes and approximate solutions. The local error estimate $\eta_{h,T}$ is computed for every element T of the mesh $\tau_{h,k}$ for some integer $k \geq 0$ and choose an appropriate tolerance ε for the error estimate η_h and choose $\mu, \nu \in (0, 1)$.

In [32], an algorithm compatible with estimate (5) was proposed for the goal-oriented problems in the following procedure (goal-adaptation)

while ($|\eta_h| > \varepsilon$) do
    sum $= 0$;
    $t = 1$;

while ($|sum| < \mu \, . |\eta_h|$) do
  $t = t - \nu$;
  if ($t \neq 0$)
    for all $T \in \tau_{h,k}$
      if (T is not marked)
        if ($|\eta_{h,T}| > t \, . |\eta_{h,T}|_{max}$) mark T ;
          sum = sum + $\eta_{h,T}$;

By choosing ν we can control how fine the procedure should work. One may choose μ depending on the complexity of f. Note that this algorithm is not expensive in its computational cost, because all local errors have already been computed.

If there are sources of error, this will affect the quality of the error estimator. Additionally, small number of marked elements does not enhance η_h. The modified algorithm tests the error between the recovered gradient and the gradient of the finite element solution in the L_2 norm sense

$$\xi_h = \|R_h u_h - \nabla u_h\|_{L_2(\Omega)} , \tag{9}$$

and provide additional marking strategy (u-adaptation), to help the goal-adaptation to achieve a good quality of the error estimator, as follow

if ($|\xi_{h,T}| > \beta \, . |\xi_{h,T}|_{max}$) mark T ;

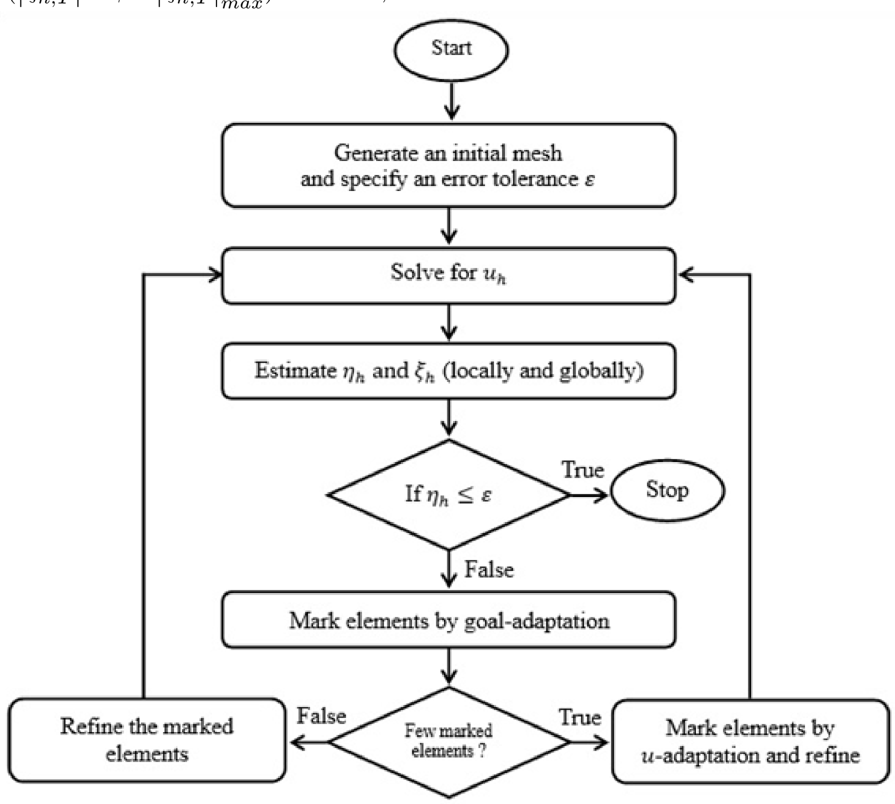

Figure 1. Flow chart for the modified algorithm

3. Numerical examples

In the following examples, we fix the following parameters

$$\nu = 0.1,\ \beta = \mu = 0.5,\ x_0 = 0.5 \text{ and } \varepsilon = 10^{-6}$$

and explore the influence of the parameter ϵ with respect to the mesh size h on the effectivity index λ_h defined by (6).

Example 1.

Consider the problem

$$-u'' = f \quad \text{in } \Omega, u = 0 \text{ on } \partial\Omega, \tag{10}$$

and the dual problem

$$-z'' = G \quad \text{in } \Omega, z = 0 \text{ on } \partial\Omega, \tag{11}$$

where Ω is $(0, 1)$. We choose f so that the exact solution is

$$u(x) = \sin(\pi x). \tag{12}$$

Figure 2. The initial mesh (16 nodes)

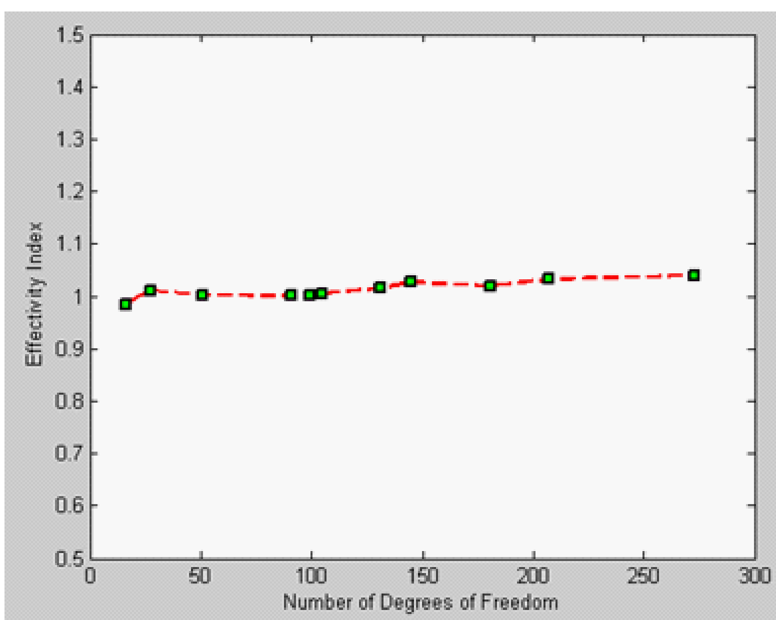

Figure 3. Effectivity index for $\epsilon = 0.1$

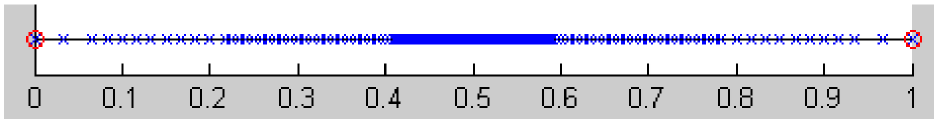

Figure 4. Final mesh for $\epsilon = 0.1$

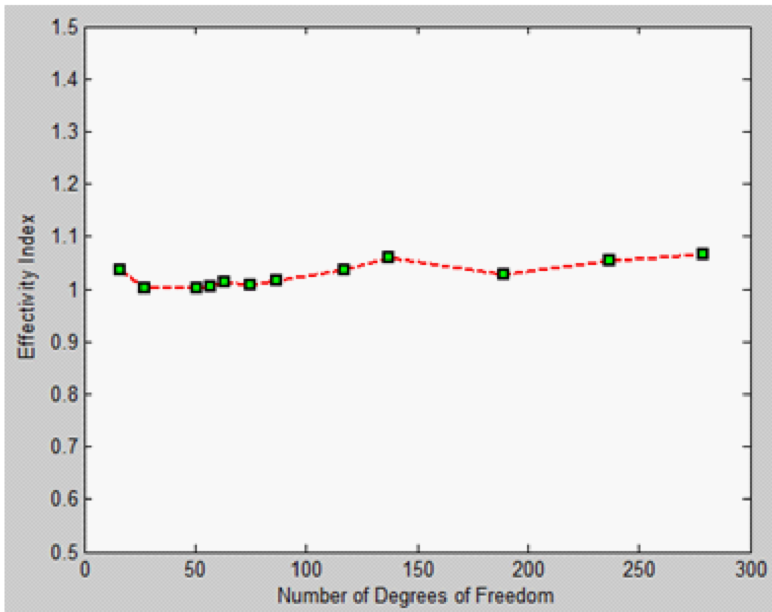

Figure 5. Effectivity index for $\epsilon = 0.15$

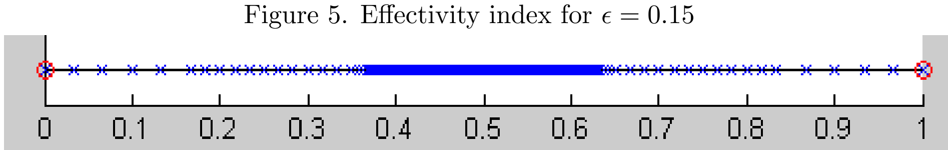

Figure 6. Final mesh for $\epsilon = 0.15$

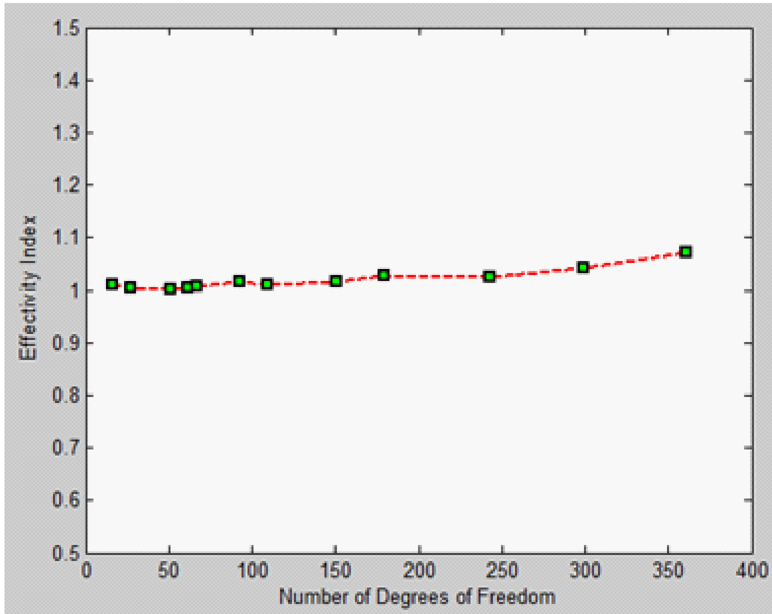

Figure 7. Effectivity index for $\epsilon = 0.2$

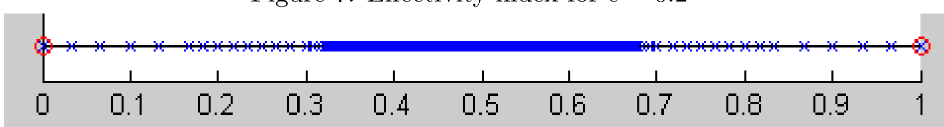

Figure 8. Final mesh for $\epsilon = 0.2$

Example 2.

For the equations in (10) and (11), we choose another f so that

$$u(x) = x^4 \ (x-1), \tag{13}$$

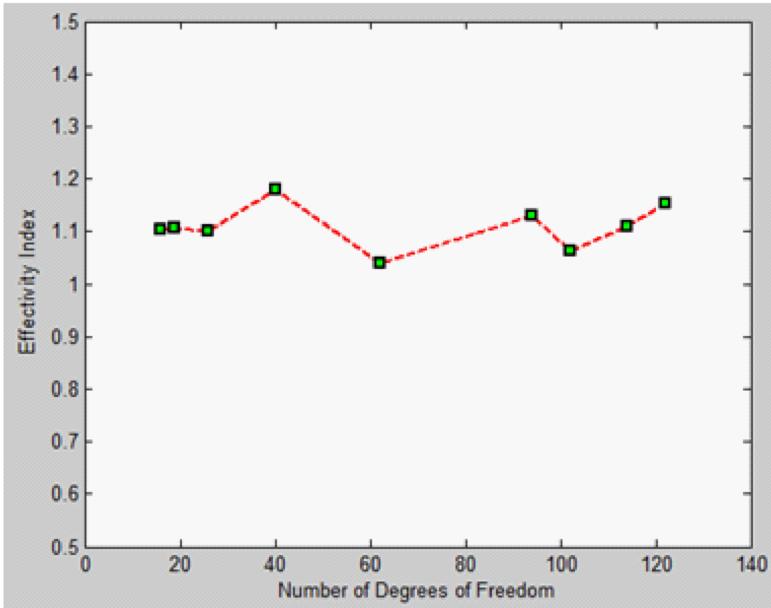

Figure 9. Effectivity index for $\epsilon = 0.1$

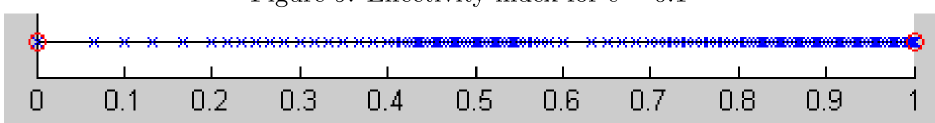

Figure 10. Final mesh for $\epsilon = 0.1$

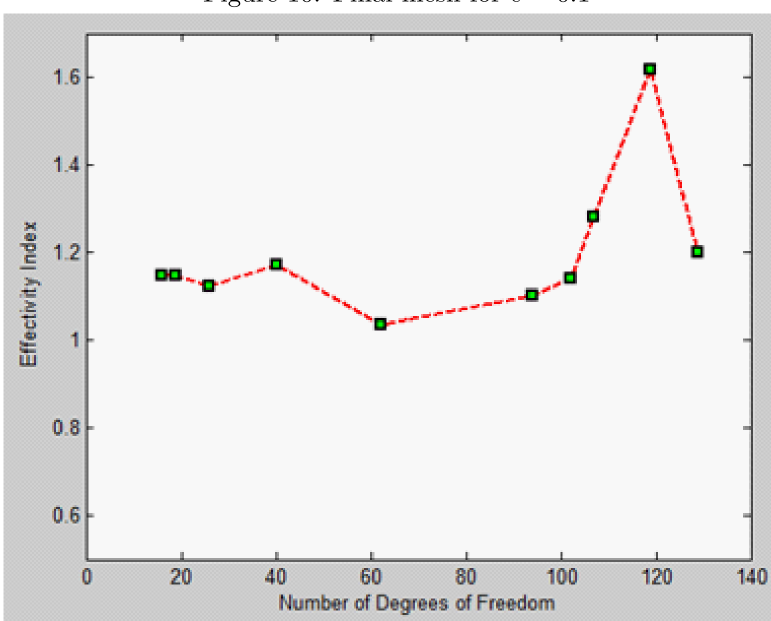

Figure 11. Effectivity index for $\epsilon = 0.15$

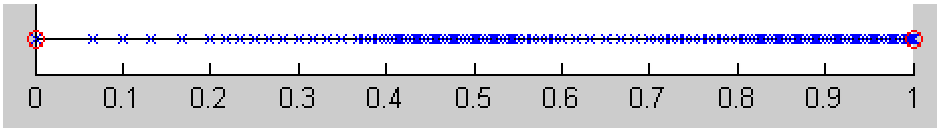

Figure 12. Final mesh for $\epsilon = 0.15$

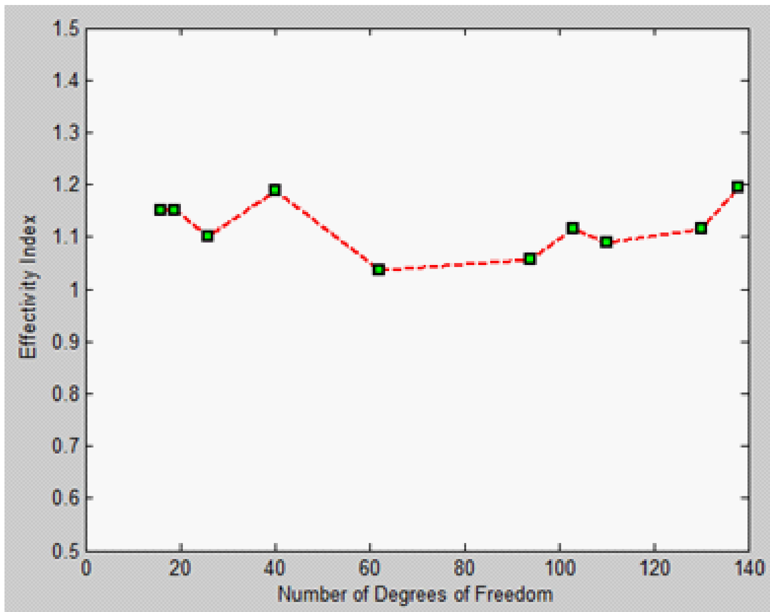

Figure 13. Effectivity index for $\epsilon = 0.2$

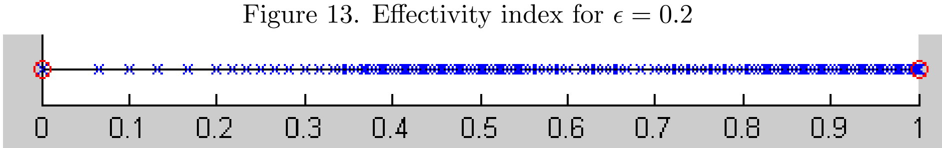

Figure 14. Final mesh for $\epsilon = 0.2$

Example 3.

Consider the problem

$$-u'' + u = f \text{ in } \Omega, \; u = 0 \text{ on } \partial\Omega, \tag{14}$$

where Ω is $(0, 1)$. Set f so that the solution of this problem given by

$$u(x) = x\ (x - 1), \tag{15}$$

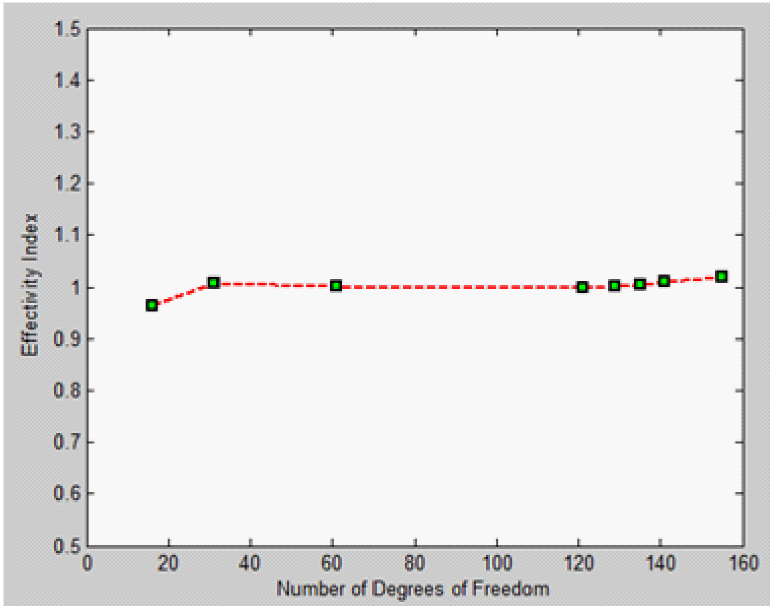

Figure 15. Effectivity index for $\epsilon = 0.1$

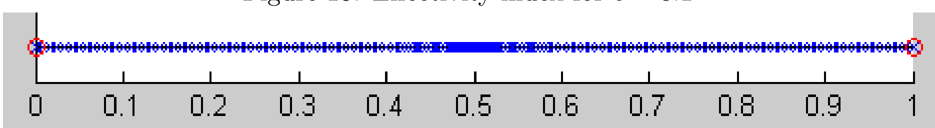

Figure 16. Final mesh for $\epsilon = 0.1$

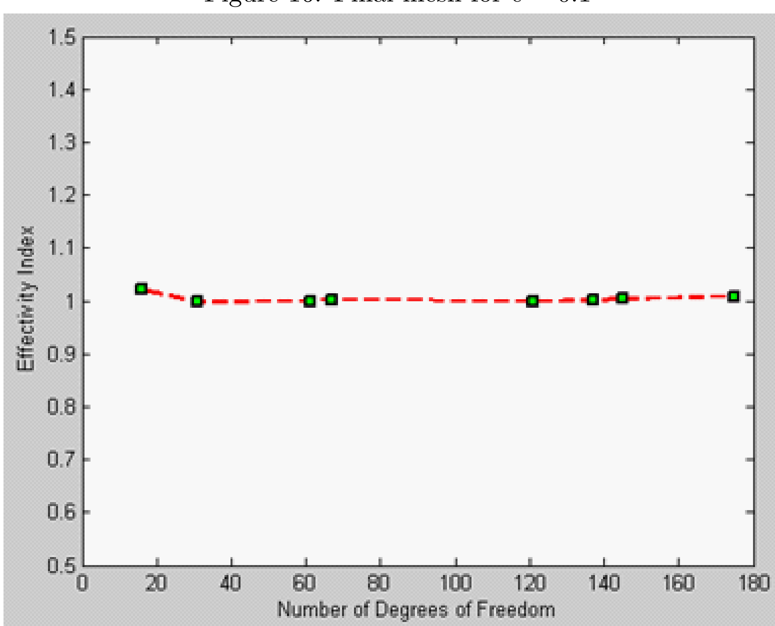

Figure 17. Effectivity index for $\epsilon = 0.15$

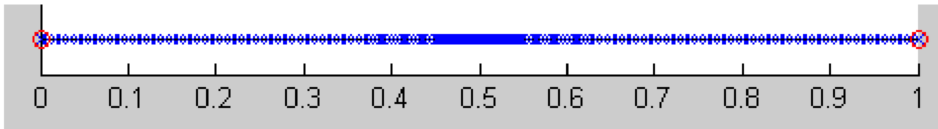

Figure 18. Final mesh for $\epsilon = 0.15$

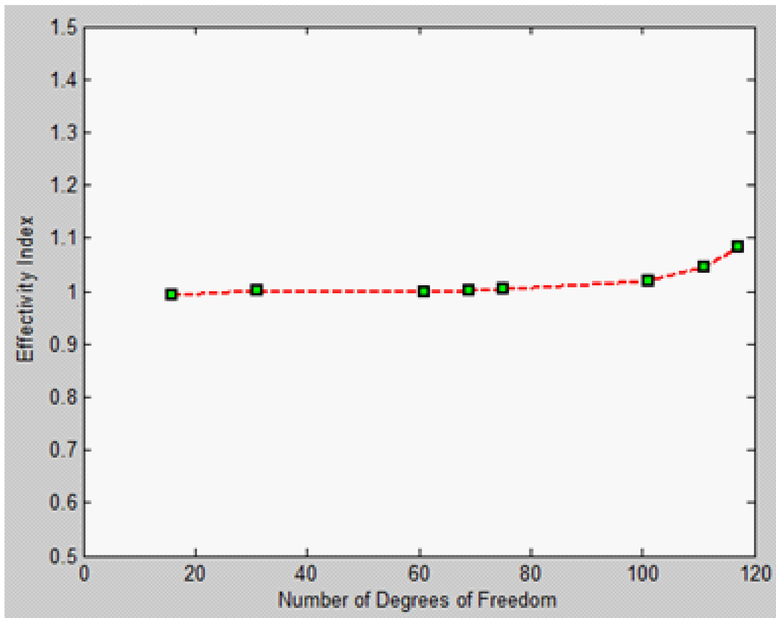

Figure 19. Effectivity index for $\epsilon = 0.2$

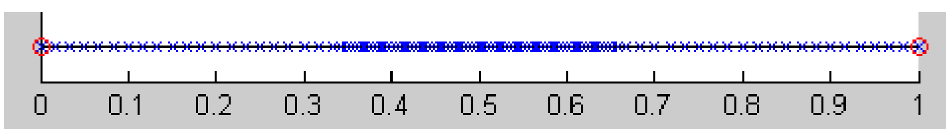

Figure 20. Final mesh for $\epsilon = 0.2$

4. Conclusion

In this work, we present a new algorithm for goal oriented problems. We proposed a mixed algorithm where we employ two error estimates, goal-oriented estimate and recovered gradient estimate, together to choose the elements to be refined. The results show that this algorithm overcome a drawback in previous algorithms as it yields an effectivity index that tends to 1 with no or minimum peaks. Yet, the degrees of freedom is larger than the one obtained in previous algorithms. This suggests that the parameters involved in the refinement algorithm should also be chosen in an adaptive way.

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