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# HANKEL DETERMINANT FOR CERTAIN SUBCLASS OF $p$-VALENT FUNCTION

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Abstract. The object of this paper is to use Toeplitz determinant to obtain a sharp upper bound of the second Hankel determinant $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for the $p$-valent functions belonging to class $\mathcal{M}_p(\alpha, \beta)$. Our main theorem unify and extend several results established earlier for special values of the parameters $p, \alpha$ and $\beta$.

## 1. Introduction and Motivation

Let $\mathcal{A}_p$ ($p$ is a fixed integer $\geq 1$) denote the class of all analytic functions $f(z)$ of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \tag{1}$$

defined on the open unit disk:

$$\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$$

and let $\mathcal{A}_1 = \mathcal{A}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$. A function $f(z) \in \mathcal{A}_p$ is said to be $p$-valent starlike function $\left(\frac{f(z)}{z} \neq 0\right)$, if it satisfies the condition

$$\Re\left\{\frac{zf'(z)}{pf(z)}\right\} > 0 \quad (z \in \mathbb{U}). \tag{2}$$

The set of all these functions is denoted by $S_p^*$. It is observed that for $p = 1$, $S_p^*$ reduces to $S^*$.
The $q$th Hankel determinant for $q \geq 1$ and $n \geq 1$ is stated by Noonan and Thomas (see [23]) as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} \cdots & a_{n+q} \\ . & \cdots & . \\ . & \cdots & . \\ . & \cdots & . \\ a_{n+q-1} & a_{n+q} \cdots & a_{n+2q-2} \end{vmatrix}. \tag{3}$$

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This determinant has been considered by several authors in the literature. For example, Noor [24] determined the rate of growth of $H_q(n)$ as $n \longrightarrow \infty$ for the functions $f$ given by (1) with bounded boundary rotation. Ehrenborg [5] studied the Hankel determinant of exponential polynomials. It is well-known [4] that for $f \in \mathcal{S}$ and given by (1), the sharp inequality $|a_3 - a_2^2| \leq 1$ holds. This corresponds to the Hankel determinant with $q = 2$ and $n = 1$. Fekete-Szegö (see [6]) then further generalized the estimate $|a_3 - \mu a_2^2|$ with $\mu$ real and $f \in \mathcal{S}$. For a given family $\mathcal{F}$ of the functions in $\mathcal{A}$, the sharp upper bound for the nonlinear functional $|a_2 a_4 - a_3^2|$ is popularly known as the second Hankel determinant. Second Hankel determinant for various subclasses of analytic functions were obtained by different researchers including Janteng et al. [11], Mishra and Gochhayat [21] and Murugusundaramoorthy and Magesh [22]. For some more recent works see [1, 2, 3, 7, 8, 9, 10, 13, 25].

For our discussion in this paper, we consider the Hankel determinant in the case of $q = 2$ and $n = p + 1$, denoted by $H_2(p + 1)$ given by

$$H_2(p+1) = \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+3} \end{vmatrix} = a_{p+1}a_{p+3} - a_{p+2}^2.$$

Motivated by the above mentioned results obtained by different researchers in this direction, in this paper, we obtain a sharp upper bound to the functional $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for the function $f$ belonging to certain subclass of $p$-valent functions, defined as follows:

**Definition 1.** A function $f(z) \in \mathcal{A}_p$ is said to be in the class $\mathcal{M}_p(\alpha, \beta)$ if it satisfies the condition

$$\Re\left[(1-\beta)\left(\frac{f(z)}{z^p}\right)^{\alpha} + \beta\left(\frac{zf'(z)}{pf(z)}\right)\left(\frac{f(z)}{z^p}\right)^{\alpha}\right] > 0 \quad (0 \leq \alpha,\ \beta \leq 1; z \in \mathbb{U}). \tag{4}$$

Note that for $\alpha = 1$, the class $\mathcal{M}_p(1, \beta)$ reduces to the class $I_p(\beta)$ studied by Krishna and Ramreddy [16]; while for $\alpha = 0$ and $\beta = 1$, the class $\mathcal{M}_p(0, 1)$ reduces to the well-known class of $p$-valent starlike function $S_p^*$ studied by Krishna and Ramreddy [14]. Furthermore, for $\alpha = \beta = 1$, the class $\mathcal{M}_p(1, 1)$ reduces to $RT_p$ studied by Krishna et al. [17]; while for $\alpha = \beta = p = 1$,the class $\mathcal{M}_1(1, 1)$ reduces to the class $RT$, the subclass of $\mathcal{S}$ consisting of functions whose derivative has a positive real part studied by Mac Gregor [20] was obtained by Janteng et al. [11].

## 2. Preliminary Lemmas

Let $\mathcal{P}$ denote the class of functions of the form

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{5}$$

which are regular in $\mathbb{U}$ and satisfy $\Re(p(z)) > 0$ for any $z \in \mathbb{U}$. Here, $p(z)$ is calledAtheòdory function (see [4].

To prove our main result, we need the following lemmas:

**Lemma 1.**(see [4]) If $p \in \mathcal{P}$, then $|c_k| \leq 2$ for each $k \in \mathbb{N}$.

**Lemma 2.**(see [18, 19]) Let $p \in \mathcal{P}$. Then

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{6}$$

and

$$4c_3 = \{c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z\} \tag{7}$$

for some values of $x,\ z$ such that $|x| \le 1$ and $|z| \le 1$ .

## 3. Main Result

We state and prove the following:

**Theorem 1.** If $f(z) \in \mathcal{M}_p(\alpha, \beta)$ $(0 < \alpha,\ \beta \le 1)$, then

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \le \frac{4p^2}{(\alpha p + 2\beta)^2}.$$

The estimate is sharp.

*Proof.* Let $f(z)$ given by (1) be in the class $\mathcal{M}_p(\alpha, \beta)$. Then from Definition 1 there exists an analytic function $p \in \mathcal{P}$ in the unit disk $\mathbb{U}$ with $p(0) = 1$ and $\Re\{p(z)\} > 0$ such that

$$(1-\beta)\left(\frac{f(z)}{z^p}\right)^{\alpha} + \beta \frac{zf'(z)}{pf(z)}\left(\frac{f(z)}{z^p}\right)^{\alpha} = p(z). \tag{8}$$

Replacing $f(z),\ f'(z)$ with their equivalent $p$-valent series expressions and the equivalent expression for $p(z)$ in series in (8), after simplification, we have

$$\begin{aligned} 1 + \frac{1}{p}(\alpha p + \beta)a_{p+1}z + \frac{1}{2p}(\alpha p + 2\beta)\{2a_{p+2} - (1-\alpha)a_{p+1}^2\}z^2 \\ + \frac{\alpha p + 3\beta}{p}\{a_{p+3} - (1-\alpha)a_{p+1}a_{p+2} + \frac{(1-\alpha)(2-\alpha)}{6}a_{p+1}^3\}z^3 + \cdots \\ = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots . \end{aligned} \tag{9}$$

Equating the coefficients of the like powers of $z,\ z^2,\ z^3$ respectively on both sides of (9). we get

$$\begin{aligned} a_{p+1} &= \frac{p}{\alpha p + \beta}c_1, \\ a_{p+2} &= \frac{p}{\alpha p + 2\beta}c_2 + (1-\alpha)\frac{p^2}{2(\alpha p + \beta)^2}c_1^2, \\ a_{p+3} &= \frac{p}{\alpha p + 3\beta}c_3 + (1-\alpha)\frac{p^2}{(\alpha p + \beta)(\alpha p + 2\beta)}c_1c_2 + \frac{(1-\alpha)(1-2\alpha)}{6(\alpha p + \beta)^3}p^3c_1^3. \end{aligned} \tag{10}$$

Substituting the values of $a_{p+1},\ a_{p+2}$ and $a_{p+3}$ from the relation (10) in the second Hankel functional $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for the function $f \in \mathcal{M}_p(\alpha, \beta)$, we obtain

$$\begin{aligned} |a_{p+1}a_{p+3} - a_{p+2}^2| = \Bigg| \left(\frac{pc_1}{\alpha p + \beta}\right)\Bigg(\frac{p}{\alpha p + 3\beta}c_3 + (1-\alpha)\frac{p^2}{(\alpha p + \beta)(\alpha p + 2\beta)}c_1c_2 \\ + \frac{(1-\alpha)(1-2\alpha)}{6(\alpha p + \beta)^3}p^3c_1^3\Bigg) - \left\{\frac{p}{\alpha p + 2\beta}c_2 + (1-\alpha)\frac{p^2}{2(\alpha p + \beta)^2}c_1^2\right\}^2 \Bigg|. \end{aligned}$$

Upon simplification, we obtain

$$\begin{aligned} |a_{p+1}a_{p+3} - a_{p+2}^2| = \frac{p^2}{12(\alpha p + \beta)^4(\alpha p + 2\beta)^2(\alpha p + 3\beta)} \big|12(\alpha p + \beta)^3(\alpha p + 2\beta)^2 c_1 c_3 \\ -12(\alpha p + \beta)^4(\alpha p + 3\beta)c_2^2 - (1-\alpha^2)(\alpha p + 2\beta)^2(\alpha p + 3\beta)p^2c_1^4\big| . \end{aligned}$$

The above expression is equivalent to

$$|a_{p+1}a_{p+3}-a_{p+2}^2| = \frac{p^2}{12(\alpha p+\beta)^4(\alpha p+2\beta)^2(\alpha p+3\beta)}|A_1c_1c_3+A_2c_2^2+A_3c_1^4|, \quad (11)$$

where

$$\begin{aligned} A_1 &= 12(\alpha p+\beta)^3(\alpha p+2\beta)^2,\\ A_2 &= -12(\alpha p+\beta)^4(\alpha p+3\beta),\\ A_3 &= -(1-\alpha^2)p^2(\alpha p+2\beta)^2(\alpha p+3\beta). \end{aligned} \quad (12)$$

Substituting the values of $c_2$ and $c_3$ from (6) and (7) respectively from Lemma 2 in the right hand side of (11), we get

$$|A_1c_1c_3+A_2c_2^2+A_3c_1^4| = |A_1c_1\times\frac{1}{4}\{c_1^3+2c_1(4-c_1^2)x-c_1(4-c_1^2)x^2+2(4-c_1^2)(1-|x|^2)z\} + A_2\times\frac{1}{4}\{c_1^2+x(4-c_1^2)\}^2+A_3c_1^4|.$$

Using the fact that $|z|<1$ and $|xa+yb|\le|x||a|+|y||b|$, where $x,\ y,\ a$ and $b$ are real numbers, after simplifying, we obtain

$$\begin{aligned} 4|A_1c_1c_3+A_2c_2^2+A_3c_1^4| &\le |(A_1+A_2+4A_3)c_1^4+2A_1c_1(4-c_1^2)\\ &+2(A_1+A_2)c_1^2(4-c_1^2)|x|-\{(A_1+A_2)c_1^2+2A_1c_1-4A_2\}(4-c_1^2)|x|^2|. \end{aligned} \quad (13)$$

Using the values of $A_1,\ A_2$ and $A_3$ from (12), upon simplification, we get

$$\begin{aligned} A_1+A_2+4A_3 &= 12\beta^2(\alpha p+\beta)^3-4(1-\alpha^2)p^2(\alpha p+2\beta)^2(\alpha p+3\beta),\\ A_1+A_2 &= 12\beta^2(\alpha p+\beta)^3, \end{aligned} \quad (14)$$

and

$$\begin{aligned} (A_1+A_2)c_1^2+2A_1c_1-4A_2 &= 12\beta^2(\alpha p+\beta)^3\left[c_1^2+2\frac{(\alpha p+2\beta)^2}{\beta^2}c_1+4\frac{(\alpha p+\beta)(\alpha p+3\beta)}{\beta^2}\right]\\ &= 12\beta^2(\alpha p+\beta)^3\left[\left\{c_1+\frac{(\alpha p+2\beta)^2}{\beta^2}\right\}^2-\left\{\frac{(\alpha p+2\beta)^4}{\beta^4}-\frac{4(\alpha p+\beta)(\alpha p+3\beta)}{\beta^2}\right\}\right]\\ &= 12\beta^2(\alpha p+\beta)^3\left[\left\{c_1+\frac{(\alpha p+2\beta)^2}{\beta^2}\right\}^2-\left\{\sqrt{\frac{\alpha^4p^4+8\alpha^3p^3\beta+20\alpha^2p^2\beta^2+16\alpha p\beta^3+4\beta^4}{\beta^4}}\right\}^2\right]\\ &= 12\beta^2(\alpha p+\beta)^3\left[c_1+\left\{\frac{(\alpha p+2\beta)^2}{\beta^2}+\sqrt{\frac{\alpha^4p^4+8\alpha^3p^3\beta+20\alpha^2p^2\beta^2+16\alpha p\beta^3+4\beta^4}{\beta^4}}\right\}\right]\\ &\times\left[c_1+\left\{\frac{(\alpha p+2\beta)^2}{\beta^2}-\sqrt{\frac{\alpha^4p^4+8\alpha^3p^3\beta+20\alpha^2p^2\beta^2+16\alpha p\beta^3+4\beta^4}{\beta^4}}\right\}\right]. \end{aligned} \quad (15)$$

By Lemma 1, $c_1\in[0,2]$. Using the result $(c_1+a)(c_1+b)\ge(c_1-a)(c_1-b)$, where $a,\ b\ge 0$ in the right hand side of (15), we get

$$(A_1+A_2)c_1^2+2A_1c_1-4A_2 \ge 12\beta^2(\alpha p+\beta)^3c_1^2-24(\alpha p+2\beta)^2(\alpha p+\beta)^3c_1+48(\alpha p+\beta)^4(\alpha p+3\beta). \quad (16)$$

Making use of (14) and (16) in (13) yield

$$\begin{aligned}4|A_1c_1c_3+A_2c_2^2+A_3c_1^4| \le |&12\beta^2(\alpha p+\beta)^3c_1^4-4(1-\alpha^2)p^2(\alpha p+2\beta)^2(\alpha p+3\beta)c_1^4\\ &+24(\alpha p+\beta)^3(\alpha p+2\beta)^2c_1(4-c_1^2)+24\beta^2(\alpha p+\beta)^3c_1^2(4-c_1^2)|x|\\ &-\left\{12\beta^2(\alpha p+\beta)^3c_1^2-24(\alpha p+\beta)^3(\alpha p+2\beta)^2c_1\right.\\ &\left.+48(\alpha p+\beta)^4(\alpha p+3\beta)\right\}(4-c_1^2)|x|^2|. \quad (17)\end{aligned}$$

Choosing $c_1=c\in[0,2]$, applying triangle inequality and replacing $|x|$ by $\mu$ in the right hand side of (17), assuming that $12\beta^2(\alpha p+\beta)^3c^2-24(\alpha p+\beta)^3(\alpha p+2\beta)^2c+48(\alpha p+\beta)^4(\alpha p+3\beta)\ge 0$, we get

$$\begin{aligned}4|A_1c_1c_3+A_2c_2^2+A_3c_1^4| \le\ &12\beta^2(\alpha p+\beta)^3c^4+4(1-\alpha^2)p^2(\alpha p+2\beta)^2(\alpha p+3\beta)c^4\\ &+24(\alpha p+\beta)^3(\alpha p+2\beta)^2c(4-c^2)+24\beta^2(\alpha p+\beta)^3c^2(4-c^2)\mu\\ &+\left\{12\beta^2(\alpha p+\beta)^3c^2-24(\alpha p+\beta)^3(\alpha p+2\beta)^2c+48(\alpha p+\beta)^4(\alpha p+3\beta)\right\}(4-c^2)\mu^2\\ &=F(c,\mu)(\text{say})\quad (0\le\mu\le1,\ 0\le c\le2), \quad (18)\end{aligned}$$

where

$$\begin{aligned}F(c,\mu)=\ &12\beta^2(\alpha p+\beta)^3c^4+4(1-\alpha^2)p^2(\alpha p+2\beta)^2(\alpha p+3\beta)c^4\\ &+24(\alpha p+\beta)^3(\alpha p+2\beta)^2c(4-c^2)+24\beta^2(\alpha p+\beta)^3c^2(4-c^2)\mu\\ &+\left\{12\beta^2(\alpha p+\beta)^3c^2-24(\alpha p+\beta)^3(\alpha p+2\beta)^2c\right.\\ &\left.+48(\alpha p+\beta)^4(\alpha p+3\beta)\right\}(4-c^2)\mu^2. \quad (19)\end{aligned}$$

Now we assume that the upper bound for (18) occurs at an interior point of the set $\{(\mu,c):\mu\in[0,1] \text{ and } c\in[0,2]\}$.
Differentiating $F(c,\mu)$ in (19) partially with respect to $\mu$, we get

$$\begin{aligned}\frac{\partial F}{\partial\mu}&=2[12\beta^2(\alpha p+\beta)^3c^2+\{12\beta^2(\alpha p+\beta)^3c^2\\ &-24(\alpha p+\beta)^3(\alpha p+2\beta)^2c+48(\alpha p+\beta)^4(\alpha p+3\beta)\}\mu](4-c^2). \quad (20)\end{aligned}$$

For $0<\mu<1$ and for fixed $c$ with $0<c<2,\ 0<\alpha,\ \beta\le1$ and $p\in\mathbb{N}$, from (20) we observe that $\frac{\partial F}{\partial\mu}>0$. Consequently, $F(c,\mu)$ is an increasing function of $\mu$ and hence it cannot have a maximum value at any point in the interior of the closed square $[0,2]\times[0,1]$. Also, for a fixed $c\in[0,2]$, we have

$$\max_{0\le\mu\le1}F(c,\mu)=F(c,1)=G(c)(\text{say}). \quad (21)$$

Therefore, replacing $\mu$ by 1 in (19), upon simplification gives

$$\begin{aligned}G(c)=\ &[-24\beta^2(\alpha p+\beta)^3+4(1-\alpha^2)p^2(\alpha p+2\beta)^2(\alpha p+3\beta)]c^4\\ &-48(\alpha p+\beta)^3(\alpha^2p^2+4\alpha p\beta)c^2+192(\alpha p+\beta)^4(\alpha p+3\beta). \quad (22)\end{aligned}$$

$$\begin{aligned}G'(c)=-4c[&\left\{24\beta^2(\alpha p+\beta)^3-4(1-\alpha^2)p^2(\alpha p+2\beta)^2(\alpha p+3\beta)\right\}c^2\\ &+24(\alpha p+\beta)^3(\alpha^2p^2+4\alpha p\beta)].\end{aligned}$$

We observe that $G'(c)\le0$ for all values of $0<c\le2$ with $p\in\mathbb{N}$ and $0<\alpha,\beta\le1$ and $G(c)$ has real critical point at $c=0$ . Hence, the maximum value of $G(c)$ occurs at $c=0$. Thus, the upper bound of $F(c,\mu)$ corresponds to $\mu=1$ and $c=0$. From (22), we obtain

$$\max_{0\le c\le2}G(c)=192(\alpha p+\beta)^4(\alpha p+3\beta). \quad (23)$$

From the relations (18) and (23), it follows that

$$|A_1c_1c_3 + A_2c_2^2 + A_3c_1^4| \leq 48(\alpha p + \beta)^4(\alpha p + 3\beta). \tag{24}$$

From the expressions (11) and (24), upon simplification, we obtain

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \frac{4p^2}{(\alpha p + 2\beta)^2}.$$

By taking $c_1 = c = 0$ and selecting $x = -1$ in (6) and (7), we obtain $c_2 = -2$ and $c_3 = 0$. Using these values in (11) we observe that equality is attained which shows that our result is sharp.
This completes the proof of Theorem 1. □

## 4. Concluding Remark and Future Enhancement

In this paper, we have obtained the sharp upper bound for the functional $|a_{p+1}a_{p+3} - a_{p+2}^2|$ for the functions $f \in \mathcal{A}_p$ belonging to the class $\mathcal{M}_p(\alpha, \beta)$. We conclude this paper by remarking that the above theorem include several previously established results for particular values of the parameters $\alpha, \beta$ and $p$. Taking $\alpha = 1$ in Theorem 1 we obtain the result due to Krishna and Ramreddy (see [16]), while for $\alpha = \beta = 1$, we get the result of Krishna et al. (see [17]). Further, choosing $\alpha = \beta = p = 1$, we obtain the result of Janteng et al. [11]. For letting $\alpha = 0, \beta = p = 1$ we get $|a_2a_4 - a_3^2| \leq 1$. This inequalities is sharp and it coincide with the result of Janteng et al.[12]( also see [14, 15]). Now we are working on to find the upper bound for the class $\mathcal{M}_p(\alpha, \beta)$ using third Hankel determinant.

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