Electronic Journal of Mathematical Analysis and Applications
Vol. 4(2) July 2016, pp. 209-226.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

ALGEBRA OF MULTISPLIT NUMBERS

FARID MESSELMI

ABSTRACT. The purpose of this paper is to contribute to the development of a general theory of multisplit numbers. We start by introducing the notion of multisplit numbers and their algebraic operations. In addition, we consider the concept of matrix representation of multisplit numbers and we prove many properties basing on the theory of circulant matrices. Moreover, we define the generalized hyperbolic functions.

1. INTRODUCTION

The concept of multicomplex numbers has been introduced by many mathematicians and Physicists. The starting point is the introduction of a generator i, such that $i^n = -1$ and create the space of multicomplex numbers of order n, $\mathbb{MC}_n$. In keeping with the case $n = 2$ of usual complex numbers and their trigonometric functions, an associated extended trigonometry follows. It is characterized by specific "angular" functions dubbed multisine (mus). A collection of useful relations exists between the mus-functions: additions, derivatives, etc, see for more details about multicomplex numbers the references [1, 6, 7, 8, 15, 18].

More recently, the theory becomes one of the important impulses for developing some new concept of quantum mechanics and cosmology.

Let us mention that the generalisation of dual numbers and their functions have been already studied in the reference [13]. The idea was to assume the existence of an element satisfying $\varepsilon^n = 0$. Several results concerning the algebra of multidual numbers have been obtained and the concept of hyperholomorphicity was generalized for their functions.

The purpose of this paper is to contribute to the development of the split (hyperbolic) analogous, by generalizing the notion of split numbers in higher dimensions. The main key point is to introduce the multisplit unit number satisfying $h^n = 1$. In details, we begin by generalizing the notion of multisplit numbers. To this end, as in multicomplex and multidual algebras, we give the definition of multisplit numbers and some properties. In order to discover more properties of multi split numbers we will focus on the concept of circulant matrices allowing us to give another representation of multisplit numbers. Also, we show the relations which exist between

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1991 *Mathematics Subject Classification.* 15A66, 33C99, 30G35.
Key words and phrases. Multisplit numbers, circulant matrices, generalized hyperbolic functions.
Submitted Feb. 20, 2016.

complex and multisplit numbers, by establishing the different isomorphisms existing between them. In addition, we will try to answer the natural question: is there a way to extended exponential function to the multisplit algebra, and if so, it is possible to build a generalized Euler formula looking like that of multicomplex and multi dual cases. As an application, we provide a generalisation of the usual hyperbolic functions in higher dimension and we show the link that exists with the generalized trigonometric functions.

The outline of the paper is as follows. In Section 2 we concentrate on the development of multisplit numbers and their algebraic properties. Section 3 is devoted to the study of the multisplit exponential function and the generalization of hyperbolic functions in higher dimensions was carried.

In this work, we only process with the pure mathematical theory and we have not tried to find physical applications of the concepts presented here. However, we will try to find future applications.

2. Multisplit Numbers

We introduce the concept of multisplit numbers, as follows.

A multisplit number z is an ordered $n-$tuple of real numbers $(x_0, x_1, ..., x_{n-1})$ associated with the real unit 1 and the powers of the multisplit unit h_{n-1}, such that h_{n-1} satisfies $h_{n-1}^n = 1$ where it differs from the real roots of the equation $s^n = 1$. A multisplit number is usually denoted in the form

$$z = \sum_{i=0}^{n-1} x_i h_{n-1}^i. \tag{2.1}$$

for which we admit that $h_{n-1}^0 = 1$.

We denote by $\mathbb{MH}_{n-1}$ the set of multisplit numbers given by

$$\mathbb{MH}_{n-1} = \left\{ z = \sum_{i=0}^{n-1} x_i h_{n-1}^i \mid x_i \in \mathbb{R} \text{ and } h_{n-1}^n = 1 \right\} \tag{2.2}$$

Furthermore, every element $z = \sum\limits_{i=0}^{n-1} x_i h_{n-1}^i$ of $\mathbb{MH}_{n-1}$ can be also written

$$z = \mathcal{V}(z)^t \mathcal{H}_{n-1}, \tag{2.3}$$

where $\mathcal{V}(z)$ is the real vector associated to the multisplit number z given by

$$\mathcal{V}(z) = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix}, \tag{2.4}$$

and $\mathcal{H}_{n-1}$ represents the following vector, said to be the unit multisplit vector,

$$\mathcal{H}_{n-1} = \begin{bmatrix} 1 \\ h_{n-1} \\ h_{n-1}^2 \\ \vdots \\ h_{n-1}^{n-1} \end{bmatrix}. \tag{2.5}$$

There are many ways to choose the multisplit unit number h_{n-1}. As simple example, we can take the real anti-diagonal matrix

$$h_{n-1}=\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{2.6}$$

We can add and multiply any two multisplit numbers by assuming that the usual rules of arithmetic apply, as well as stipulating that $h_{n-1}^{n}=1$. Doing so enables us to write down immediately the rules for addition and multiplication

$$\sum_{i=0}^{n-1} x_i h_{n-1}^{i}+\sum_{i=0}^{n-1} y_i h_{n-1}^{i}=\sum_{i=0}^{n-1}\left(x_i+y_i\right)h_{n-1}^{i}, \tag{2.7}$$

$$\left(\sum_{i=0}^{n-1} x_i h_{n-1}^{i}\right)\cdot\left(\sum_{i=0}^{n-1} y_i h_{n-1}^{i}\right)=\left(x_0y_0+\sum_{i=1}^{n-1} x_{n-i}y_i\right)+$$
$$\sum_{i=1}^{n-2}\left(\sum_{j=1}^{i} x_{i-j}y_j+\sum_{j=i+1}^{n-1} x_{n+i-j}y_j\right)h_{n-1}^{i}+\left(\sum_{j=0}^{n-1} x_{n-j-1}y_j\right)h_{n-1}^{n-1}. \tag{2.8}$$

If $z=\sum\limits_{i=0}^{n-1} x_i h_{n-1}^{i}$ is a multisplit number, we will denote by $\mathrm{real}\,(z)$ the real part of z given by

$$\mathrm{real}\,(z)=x_0, \tag{2.9}$$

and $x_0, x_1, ..., x_{n-1}$ are called the multisplit components of z.

Thus, the multisplit numbers form a commutative ring with characteristic 0. Moreover the inherited multiplication gives the multisplit numbers the structure of $n-$dimensional associative, commutative and unitary generalized Clifford Algebra.

In abstract algebra terms, the algebra of multisplit numbers can be described as the quotient of the polynomial ring $\mathbb{R}\left[X\right]$ by the ideal generated by the polynomial X^n-1, meaning that

$$\mathbb{MH}_{n-1}\approx\mathbb{R}\left[X\right]/X^n-1. \tag{2.10}$$

If $n=1$, $\mathbb{MH}_0=\mathbb{R}$ and if $n=2$, $\mathbb{MH}_1$ is the Clifford algebra of hyperbolic numbers or split numbers, see for more details regarding split numbers the references [2, 16, 19].

It is also interesting to show that every multisplit number has another representation, using circulant matrices.

To this aim, let us denote by $\mathcal{C}_n(\mathbb{R})$ the subset of $\mathcal{M}_n(\mathbb{R})$ constituted of circulant matrices, it means that

$$\mathcal{C}_n(\mathbb{R}) = \left\{ A = (a_{ij}) \in \mathcal{M}_n(\mathbb{R}) \mid A = \begin{bmatrix} a_1 & a_n & a_{n-1} & \dots & a_2 \\ a_2 & a_1 & a_n & \dots & a_3 \\ a_3 & a_2 & a_1 & \dots & a_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \dots & a_1 \end{bmatrix} \text{ where } a_{ij} \in \mathbb{R},\ 1 \le i, j \le n \right\}. \quad (2.11)$$

It is well known that $\mathcal{C}_n(\mathbb{R})$ is a subring of $\mathcal{M}_n(\mathbb{R})$ which forms a $n-$dimensional associative, commutative and unitary Algebra, see [11].

Introducing now the map

$$\begin{cases} \mathcal{R} : \mathbb{MH}_{n-1} \longrightarrow \mathcal{C}_n(\mathbb{R}), \\ \mathcal{R}\left(\sum\limits_{i=0}^{n-1} x_i h_{n-1}^i\right) = \begin{bmatrix} x_0 & x_{n-1} & x_{n-2} & \dots & x_1 \\ x_1 & x_0 & x_{n-1} & \dots & x_2 \\ x_2 & x_1 & x_0 & \dots & x_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & x_{n-2} & x_{n-3} & \dots & x_0 \end{bmatrix} \end{cases} \quad (2.12)$$

This map will be called the matrix representation of multisplit numbers. The following results are immediate consequences of the definitions of $\mathcal{C}_n(\mathbb{R})$ and $\mathbb{MH}_{n-1}$.

Theorem 1 $\mathcal{R}$ is an isomorphism of Algebras.

Denoting by $\mathcal{C}_n^0(\mathbb{R})$ the subset of $\mathcal{C}_n(\mathbb{R})$ defined as

$$\mathcal{C}_n^0(\mathbb{R}) = \{A \in \mathcal{C}_n(\mathbb{R}) \mid \det(A) = 0\}. \quad (2.13)$$

Hence, $\mathcal{C}_n(\mathbb{R}) - \mathcal{C}_n^0(\mathbb{R})$ is a multiplicative subgroup of $GL(n)$.

Let us consider the subset $\mathcal{D}_{n-1}$, said to be the null part of $\mathbb{MH}_{n-1}$, given by

$$\mathcal{D}_{n-1} = \mathcal{R}^{-1}\left(\mathcal{C}_n^0(\mathbb{R})\right). \quad (2.14)$$

Denoting by $\mathbb{MH}_{n-1}^*$ the set $\mathbb{MH}_{n-1} - \mathcal{D}_{n-1}$. Hence, $\mathbb{MH}_{n-1}^*$ is a multiplicative group. The following result holds.

Theorem 2 There exists $\left\lfloor \frac{n}{2} \right\rfloor + 1$ prime ideals of the ring $\mathbb{MH}_{n-1}$, denoted by J_k, where $\lfloor . \rfloor$ represents the floor function, such that

$$\mathcal{D}_{n-1} = \bigcup_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} J_k, \quad (2.15)$$

Proof. It is well known, see [11], that the determinant of the circulant matrix $\mathcal{R}(z)$ can be computed by the formula

$$\det(\mathcal{R}(z)) = \prod_{k=0}^{n-1} \left(\sum_{i=0}^{n-1} x_i \exp\left(\frac{2ik\pi j}{n}\right) \right). \quad (2.16)$$

where j is the usual complex imaginary unit such that $j^2 = -1$.

Then an element $z = \sum\limits_{i=0}^{n-1} x_i h_{n-1}^i$ of $\mathbb{MH}_{n-1}$ belongs to $\mathcal{D}_{n-1}$ if and only if the following equation holds

$$\prod_{k=0}^{n-1}\left(\sum_{i=0}^{n-1} x_i \exp\left(\frac{2ik\pi j}{n}\right)\right) = 0. \tag{2.17}$$

If we admit at present that n is odd, then $\mathcal{D}_{n-1} = \bigcup\limits_{k=0}^{n-1} J_k$ such that

$$J_0 = \left\{ z = \sum_{i=0}^{n-1} x_i h_{n-1}^i \in \mathbb{MH}_{n-1} \mid \sum_{i=0}^{n-1} x_i = 0 \right\}. \tag{2.18}$$

and for $k = 1, ..., n-1$

$$J_k = \left\{ z = \sum_{i=0}^{n-1} x_i h_{n-1}^i \in \mathbb{MH}_{n-1} \mid \sum_{i=0}^{n-1} x_i \operatorname{Re}\left(\omega^{ki}\right) = 0 \right. \\ \left. \text{and } \sum_{i=0}^{n-1} x_i \operatorname{Re}\left(\omega^{ki}\right) = 0 \right\}. \tag{2.19}$$

whither $\omega_n = \exp\left(\frac{2\pi}{n} j\right)$.

Indeed, since $\det\left(\mathcal{R}\left(z\right)\right) = \prod\limits_{k=0}^{n-1}\left(\sum\limits_{i=0}^{n-1} x_i \omega_n^{ki}\right) = 0$, it follows that

$$\left(\sum_{i=0}^{n-1} x_i\right) \prod_{k=1}^{\frac{n-1}{2}} \left(\sum_{i=0}^{n-1} x_i \omega_n^{ki}\right)\left(\sum_{i=0}^{n-1} x_i \overline{\omega_n}^{ki}\right) = 0, \tag{2.20}$$

So,

$$\mathcal{D}_{n-1} = \bigcup_{k=0}^{\frac{n-1}{2}} J_k. \tag{2.21}$$

Suppose now that n is even, then $\mathcal{D}_{n-1} = \bigcup\limits_{k=0}^{n-1} J_k$ where J_k are defined by

$$J_0 = \left\{ z = \sum_{i=0}^{n-1} x_i h_{n-1}^i \in \mathbb{H}_{n-1} \mid \sum_{i=0}^{n-1} x_i = 0 \right\}. \tag{2.22}$$

$$J_{\frac{n}{2}} = \left\{ z = \sum_{i=0}^{n-1} x_i h_{n-1}^i \in \mathbb{H}_{n-1} \mid \sum_{i=0}^{n-1} (-1)^{n-1} x_i = 0 \right\}. \tag{2.23}$$

and for $k = 1, ..., \frac{n}{2}-1, \frac{n}{2}+1, ..., n-1$

$$J_k = \left\{ z = \sum_{i=0}^{n-1} x_i h_{n-1}^i \in \mathbb{MH}_{n-1} \mid \sum_{i=0}^{n-1} \operatorname{Re}\left(\omega_n^{ki}\right) x_k = 0 \right. \\ \left. \text{and } \sum_{i=0}^{n-1} \operatorname{Im}\left(\omega_n^{ki}\right) x_k = 0 \right\}. \tag{2.24}$$

Moreover, since $\det(\mathcal{R}(z)) = \prod_{k=0}^{n-1}\left(\sum_{i=0}^{n-1} x_i^{ki}\omega_n\right) = 0$, we find

$$\left(\sum_{i=0}^{n-1} x_i\right)\left[\prod_{k=1,\ k\neq\frac{n}{2}}^{\frac{n}{2}-1}\left(\sum_{i=0}^{n-1} x_i\omega_n^{ki}\right)\left(\sum_{i=0}^{n-1} x_i\overline{\omega_n}^{ki}\right)\right]\left(\sum_{i=0}^{n-1}(-1)^i x_i\right) = 0. \tag{2.25}$$

Then

$$\mathcal{D}_{n-1} = \bigcup_{k=0}^{\frac{n}{2}} J_k. \tag{2.26}$$

Thus, from (2.21) and (2.26), (2.15) follows

Our next task is to verify that the sets J_k absorb multiplication by elements of the ring $\mathbb{MH}_{n-1}$. For this purpose, let us consider two numbers $z = \sum_{i=0}^{n-1} x_i h_{n-1}^i \in J_k$ and $t = \sum_{i=0}^{n-1} y_i h_{n-1}^i \in \mathbb{MH}_{n-1}$.

We remark from the definition of J_k that $z \in J_k$ if and only if $\mathcal{V}(z)\cdot W_k = 0$, where W_k is the complex vector

$$W_k = \begin{bmatrix} 1 \\ \omega_n^k \\ \omega_n^{2k} \\ \vdots \\ \omega_n^{(n-1)k} \end{bmatrix}. \tag{2.27}$$

Let e_1 be the first element in the canonical basis of $\mathbb{R}^n$. Indeed, to prove that $zt = 0$ it is enough to verify that $\mathcal{V}(zt)\cdot W_k = 0$. For this , we proceed as follows

$$\begin{aligned}
\mathcal{V}(zt)\cdot W_k &= (\mathcal{R}(zt)\,e_1)\cdot W_k \\
&= (\mathcal{R}(z)\,\mathcal{R}(t)\,e_1)\cdot W_k \\
&= \sum_{m=0}^{n-1}\left(\sum_{l=0}^{n-1}\mathcal{R}_{ml}(z)\,\mathcal{R}_{l0}(t)\right)\omega_n^{mk} \\
&= \sum_{l=0}^{n-1}\left(\sum_{m=0}^{n-1}\mathcal{R}_{ml}(z)\,\omega_n^{mk}\right)\mathcal{R}_{l0}(t) \\
&= \sum_{l=0}^{n-1}\left(\sum_{m=0}^{l-1} x_{n+m-l}\omega_n^{mk} + \sum_{m=l}^{n-1} x_{m-l}\omega_n^{mk}\right) y_l \\
&= \sum_{l=0}^{n-1}\left(\sum_{m'=n-l}^{n-1} x_{m'}\omega_n^{(m'-n+l)k} + \sum_{m'=0}^{n-l-1} x_{m'}\omega_n^{(m'+l)k}\right) y_l.
\end{aligned}$$

Since $\omega_n^{nk} = 1$, we can infer that

$$\mathcal{V}(zt)\cdot W_k = \sum_{l=0}^{n-1}\left(\sum_{m'=0}^{n-1} x_{m'}\omega_n^{m'k}\right) y_l\omega_n^{lk}.$$

Hence, the fact that $z \in J_k$ leads to

$$\mathcal{V}(zt)\cdot W_k = 0. \tag{2.28}$$

Which permits us to deduce that $zt \in J_k$. Consequently, J_k, $k = 0, ..., n-1$, is an ideal of $\mathbb{MH}_{n-1}$.

Moreover, if we suppose that there exists $z = \sum_{i=0}^{n-1} x_i h_{n-1}^i \in \bigcap_{k=0}^{\lfloor \frac{n}{2} \rfloor} J_k$, then it verifies the system

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \dots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)^2} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} = 0. \tag{2.29}$$

(The above matrix is called the Fourier matrix). We know that his determinant is not equal to zero. Thus, $z = 0$ and so

$$\bigcap_{k=0}^{\lfloor \frac{n}{2} \rfloor} J_k = \{0\}. \tag{2.30}$$

Also, it is easy to see that

$$J_k \nsubseteq J_{k'} \quad \forall k, k' = 0, ..., n-1. \tag{2.31}$$

From another side, if $zt \in J_k$ for $k = 0, ..., n-1$, we get $\det(\mathcal{R}(z)) \det(\mathcal{R}(t)) = 0$, and so $\det(\mathcal{R}(z)) = 0$ or $\det(\mathcal{R}(t)) = 0$. Suppose that $\det(\mathcal{R}(z)) = 0$, then there exists $k' = 0, ..., n-1$ such that $z \in J_{k'}$. Thus, $zt \in J_{k'}$, which gives, taking into account (2.31), $k = k'$. We conclude that J_k, $k = 0, ..., n-1$, is prime. In particular, we can assert that $\mathbb{MH}_{n-1}/J_k$, $k = 0, ..., n-1$ is an integral domain. Which allows us to achieve the proof of the Theorem.

In addition and as consequence it is straightforward to see that the ideals J_k are pairwise coprime. Thus, in view of the Chinese remainder Theorem, it becomes

$$\mathbb{MH}_{n-1} \approx \bigotimes_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathbb{MH}_{n-1}/J_k, \tag{2.32}$$

via the following map

$$z \longmapsto \left(z + J_0, ..., z + J_{\lfloor \frac{n}{2} \rfloor}\right). \tag{2.33}$$

Further, the concept of conjugation can be also generalized to multisplit numbers. This is the object of the following definition.

Proposition 3 For all $z \in \mathbb{MH}_{n-1}$, the conjugate of z denoted by $\bar{z}$ is the multisplit number that verifies

$$z\bar{z} = \det(\mathcal{R}(z)). \tag{2.34}$$

The below result shows some useful properties of the conjugate.

Proposition 4 The following statements hold

1. $\det(\mathcal{R}(\bar{z})) = \det(\mathcal{R}(z))^{n-1}$.

2. $\mathcal{R}(\bar{z}) = adj(\mathcal{R}(z))$, where $adj(\mathcal{R}(z))$ is the adjugate matrix of $\mathcal{R}(z)$.

3. $\bar{z}$ can be calculated via the formula

$$\bar{z} = \frac{1}{n}\left(\frac{\partial \det(\mathcal{R}(z))}{\partial x_0} + \sum_{i=1}^{n-1} \frac{\partial \det(\mathcal{R}(z))}{\partial x_{n-i}} h_{n-1}^i\right). \tag{2.35}$$

4. The mapping

$$\begin{aligned} \left(\mathbb{MH}_{n-1}^{*},\times\right) &\longrightarrow \left(\mathbb{MH}_{n-1}^{*},\times\right), \\ z &\longmapsto \bar{z}, \end{aligned} \tag{2.36}$$

is an automorphism.

The third statement is an immediate consequence of Jacobi's formula regarding the derivative of determinants.

The concept of conjugate enables us to construct a structure of modulus over the multisplit algebra $\mathbb{MH}_{n-1}$, given by $n-$ubic form

$$\left\{ \begin{array}{c} \mathcal{P}:\mathbb{MH}_{n-1}\longrightarrow\mathbb{R}_{+}, \\ \mathcal{P}\left(z\right)=\det\left(\mathcal{R}\left(z\right)\right)=z\bar{z}. \end{array}\right. \tag{2.37}$$

There is no chance that the modulus $\mathcal{P}$ induces a norm over the algebra $\mathbb{MH}_{n-1}$. However, we can, basing on the formula (2.37), build a seminorm as

$$\|z\|_{\mathbb{MH}_{n-1}}=|z\bar{z}|^{\frac{1}{n}} \tag{2.38}$$

It is obvious that $\|.\|_{\mathbb{MH}_{n-1}}$ satisfies the following properties

$$\left\{ \begin{array}{c} \|z\|_{\mathbb{MH}_{n-1}}=\left|\det\left(\mathcal{R}\left(z\right)\right)\right|^{\frac{1}{n}} \quad \forall z\in\mathbb{MH}_{n-1}. \\ \|z_1z_2\|_{\mathbb{MH}_{n-1}}=\|z_1\|_{\mathbb{MH}_{n-1}}\|z_2\|_{\mathbb{MH}_{n-1}} \quad \forall z_1,z_2\in\mathbb{MH}_{n-1}, \\ \|z\|_{\mathbb{MH}_{n-1}}=0 \text{ iff } z\in\mathcal{D}_{n-1}. \end{array}\right. \tag{2.39}$$

It induces, in particular, a structure of seminormed space over the algebra $\mathbb{MH}_{n-1}$.

Thus, we can define the multisplit disk and multisplit sphere of centre $t=\sum\limits_{i=0}^{n-1}y_ih_{n-1}^{i}\in\mathbb{H}_{n-1}$ and radius $r>0$, respectively, by

$$D_{\mathbb{MH}_{n-1}}\left(t,r\right)=\left\{z=\sum_{i=0}^{n-1}x_ih_{n-1}^{i}\in\mathbb{MH}_{n-1}\mid\|z-t\|_{\mathbb{MH}_{n-1}}<r\right\}, \tag{2.40}$$

$$S_{\mathbb{MH}_{n-1}}\left(t,r\right)=\left\{z=\sum_{i=0}^{n-1}x_ih_{n-1}^{i}\in\mathbb{MH}_{n}\mid\|z-t\|_{\mathbb{MH}_{n-1}}=r\right\}. \tag{2.41}$$

$S_{\mathbb{MH}_{n-1}}\left(t,r\right)$ can be also called the generalized hyperbolic sphere, see the references [16, 19] for the particular case $n=2$.

It is worth noting that the algebra $\mathbb{MH}_{n-1}$ endowed with the topology generated by the seminorm $\|.\|_{\mathbb{MH}_{n-1}}$ is not Hausdorff space. The vanishing of the seminorm induces an identification equivalence relation that converts the seminormed space into a full-fledged normed space. This is done by defining

$$\forall z,t\in\mathbb{MH}_{n-1},\ z\backsim t \text{ iff } \exists i=0,...,\left\lfloor\frac{n}{2}\right\rfloor:z-t\in J_i. \tag{2.42}$$

Denoting by $\|.\|_{\mathbb{MH}_{n-1}}^{*}$ the map defined in the topological space $\mathbb{MH}_{n-1}/\backsim$ by $\|[z]\|_{\mathbb{MH}_{n-1}/\backsim}^{*}=\|z\|_{\mathbb{MH}_{n-1}}$, then defines $\|.\|_{\mathbb{MH}_{n-1}/\backsim}^{*}$ is a norm over $\mathbb{MH}_{n-1}/\backsim$ and $\left(\mathbb{MH}_{n-1}/\backsim,\|.\|_{\mathbb{MH}_{n-1}/\backsim}^{*}\right)$ is a well-defined normed space.

By virtue of the definition (2.38), we can affirm the following.

Lemma 5

1. $\left\|h_{n-1}^{i}\right\|_{\mathbb{MH}_{n-1}}=1\ \forall i=0,...,n-1.$

2. $\|z\|_{\mathbb{MH}_{n-1}} \leq \sum\limits_{i=0}^{n-1} |x_i| \ \forall z \in \mathbb{MH}_{n-1}$.

3. The map

$$\begin{cases} \left(\mathbb{MH}_{n-1}^{*}, \times\right) \longrightarrow \left(\mathbb{R}_{+}^{*}, \times\right), \\ z \longmapsto \|z\|_{\mathbb{MH}_{n-1}}, \end{cases} \tag{2.43}$$

is a homomorphism of groups where his kernel is given by

$$\ker\left(\|.\|_{\mathbb{MH}_{n-1}}\right) = S_{\mathbb{MH}_{n-1}}(0,1). \tag{2.44}$$

We introduce now the concepts of positive and negatives parts of $\mathbb{MH}_{n-1}$ as follows.

$$\mathbb{MH}_{n-1}^{+} = \left\{z \in \mathbb{MH}_{n-1}^{*} \mid \det\left(\mathcal{R}(z)\right) > 0\right\}. \tag{2.45}$$

$$\mathbb{MH}_{n-1}^{-} = \left\{z \in \mathbb{MH}_{n-1}^{*} \mid \det\left(\mathcal{R}(z)\right) < 0\right\}. \tag{2.46}$$

Note that these definitions are different from the standard notions of positive and negative numbers in the real case. After above notations, we will state the following proposition, which can be easily obtained.

Proposition 6

1. $\mathbb{MH}_{n-1}$ has the decomposition

$$\mathbb{MH}_{n-1} = \mathbb{MH}_{n-1}^{-} \cup \mathcal{D}_{n-1} \cup \mathbb{MH}_{n-1}^{+}. \tag{2.47}$$

2. $\left(\mathbb{MH}_{n-1}^{+}, \times\right)$ is a subgroup of $\left(\mathbb{MH}_{n-1}^{*}, \times\right)$.

3. If $z_1, z_2 \in \mathbb{MH}_{n-1}^{-}$ then $z_1 z_2 \in \mathbb{MH}_{n-1}^{+}$.

4. If n is even then $z \in \mathbb{MH}_{n-1}^{+}$ implies $-z \in \mathbb{MH}_{n-1}^{+}$ and if n is odd $z \in \mathbb{MH}_{n-1}^{+}$ implies $-z \in \mathbb{MH}_{n-1}^{-}$.

Otherwise, it is clear that if $z = \sum\limits_{i=0}^{n-1} x_i h_{n-1}^{i} \in \mathbb{MH}_{n-1}$, the eigenvalues of the matrix $\mathcal{R}(z)$ are solution of the system

$$\begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \dots & \omega_n^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)^2} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix}. \tag{2.48}$$

Thus, the Jordan canonical form of the matrix $\mathcal{R}(z)$ will be given by

$$
\mathcal{M}(z)=\begin{cases}\begin{bmatrix}\sum\limits_{i=0}^{n-1}x_i & & & \\ & \mathcal{M}_1(z) & 0 & \\ & 0 & \ddots & \\ & & & \mathcal{M}_{\frac{n-1}{2}}(z)\end{bmatrix} & \text{if } n \text{ is odd,}\\ \begin{bmatrix}\sum\limits_{i=0}^{n-1}x_i & & & & \\ & \mathcal{M}_1(z) & & 0 & \\ & & \ddots & & \\ & 0 & & \mathcal{M}_{\frac{n}{2}-1}(z) & \\ & & & & \sum\limits_{i=0}^{n-1}(-1)^i x_i\end{bmatrix} & \text{if } n \text{ is even.}\end{cases} \tag{2.49}
$$

whither the matrix $\mathcal{M}_k(z)$ is such that

$$
\mathcal{M}_k(z)=\begin{bmatrix}\sum\limits_{i=0}^{n-1}x_i\operatorname{Re}\left(\omega_n^{ki}\right) & 0\\ 0 & \sum\limits_{i=0}^{n-1}x_i\operatorname{Im}\left(\omega_n^{ki}\right)\end{bmatrix}. \tag{2.50}
$$

As an immediate consequence, we have all the necessary tools to present the following.

Proposition 7 There are $\left\lfloor \frac{n}{2}\right\rfloor+1$ nontrivial elements $e_i\in\mathbb{MH}_{n-1}$, satisfying

$$
e_i\in\mathcal{D}_{n-1} \text{ and } e_i^n=e_i, \tag{2.51}
$$

and for which if $z=\sum\limits_{i=0}^{n-1}x_ih_{n-1}^i$ is a multisplit number then z can be written in the basis $\left\{e_0,...e_{\left\lfloor \frac{n}{2}\right\rfloor}\right\}$

$$
\begin{cases} z=\sum\limits_{i=0}^{n-1}x_ie_0+\sum\limits_{k=1}^{\frac{n-1}{2}}\left(\sum\limits_{i=0}^{n-1}x_i\omega_n^{ki}\right)e_k \text{ if } n \text{ is odd,}\\ z=\sum\limits_{i=0}^{n-1}x_ie_0+\sum\limits_{k=1}^{\frac{n}{2}-1}\left(\sum\limits_{i=0}^{n-1}x_i\omega_n^{ki}\right)e_k+\sum\limits_{i=0}^{n-1}(-1)^i x_ie_{\frac{n}{2}} \text{ if } n \text{ is even.}\end{cases} \tag{2.52}
$$

Here $\left\{e_0,...e_{\left\lfloor \frac{n}{2}\right\rfloor}\right\}$ is called the diagonal null basis.

For example, if $n=3$, it comes that $e_0=\frac{1}{3}\left(1+h+h^2\right)$ and $e_1=\frac{1}{3}\left(2-h-h^2\right)$. If $n=4$, one finds $e_0=\frac{1}{4}\left(1+h+h^2+h^3\right)$, $e_1=\frac{1}{4}\left(3-h-h^2-h^3\right)$ and $e_1=\frac{1}{4}\left(1-h+h^2-h^3\right)$.

Suppose that n is odd, if we denote by $z=\alpha e_0+\sum\limits_{i=1}^{\frac{n-1}{2}}\beta_ie_i$ for a real number α and complex numbers β_i, then multisplit multiplication is given by

$$
\left(\alpha,\beta_1,...,\beta_{\frac{n-1}{2}}\right)\left(\alpha',\beta_1',...,\beta_{\frac{n-1}{2}}'\right)=\left(\alpha\alpha',\beta_1\beta_1',...,\beta_{\frac{n-1}{2}}\beta_{\frac{n-1}{2}}'\right). \tag{2.53}
$$

Thus, the algebra $\mathbb{MH}_{n-1}$ may be considered as the direct sum

$$
\mathbb{MH}_{n-1}\approx\mathbb{R}\oplus\mathbb{C}^{\frac{n-1}{2}},\ \text{(ring-isomorphic)}. \tag{2.54}
$$

In the same way if we suppose that n is even and denoting by $z = \alpha_1 e_0 + \sum_{i=1}^{\frac{n}{2}-1} \beta_i e_i + \alpha_2 e_{\frac{n}{2}}$ for a real numbers α_1 and α_2 and complex numbers β_i, then multisplit multiplication is given by

$$\left(\alpha_1, \beta_1, ..., \beta_{\frac{n}{2}-1}, \alpha_2\right)\left(\alpha_1', \beta_1', ..., \beta_{\frac{n}{2}-1}', \alpha_2'\right) = \left(\alpha_1\alpha_1', \beta_1\beta_1', ..., \beta_{\frac{n}{2}-1}\beta_{\frac{n}{2}-1}', \alpha_2\alpha_2'\right). \tag{2.55}$$

It becomes clear that

$$\mathbb{MH}_{n-1} \approx \mathbb{R} \oplus \mathbb{C}^{\frac{n}{2}-1} \oplus \mathbb{R}, \text{ (ring-isomorphic)}. \tag{2.56}$$

It is also interest to see that in the diagonal null basis the modulus can computed by

$$\begin{cases} \mathcal{P}\left(\alpha, \beta_1, ..., \beta_{\frac{n-1}{2}}\right) = \alpha \prod_{i=1}^{\frac{n-1}{2}} \|\beta_i\|^2 \text{ if } n \text{ is odd}, \\ \mathcal{P}\left(\alpha_1, \beta_1, ..., \beta_{\frac{n}{2}-1}, \alpha_2\right) = \alpha_1\alpha_2 \prod_{i=1}^{\frac{n}{2}-1} \|\beta_i\|^2 \text{ if } n \text{ is even}. \end{cases} \tag{2.57}$$

Some additional properties result. We list them in the below Corollary.

Corollary 8

1. $\mathcal{D}_{n-1} \approx \begin{cases} \left(\{0_{\mathbb{R}}\} \times \mathbb{C}^{\frac{n-1}{2}}\right) \cup \left(\mathbb{R} \times \left\{0_{\mathbb{C}^{\frac{n-1}{2}}}\right\}\right) \text{ if } n \text{ is odd} \\ \left(\{0_{\mathbb{R}}\} \times \mathbb{C}^{\frac{n}{2}-1} \times \mathbb{R}\right) \cup \left(\mathbb{R} \times \left\{0_{\mathbb{C}^{\frac{n}{2}-1}}\right\} \times \mathbb{R}\right) \cup \\ \left(\mathbb{R} \times \mathbb{C}^{\frac{n}{2}-1} \times \{0_{\mathbb{R}}\}\right) \text{ if } n \text{ is even} \end{cases}$

2. If n is odd then $S_{\mathbb{MH}_{n-1}}(0,1) \approx S$ where S is the hypersurface of $\mathbb{C}^{\frac{n-1}{2}} \times \mathbb{R}$ given by the equation

$$\left(\prod_{i=1}^{\frac{n-1}{2}} \|\beta_i\|^2\right)|\alpha| = 1. \tag{2.58}$$

If n is even then $S_{\mathbb{MH}_{n-1}}(0,1) \approx S$ where S is the hypersurface of $\mathbb{R} \times \mathbb{C}^{\frac{n-1}{2}} \times \mathbb{R}$ satisfying the equation

$$|\alpha_1|\left(\prod_{i=1}^{\frac{n-1}{2}} \|\beta_i\|^2\right)|\alpha_2| = 1. \tag{2.59}$$

3. Exponential Function

The multisplit exponential function can be obtained as extension of the exponential real function to the algebra of multisplit numbers. Nevertheless, is some technical difficulties to work with such definition. To this end, we prefer to use the exponential of matrices.

Let $A \in \mathcal{C}_n(\mathbb{R})$ and suppose that $\|A\| < +\infty$ for some norm. It is well known that the exponential of A can be defined by the series

$$\exp(A) = e^A = \sum_{m=0}^{+\infty} \frac{1}{m!} A^m. \tag{3.1}$$

In addition, the series converges normally in each bounded domain of $\mathcal{C}_n(\mathbb{R})$.

Since $A \in \mathcal{C}_n(\mathbb{R})$, we can affirm that for all $m \in \mathbb{N}$ we have $A^m \in \mathcal{C}_n(\mathbb{R})$. Thus, by passage to the limit it comes

$$\exp(A) \in \mathcal{C}_n(\mathbb{R}). \tag{3.2}$$

Introducing now the function E of multisplit variable defined for each $z \in \mathbb{MH}_{n-1}$ by

$$E(z) = \left(\mathcal{R}^{-1} \circ \exp \circ \mathcal{R}\right)(z). \tag{3.3}$$

Thanks to this definition, one has

$$\mathcal{R}(E(z)) = \exp(\mathcal{R}(z)), \tag{3.4}$$

Definiton 1 The multisplit exponential function is well-defined by

$$\begin{cases} E : \mathbb{MH}_{n-1} \longrightarrow \mathbb{MH}_{n-1} \\ z \longmapsto E(z) \quad \forall z \in \mathbb{MH}_{n-1}. \end{cases} \tag{3.5}$$

Some properties of the multisplit exponential function are collected in the following.

Proposition 9

1. $E(z_1 + z_2) = E(z_1) E(z_2) \ \forall z_1, z_2 \in \mathbb{MH}_{n-1}$.
2. $E(0) = 1$.
3. $E(-z) = \frac{1}{E(z)} \ \forall z \in \mathbb{MH}_{n-1}$.
4. $E(z) = e^z \quad \forall z \in \mathbb{R}$.
5. $E(z) \in \mathbb{MH}^+_{n-1} \quad \forall z \in \mathbb{MH}_{n-1}$.

Proof. 1. For all $z_1, z_2 \in \mathbb{MH}_{n-1}$ we get, keeping in mind formula (3.4)

$$\mathcal{R}(E(z_1 + z_2)) = e^{\mathcal{R}(z_1+z_2)}.$$

Since, $\mathcal{R}(z_1)\mathcal{R}(z_2) = \mathcal{R}(z_2)\mathcal{R}(z_1)$, we find

$$\begin{aligned} \mathcal{R}(E(z_1 + z_2)) &= e^{\mathcal{R}(z_1)} e^{\mathcal{R}(z_2)} \\ &= \mathcal{R}(E(z_1))\mathcal{R}(E(z_2)) \\ &= \mathcal{R}(E(z_1) E(z_2)). \end{aligned}$$

Therefore, the proof of the first statement is completed.
2. We have

$$\begin{aligned} \mathcal{R}(E(0)) &= e^{\mathcal{R}(0)} \\ &= I_n = \mathcal{R}(1). \end{aligned}$$

3. For every $z \in \mathbb{MH}_{n-1}$, it follows by the use of the previous statements

$$1 = E(z) E(-z).$$

Which allows us to deduce the third one.
4. If $z \in \mathbb{R}$, one can find

$$\begin{aligned} \mathcal{R}(E(z)) &= e^{zI_n} \\ &= e^z I_n \\ &= e^z \mathcal{R}(1) \\ &= \mathcal{R}(e^z). \end{aligned}$$

So, $E(z) = e^z$.
5. Since $\det \mathcal{R}(E(z)) = \det e^{\mathcal{R}(z)}$, we get

$$\begin{aligned} \det \mathcal{R}(E(z)) &= e^{tr(\mathcal{R}(z)} \\ &= e^{n \operatorname{real}(z)} > 0. \end{aligned}$$

Thus, $E(z) \in \mathbb{MH}^+_{n-1}$.

From now on we will denote by e^z the multisplit exponential of z instead of $E(z)$.

We introduce now the concept of generalized hyperbolic functions, generalizing the hyperbolic functions cosh and sinh in higher dimensions. To do this, we write the exponential of $z = \sum\limits_{i=0}^{n-1} x_i h_{n-1}^i \in \mathbb{MH}_{n-1}$ in term of its real and multisplit components as

$$e^{\sum\limits_{i=0}^{n-1} x_i h_{n-1}^i} = \sum_{i=0}^{n-1} \theta_i (x_0, ..., x_{n-1}) h_{n-1}^i. \tag{3.6}$$

Proposition 10 The following properties are fulfilled

1. $\nabla \left(\mathcal{V}(e^z)^t \right) = \mathcal{R}(e^z)$.

2. $\sum\limits_{i=0}^{n-1} \theta_i (x_0, ..., x_{n-1}) = e^{\sum\limits_{i=0}^{n-1} x_i}$.

Proof. 1. By differentiating the formula (3.6) with respect to the real variable x_j, $j = 0, ..., n-1$ we get

$$h^j e^{\sum\limits_{i=0}^{n-1} x_i h_{n-1}^i} = \sum_{i=0}^{n-1} \frac{\partial \theta_i}{\partial x_j} (x_0, ..., x_{n-1}) h_{n-1}^i.$$

So

$$\sum_{i=0}^{n-1} \theta_i (x_0, ..., x_{n-1}) h_{n-1}^{i+j} = \sum_{i=0}^{n-1} \frac{\partial \theta_i}{\partial x_j} (x_0, ..., x_{n-1}) h_{n-1}^i.$$

Hence, if follows that

$$\begin{cases} \frac{\partial \theta_i}{\partial x_j} = \theta_{i-j} \text{ if } i \geq j, \\ \frac{\partial \theta_i}{\partial x_j} = \theta_{n+i-j} \text{ if } \ i < j. \end{cases}$$

This can be written in matrix form

$$\begin{bmatrix} \frac{\partial \theta_0}{\partial x_0} & \frac{\partial \theta_0}{\partial x_1} & \frac{\partial \theta_0}{\partial x_2} & \cdots & \frac{\partial \theta_0}{\partial x_{n-1}} \\ \frac{\partial \theta_1}{\partial x_0} & \frac{\partial \theta_1}{\partial x_1} & \frac{\partial \theta_1}{\partial x_2} & \cdots & \frac{\partial \theta_1}{\partial x_{n-1}} \\ \frac{\partial \theta_2}{\partial x_0} & \frac{\partial \theta_2}{\partial x_1} & \frac{\partial \theta_2}{\partial x_2} & \cdots & \frac{\partial \theta_2}{\partial x_{n-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \theta_{n-1}}{\partial x_0} & \frac{\partial \theta_{n-1}}{\partial x_1} & \frac{\partial \theta_{n-1}}{\partial x_2} & \cdots & \frac{\partial \theta_{n-1}}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \theta_0 & \theta_{n-1} & \theta_{n-2} & \cdots & \theta_1 \\ \theta_1 & \theta_0 & \theta_{n-1} & \cdots & \theta_2 \\ \theta_2 & \theta_1 & \theta_0 & \cdots & \theta_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \theta_{n-1} & \theta_{n-2} & \theta_{n-3} & \cdots & \theta_0 \end{bmatrix}.$$

This gives

$$\nabla \left(\mathcal{V}(e^z)^t \right) = \mathcal{R}(e^z).$$

2. We deduce using the previous property that

$$\frac{\partial}{\partial x_j} \sum_{i=0}^{n-1} \theta_i = \sum_{i=0}^{n-1} \theta_i \quad \forall j = 0, ..., n-1.$$

Which permits us to achieve the proof, using some algebraic manipulations and taking into account the fact that $\theta_0 (0, ..., 0) = 1$ and $\theta_i (0, ..., 0) = 1$, $i = 1, ..., n-1$.

We are ready now to introduce the concept of generalized hyperbolic functions. For this purpose, let us denote by $muh_{n,i}$, $i = 0, ..., n-1$ the real functions defined on $\mathbb{R}^{n-1}$ by

$$muh_{n,i} (x_1, ..., x_{n-1}) = \theta_i (0, x_1, ..., x_{n-1}) \quad \forall (x_1, ..., x_{n-1}) \in \mathbb{R}^{n-1}. \tag{3.7}$$

The functions $muh_{n,i}$ are called the generalized hyperbolic functions.
In particular, if $n = 2$ we find the usual real hyperbolic functions, i.e.

$$muh_{2,0}(x) = \cosh x \text{ and } muh_{2,1}(x) = \sinh x \quad \forall x \in \mathbb{R}. \tag{3.8}$$

Employing the formula (3.6) one can write

$$e^{\sum\limits_{i=0}^{n-1} x_i h_{n-1}^i} = e^{x_0} \sum_{i=0}^{n-1} muh_{n,i}(x_1, ..., x_{n-1}) h_{n-1}^i. \tag{3.9}$$

Of course, all that has been done for the function θ_i is equally valid for an arbitrary the generalized hyperbolic functions and we can cite the following results listing the main properties of the generalized hyperbolic functions. The first one is an immediate consequence of from the proposition 10.

Proposition 11 The generalized hyperbolic functions satisfy

1. $\begin{cases} \frac{\partial muh_{n,i}}{\partial x_j} = muh_{n,i-j} \text{ if } i \geq j \geq 1, \\ \frac{\partial muh_{n,i}}{\partial x_j} = muh_{n,n+i-j} \text{ if } 0 \leq i < j. \end{cases}$

2. $\sum\limits_{i=0}^{n-1} muh_{n,i}(x_1, ..., x_{n-1}) = e^{\sum\limits_{i=1}^{n-1} x_i}$.

The second one is given by.

Proposition 12

1. $\frac{\partial^n muh_{n,i}}{\partial x_j^n} = muh_{n,i}, \; i = 0, ..., n-1$ and $j = 1, ..., n-1$.

2. If $n = pj$, then $\frac{\partial^p muh_{n,i}}{\partial x_j^p} = muh_{n,i}, \; i = 0, ..., n-1$.

Proof. 1. Making use the formula (3.9) we get

$$e^{\sum\limits_{i=1}^{n-1} x_i h_{n-1}^i} = \sum_{i=0}^{n-1} muh_{n,i}(x_1, ..., x_{n-1}) h_{n-1}^i. \tag{3.10}$$

By taking n times the derivate of the two hand side of the expression (3.10) with respect to the variable x_j we deduce that

$$h^{nj} e^{\sum\limits_{i=1}^{n-1} x_i h_{n-1}^i} = \sum_{i=0}^{n-1} \frac{\partial^n muh_{n,i}}{\partial x_j^n}(x_1, ..., x_{n-1}) h_{n-1}^i.$$

Which implies, using again (3.10), that

$$\sum_{i=0}^{n-1} \frac{\partial^n muh_{n,i}}{\partial x_j^n}(x_1, ..., x_{n-1}) h_{n-1}^i = \sum_{i=0}^{n-1} muh_{n,i}(x_1, ..., x_{n-1}) h_{n-1}^i \quad \forall i = 0, ..., n-1.$$

Hence, the first statement done.

2. Suppose that $n = pj$, differentiating p times the two hand side of the expression (3.10) with respect to the variable x_j one obtains

$$h^{pj} e^{\sum\limits_{i=1}^{n-1} x_i h_{n-1}^i} = \sum_{i=0}^{n-1} \frac{\partial^p muh_{n,i}}{\partial x_j^p}(x_1, ..., x_{n-1}) h_{n-1}^i.$$

This permits us to conclude the proof.

Elsewhere, since

$$\det\left(\mathcal{R}\left(\sum_{i=0}^{n-1} muh_{n,i}(x_1, ..., x_{n-1}) h_{n-1}^i\right)\right) = e^{tr\left(\mathcal{R}\left(\sum\limits_{i=1}^{n-1} x_i h_{n-1}^i\right)\right)}.$$

We can infer

$$\det\left(\mathcal{R}\left(\sum_{i=0}^{n-1} muh_{n,i}\left(x_1,...,x_{n-1}\right)h_{n-1}^i\right)\right)=1\ \ \forall\left(x_1,...,x_{n-1}\right)\in\mathbb{R}^{n-1}. \quad (3.11)$$

This suggests us a generalization of the standard hyperbolic identity $\cosh^2 x - \sinh^2 x = 1$ in higher dimensions.

For the particular case $n=3$, one can find

$$muh_{3,0}^3\left(x_1,x_2,x_3\right)+muh_{3,1}^3\left(x_1,x_2,x_3\right)+muh_{3,2}^3\left(x_1,x_2,x_3\right)- \\ 3muh_{3,0}\left(x_1,x_2,x_3\right)muh_{3,1}\left(x_1,x_2,x_3\right)muh_{3,2}\left(x_1,x_2,x_3\right)=1.$$

Particularly, we can affirm that for all $\left(x_1,...,x_{n-1}\right)\in\mathbb{R}^{n-1}$ we have

$$\mathcal{R}\left(e^{\sum\limits_{i=1}^{n-1} x_i h_{n-1}^i}\right)\in SL\left(n,\mathbb{R}\right). \quad (3.12)$$

The below result also holds.

Proposition 13 The mapping defined by

$$\left\{\begin{array}{c}\left(\mathbb{R}^{n-1},+\right)\longrightarrow\left(S_{\mathbb{MH}_{n-1}}\left(0,1\right),\times\right), \\ \left(x_1,...,x_{n-1}\right)\longmapsto e^{\sum\limits_{i=1}^{n-1} x_i h_{n-1}^i}=\sum\limits_{i=0}^{n-1} muh_{n,i}\left(x_1,...,x_{n-1}\right)h_{n-1}^i,\end{array}\right. \quad (3.13)$$

is an injective homomorphism of groups.

Furthermore, since

$$e^{\sum\limits_{k=1}^{n-1} x_i h_{n-1}^i}=\prod_{i=1}^{n-1}\sum_{m\geq 0}\frac{1}{m}x_i^m h_{n-1}^{mi}, \quad (3.14)$$

it is easy to verify that the generalized hyperbolic functions can be represented only using suitable combinations of the real functions

$$\pi_{n,q}\left(t\right)=\sum_{m\geq 0}\frac{t^{nm+q}}{(nm+q)!},\quad n\geq 2 \text{ and } q=0,...,n-1. \quad (3.15)$$

As example, if $n=2$, we have

$$\left\{\begin{array}{l} muh_{2,0}\left(x\right)=\pi_{2,0}\left(x\right), \\ muh_{2,1}\left(x\right)=\pi_{2,1}\left(x\right),\end{array}\right. \quad (3.16)$$

and if $n=3$, we find

$$\left\{\begin{array}{l} muh_{3,0}\left(x_1,x_2\right)=\pi_{3,0}\left(x_1\right)\pi_{3,0}\left(x_2\right)+\pi_{3,1}\left(x_1\right)\pi_{3,1}\left(x_2\right)+\pi_{3,2}\left(x_1\right)\pi_{3,2}\left(x_2\right), \\ muh_{3,1}\left(x_1,x_2\right)=\pi_{3,0}\left(x_1\right)\pi_{3,2}\left(x_2\right)+\pi_{3,1}\left(x_1\right)\pi_{3,0}\left(x_2\right)+\pi_{3,2}\left(x_1\right)\pi_{3,1}\left(x_2\right), \\ muh_{3,2}\left(x_1,x_2\right)=\pi_{3,0}\left(x_1\right)\pi_{3,1}\left(x_2\right)+\pi_{3,1}\left(x_1\right)\pi_{3,2}\left(x_2\right)+\pi_{3,2}\left(x_1\right)\pi_{3,0}\left(x_2\right).\end{array}\right. \quad (3.17)$$

Various properties of the functions $\pi_{n,q}$ have been given in the references [3, 18], using standard arguments of real analysis. Additionally, the theory of multisplit numbers allows us to show that the functions $\pi_{n,q}$ possess some other interesting properties analogous to those of the real hyperbolic functions cosh and sinh. This is illustrated in the following**.**

Proposition 14

1. For every $t \in \mathbb{R}$, we have

$$\begin{vmatrix} \pi_{n,0}(t) & \pi_{n,n-1}(t) & \pi_{n,n-2}(t) & \dots & \pi_{n,1}(t) \\ \pi_{n,1}(t) & \pi_{n,0}(t) & \pi_{n,n-1}(t) & \dots & \pi_{n,2}(t) \\ \pi_{n,2}(t) & \pi_{n,1}(t) & \pi_{n,0}(t) & \dots & \pi_{n,3}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \pi_{n,n-1}(t) & \pi_{n,n-2}(t) & \pi_{n,n-3}(t) & \dots & \pi_{n,0}(t) \end{vmatrix} = 1. \tag{3.18}$$

2. For every $t, s \in \mathbb{R}$, we have

$$\pi_{n,0}(t+s) = \pi_{n,0}(t)\,\pi_{n,0}(s) + \sum_{i=1}^{n-1} \pi_{n,n-i}(t)\,\pi_{n,i}(s).$$

$$\pi_{n,i}(t+s) = \sum_{j=1}^{i} \pi_{n,i-j}(t)\,\pi_{n,j}(s) + \sum_{j=i+1}^{n-1} \pi_{n,n+i-j}(t)\,\pi_{n,j}(s) \text{ if } i = 1, ..., n-2.$$

$$\pi_{n,n-1}(t+s) = \sum_{j=0}^{n-1} \pi_{n,n-j-1}(t)\,\pi_{n,j}(s).$$

Proof. 1. Denoting by π the function defined by

$$\begin{cases} \pi : \mathbb{R} \longrightarrow \mathbb{MH}_{n-1}, \\ t \longmapsto \pi(t) = \sum\limits_{i=0}^{n-1} \pi_{n,i}(t)\, h_{n-1}^{i}. \end{cases}$$

Observing preliminarily, by construction of the functions $\pi_{n,q}$, that

$$\begin{cases} \frac{d\pi_{n,q}}{dt} = \pi_{n,p-1}(t) \text{ if } q = 1, ..., n-1, \\ \frac{d\pi_{n,0}}{dt} = \pi_{n,n-1}(t). \end{cases}$$

This yields

$$\frac{d\pi}{dt} = \pi_{n,n-1}(t) + \sum_{i=1}^{n-1} \pi_{n,i-1}(t)\, h_{n-1}^{i}.$$

So,

$$\begin{aligned} \frac{d\pi}{dt} &= h\left(h_{n-1}^{n-1}\pi_{n,n-1}(t) + h_{n-1}\sum_{i=1}^{n-1} \pi_{n,i-1}(t)\, h_{n-1}^{i-1}\right) \\ &= h_{n-1}\pi(t). \end{aligned}$$

Which leads, taking into account the fact that $\pi(0) = 1$, to

$$\pi(t) = e^{th_{n-1}}.$$

Then, we can infer

$$\begin{aligned} \det \mathcal{R}(\pi(t)) &= \det \mathcal{R}\left(e^{th_{n-1}}\right) \\ &= \det e^{\mathcal{R}(th_{n-1})} \\ &= e^{tr\mathcal{R}(th_{n-1})} \\ &= 1. \end{aligned}$$

Consequently, (3.18) follows.

2. Let $t, s \in \mathbb{R}$, it is easy to check that

$$\begin{aligned}\sum_{i=0}^{n-1} \pi_{n,i}(t+s)\, h_{n-1}^{i} &= \pi(t+s)\\ &= \pi(t)\,\pi(s)\\ &= \left(\sum_{i=0}^{n-1} \pi_{n,i}(t)\, h_{n-1}^{i}\right)\left(\sum_{i=0}^{n-1} \pi_{n,i}(s)\, h_{n-1}^{i}\right).\end{aligned}$$

So, thanks to (2.8), the second statement follows.

We present in the following result the link existing between the generalized hyperbolic functions and the generalized trigonometric functions. To this aim, we shall start by recalling, according to the references [1, 6, 7, 8, 15, 18], the concept of multicomplex numbers. Denoting by i_{n-1} the unit multicomplex number satisfying $i_{n-1}^{n} = -1$ and by $\mathbb{MC}_{n-1}$ the algebra of multicomplex numbers given by

$$\mathbb{MC}_{n-1} = \left\{ z = \sum_{k=0}^{n-1} x_k i_{n-1}^{k} \mid x_k \in \mathbb{R} \text{ and } i_{n-1}^{n} = 1 \right\}. \tag{3.19}$$

We also remember the generalized trigonometric functions defined via the generalized Euler formula

$$e^{\sum\limits_{k=1}^{n-1} x_i h_{n-1}^{i}} = \sum_{k=1}^{n-1} mus_{n-1,k}(x_1, ..., x_{n-1})\, i_{n-1}^{k}. \tag{3.20}$$

The desired result is the content of the following

Proposition 15 The following formula holds.

$$muh_k\left(i_{n-1}x_1, i_{n-1}^{2}x_2, ..., i_{n-1}^{n-1}x_{n-1}\right) = i_{n-1}^{k} mus_k(x_1, x_2, ..., x_{n-1}), \quad k = 0, ..., n-1. \tag{3.21}$$

Proof. We get keeping in mind formula (3.10)

$$e^{\sum\limits_{k=1}^{n-1} x_k i_{n-1}^{k} h_{n-1}^{k}} = \sum_{k=1}^{n-1} muh_k\left(i_{n-1}x_1, i_{n-1}^{2}x_2, ..., i_{n-1}^{n-1}x_{n-1}\right) h_{n-1}^{k}. \tag{3.22}$$

On the other hand, one can easily check that $i_{n-1}^{n} h_{n-1}^{n} = -1$, then $i_{n-1}h_{n-1}$ is another unit multicomplex number. So, from formula (3.20), we can infer

$$e^{\sum\limits_{k=1}^{n-1} x_k i_{n-1}^{k} h_{n-1}^{k}} = \sum_{k=1}^{n-1} mus_k(x_1, ..., x_{n-1})\, i_{n-1}^{k} h_{n-1}^{k}. \tag{3.23}$$

Consequently, by (3.22) and (3.23), formula (3.23) immediately follows

This formula generalizes the usual identities

$$\cosh(ix) = \cos x \text{ if } \sinh(ix) = i \sin x. \tag{3.24}$$

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Farid Messelmi, Department of Mathematics and Computer Science and LDMM Laboratory, Faculty of Sciences and Technology, University Ziane Achour of Djelfa, Djelfa, Algeria

E-mail address: foudimath@yahoo.fr