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GENERALIZATIONS OF SHERMAN'S INEQUALITY BY MONTGOMERY IDENTITY AND GREEN FUNCTION

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Abstract. In this paper, we give generalization of Sherman inequality by using Green function and Montgomery identity. We present Grüss and Ostrowski-type inequalities related to generalized Sherman inequality. We give mean value theorems and n-exponential convexity for the functional associated to generalized inequality. We also give a family of functions which support our results for exponentially convex functions and construct a class of means.

1. Introduction

For fixed $m \geq 2$, let $\mathbf{x} = (x_1, ..., x_m)$ and $\mathbf{y} = (y_1, ..., y_m)$ denote two m-tuples. Let

$$x_{[1]} \geq x_{[2]} \geq ... \geq x_{[m]}, \quad y_{[1]} \geq y_{[2]} \geq ... \geq y_{[m]},$$
$$x_{(1)} \leq x_{(2)} \leq ... \leq x_{(m)}, \quad y_{(1)} \leq y_{(2)} \leq ... \leq y_{(m)}$$

be their ordered components. We say that $\mathbf{x}$ *majorizes* $\mathbf{y}$ or $\mathbf{y}$ is *majorized* by $\mathbf{x}$ and write

$$\mathbf{y} \prec \mathbf{x}$$

if

$$\sum_{i=1}^{k} y_{[i]} \leq \sum_{i=1}^{k} x_{[i]}, \quad k = 1, ..., m-1, \tag{1}$$
$$\sum_{i=1}^{m} y_i = \sum_{i=1}^{m} x_i.$$

Note that (1) is equivalent to

$$\sum_{i=m-k+1}^{m} y_{(i)} \leq \sum_{i=m-k+1}^{m} x_{(i)}, \quad k = 1, ..., m-1.$$

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The following notion of Schur-convexity generalizes the definition of convex function via the notion of majorization.

A function $F : S \subseteq \mathbb{R}^m \to \mathbb{R}$ is called *Schur-convex* on S if

$$F(\mathbf{y}) \leq F(\mathbf{x}) \tag{2}$$

for every $\mathbf{x}$, $\mathbf{y} \in S$ such that

$$\mathbf{y} \prec \mathbf{x}.$$

A relation between one-dimensional convex function and m-dimensional Schur-convex function is included in the following *Majorization theorem* proved by G. H. Hardy, J. E. Littlewood and G. Pólya (see [10], [15, p. 333]).

Theorem 1 [Majorization theorem] Let $I \subset \mathbb{R}$ be an interval and $\mathbf{x} = (x_1, ..., x_m)$, $\mathbf{y} = (y_1, ..., y_m) \in I^m$. Let $\phi : I \to \mathbb{R}$ be continuous function. Then a function $F : I^m \to \mathbb{R}$, defined by

$$F(\mathbf{x}) = \sum_{i=1}^{m} \phi(x_i),$$

is Schur-convex on I^m if and only if ϕ is convex on I.

The following theorem gives weighted generalization of Majorization theorem (see [9], [15, p. 323]).
Theorem 2 [Fuchs's theorem] Let $\mathbf{x} = (x_1, ..., x_m)$, $\mathbf{y} = (y_1, ..., y_m) \in I^m$ be two decreasing m-tuples and $\mathbf{p} = (p_1, ..., p_m)$ be a real m-tuple such that

$$\sum_{i=1}^{k} p_i y_i \leq \sum_{i=1}^{k} p_i x_i, \quad k = 1, ..., m-1,$$
$$\sum_{i=1}^{m} p_i y_i = \sum_{i=1}^{m} p_i x_i.$$

Then for every continuous convex function $\phi : I \to \mathbb{R}$, we have

$$\sum_{i=1}^{m} p_i \phi(y_i) \leq \sum_{i=1}^{m} p_i \phi(x_i).$$

The Jensen inequality in the form

$$\phi\left(\frac{1}{P_m}\sum_{i=1}^{m} p_i x_i\right) \leq \frac{1}{P_m}\sum_{i=1}^{m} p_i \phi\left(x_i\right) \tag{3}$$

for convex function ϕ, where $\mathbf{p} = (p_1, \ldots, p_m)$ is a nonnegative m-tuple such that $P_m = \sum_{i=1}^{m} p_i > 0$, can be obtained as a special case of the previous result putting $y_1 = y_2 = ... = y_m = \frac{1}{P_m}\sum_{i=1}^{m} p_i x_i$.
A natural problem of interest is extension of notation from m-tuples (vectors) to $m \times l$ matrices $\mathbf{A} = (a_{ij}) \in \mathcal{M}_{ml}(\mathbb{R})$. In order that, we introduce the notion of row stochastic and double stochastic matrices.

A matrix $\mathbf{A} = (a_{ij}) \in \mathcal{M}_{ml}(\mathbb{R})$ is called *row stochastic* if all of its entries are greater or equal to zero, i.e. $a_{ij} \geq 0$ for $i = 1, ..., m,\ j = 1, ..., l$ and the sum of the entries in each row is equal to 1, i.e. $\sum_{j=1}^{l} a_{ij} = 1$ for $i = 1, ..., m$. If in addition the transpose $\mathbf{A}^T = (a_{ji})$ of $\mathbf{A} = (a_{ij})$ is row stochastic, then $\mathbf{A}$ is called *doubly stochastic*. In other words, $\mathbf{A} = (a_{ij}) \in \mathcal{M}_{ml}(\mathbb{R})$ is called *double stochastic* if all of its entries are greater or equal to zero (nonnegative), i.e. $a_{ij} \geq 0$ for $i = 1, ..., m$, $j = 1, ..., l$ and the sum of the entries in each column and each row is equal to 1, i.e. $\sum_{i=1}^{m} a_{ij} = 1$ for $j = 1, ..., l$ and $\sum_{j=1}^{l} a_{ij} = 1$ for $i = 1, ..., m$.

It is well known that for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^l$ is valid

$$\mathbf{y} \prec \mathbf{x} \quad \text{if and only if} \quad \mathbf{y} = \mathbf{x}\mathbf{A}$$

for some double stochastic matrix $\mathbf{A} \in \mathcal{M}_{ml}(\mathbb{R})$.

The concept of majorization has many important applications in different areas of research. Some recent results, which give generalizations of majorization theorem with applications, can be found in [1]-[5], [12].

The next generalization is obtained by S. Sherman (see [13], [17]).

Theorem 3 [Sherman's theorem] Let $\mathbf{x} = (x_1, ..., x_l) \in [\alpha, \beta]^l$, $\mathbf{y} = (y_1, ..., y_m) \in [\alpha, \beta]^m$, $\mathbf{u} = (u_1, ..., u_l) \in [0, \infty)^l$, $\mathbf{v} = (v_1, ..., v_m) \in [0, \infty)^m$ and

$$\mathbf{y} = \mathbf{x}\mathbf{A}^T \text{ and } \mathbf{u} = \mathbf{v}\mathbf{A} \tag{4}$$

for some row stochastic matrix $\mathbf{A} = (a_{ij}) \in \mathcal{M}_{ml}(\mathbb{R})$. Then for every convex function $\phi : [\alpha, \beta] \to \mathbb{R}$ we have

$$\sum_{i=1}^{m} v_i \phi(y_i) \leq \sum_{j=1}^{l} u_j \phi(x_j). \tag{5}$$

Remark 1 In a special case, from Sherman's theorem we get Fuchs's theorem. When $m = l$, and all weights v_i and u_j are equal and nonnegative, the condition $\mathbf{u} = \mathbf{v}\mathbf{A}$ assures the stochasticity on columns, so in that case we deal with doubly stochastic matrices.

Consider the Green function G defined on $[\alpha, \beta] \times [\alpha, \beta]$ by

$$G(t, s) = \begin{cases} \frac{(t-\beta)(s-\alpha)}{\beta-\alpha}, & \alpha \leq s \leq t; \\ \frac{(s-\beta)(t-\alpha)}{\beta-\alpha}, & t \leq s \leq \beta. \end{cases} \tag{6}$$

The function G is convex in s, it is symmetric, so it is also convex in t. The function G is continuous in s and continuous in t.

For any function $\phi : [\alpha, \beta] \to \mathbb{R}$, $\phi \in C^2([\alpha, \beta])$, we can easily show by integrating by parts that the following is valid

$$\phi(x) = \frac{\beta - x}{\beta - \alpha}\phi(\alpha) + \frac{x - \alpha}{\beta - \alpha}\phi(\beta) + \int_{\alpha}^{\beta} G(x, s)\phi''(s)ds, \tag{7}$$

where the function G is defined as above in (6) ([18]).

In order to obtain our main results in the present paper, we use the following generalized Montgomery identity given in paper [7].

Theorem 4. Let $n \in \mathbb{N}$, $\phi : I \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $\alpha, \beta \in I$ and $\alpha \leq \beta$. Then the following identity holds

$$\phi(x) = \frac{1}{\beta - \alpha}\int_{\alpha}^{\beta}\phi(t)\,dt + \sum_{k=0}^{n-2}\frac{\phi^{(k+1)}(\alpha)}{k!\,(k+2)}\frac{(x-\alpha)^{k+2}}{\beta-\alpha} - \sum_{k=0}^{n-2}\frac{\phi^{(k+1)}(\beta)}{k!\,(k+2)}\frac{(x-\beta)^{k+2}}{\beta-\alpha}$$
$$+ \frac{1}{(n-1)!}\int_{\alpha}^{\beta}T_n(x,s)\,\phi^{(n)}(s)\,ds, \tag{8}$$

where

$$T_n(x,s) = \begin{cases} -\frac{(x-s)^n}{n(\beta-\alpha)} + \frac{x-\alpha}{\beta-\alpha}(x-s)^{n-1}, & \alpha \leq s \leq x, \\ \\ -\frac{(x-s)^n}{n(\beta-\alpha)} + \frac{x-\beta}{\beta-\alpha}(x-s)^{n-1}, & x < s \leq \beta. \end{cases} \tag{9}$$

To complete the Introduction, we state definition of divided differences and n-convexity (see for example [15]).

Definition 1 The divided difference of order n, $n \in \mathbb{N}$, of the function $\phi : [\alpha, \beta] \to \mathbb{R}$ at mutually different points $x_0, x_1, ..., x_n \in [\alpha, \beta]$ is defined recursively by

$$[x_i; \phi] = \phi(x_i), \quad i = 0, ..., n$$
$$[x_0, ..., x_n; \phi] = \frac{[x_1, ..., x_n; \phi] - [x_0, ..., x_{n-1}; \phi]}{x_n - x_0}.$$

The value $[x_0, ..., x_n; \phi]$ is independent of the order of the points $x_0, ..., x_n$.

This definition may be extended to include the case in which some or all the points coincide. Assuming that $\phi^{(j-1)}(x)$ exists, we define

$$[\underbrace{x, ..., x}_{j\text{-times}}; \phi] = \frac{\phi^{(j-1)}(x)}{(j-1)!}. \tag{10}$$

Definition 2 A function $\phi : [\alpha, \beta] \to \mathbb{R}$ is n-convex, $n \geq 0$, if for all choices of $(n+1)$ distinct points $x_i \in [\alpha, \beta]$, $i = 0, ..., n$, the inequality

$$[x_0, x_1, ..., x_n; \phi] \geq 0$$

holds.
From Definition 2, it follows that 2-convex functions are just convex functions. Furthermore, 1-convex functions are increasing functions and 0-convex functions are nonnegative functions.

2. Main results

In the following theorem we give general identity for Sherman's inequality.

Theorem 5 Let $n \in \mathbb{N}, n \geq 4$, $\phi : I \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $\alpha, \beta \in I$, $\alpha < \beta$. Suppose that $\mathbf{x} = (x_1, ..., x_l) \in [\alpha, \beta]^l$, $\mathbf{y} = (y_1, ..., y_m) \in [\alpha, \beta]^m$, $\mathbf{u} = (u_1, ..., u_l) \in \mathbb{R}^l$, $\mathbf{v} = (v_1, ..., v_m) \in \mathbb{R}^m$ be such that (4) holds for some matrix $\mathbf{A} = (a_{ij}) \in \mathcal{M}_{ml}(\mathbb{R})$ satisfying the condition

$\sum\limits_{j=1}^{l} a_{ij} = 1,\ i = 1, 2, ..., m$. Let G and T_n be as defined in (6) and (9) respectively. Then

$$\begin{aligned}
&\sum_{p=1}^{l} u_p \phi(x_p) - \sum_{q=1}^{m} v_q \phi(y_q) \\
&= \int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{m} v_q G(y_q, t) \right] \times \\
&\left[\frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \phi(t)\, dt + \sum_{k=0}^{n-4} \frac{\phi^{(k+1)}(\alpha)}{k!\,(k+2)} \frac{(t-\alpha)^{k+2}}{\beta - \alpha} - \sum_{k=0}^{n-4} \frac{\phi^{(k+1)}(\beta)}{k!\,(k+2)} \frac{(t-\beta)^{k+2}}{\beta - \alpha} \right] dt \\
&+ \frac{1}{(n-1)!} \int_{\alpha}^{\beta} \int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{m} v_q G(y_q, t) \right] T_{n-2}(t, s)\, \phi^{(n)}(s)\, ds dt. \quad (11)
\end{aligned}$$

Proof. By an easy calculation, using (7) in $\sum\limits_{p=1}^{l} u_p \phi(x_p) - \sum\limits_{q=1}^{m} v_q \phi(y_q)$ and applying (4), we have

$$\begin{aligned}
&\sum_{p=1}^{l} u_p \phi(x_p) - \sum_{q=1}^{m} v_q \phi(y_q) \\
&= \int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{m} v_q G(y_q, t) \right] \phi''(t) dt. \quad (12)
\end{aligned}$$

By Theorem 4 the function $\phi''(t)$ can be expressed as

$$\begin{aligned}
\phi''(t) &= \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} \phi(t)\, dt + \sum_{k=0}^{n-4} \frac{\phi^{(k+1)}(\alpha)}{k!\,(k+2)} \frac{(t-\alpha)^{k+2}}{\beta - \alpha} - \sum_{k=0}^{n-4} \frac{\phi^{(k+1)}(\beta)}{k!\,(k+2)} \frac{(t-\beta)^{k+2}}{\beta - \alpha} \\
&+ \frac{1}{(n-1)!} \int_{\alpha}^{\beta} T_{n-2}(t, s)\, \phi^{(n)}(s)\, ds. \quad (13)
\end{aligned}$$

Now, combining (12) and (13), we get (11).

Now we state the main generalization of the Sherman inequality by using the above obtained identity.

Theorem 6 Suppose that all the assumptions of Theorem 5 hold. Additionally, if for any even n the function $\phi : I \to \mathbb{R}$ is n-convex and

$$\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{m} v_q G(y_q, t) \geq 0, t \in [\alpha, \beta].$$

Then the following inequality holds

$$\begin{aligned}
&\sum_{p=1}^{l} u_p \phi(x_p) - \sum_{q=1}^{m} v_q \phi(y_q) \\
&\geq \int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{m} v_q G(y_q, t) \right] \times \\
&\left[\frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \phi(t)\, dt + \sum_{k=0}^{n-4} \frac{\phi^{(k+1)}(\alpha)}{k!\,(k+2)} \frac{(t-\alpha)^{k+2}}{\beta-\alpha} - \sum_{k=0}^{n-4} \frac{\phi^{(k+1)}(\beta)}{k!\,(k+2)} \frac{(t-\beta)^{k+2}}{\beta-\alpha} \right] dt.
\end{aligned} \tag{14}$$

Proof. Since the function ϕ is n-convex so we have $\phi^{(n)} \geq 0$. Also it is obvious that if n is even then $T_n \geq 0$ because

Case I: If $\alpha \leq s \leq x$, then $x - a \geq x - s$ and hence $(x-\alpha)(x-s)^{n-1} \geq (x-s)^n$ that is

$$\frac{(x-\alpha)(x-s)^{n-1}}{\beta-\alpha} \geq \frac{(x-s)^n}{\beta-\alpha} \geq \frac{(x-s)^n}{n(\beta-\alpha)}.$$

So in this case from (9) we have $T_n \geq 0$.

Case II: If $x < s \leq \beta$, then $x - s < 0$ and $s - \beta \leq 0$. As n is even so we have $(x-s)^{n-1} < 0$, therefore we have

$$\begin{aligned}
& (x-s)^{n-1}(s-\beta) \geq 0 \\
\Rightarrow\ & (x-s)^{n-1}(x-\beta+s-x) \geq 0 \\
\Rightarrow\ & (x-s)^{n-1}(x-\beta) - (x-s)^n \geq 0
\end{aligned}$$

that is

$$\frac{(x-s)^{n-1}(x-\beta)}{\beta-\alpha} \geq \frac{(x-s)^n}{\beta-\alpha} \geq \frac{(x-s)^n}{n(\beta-\alpha)}.$$

So in this case from (9) we have $T_n \geq 0$.

Now using (6) and the positivity of T_n and $\phi^{(n)}$ in (11) we get (14).

In the following theorem we prove generalization of Sherman's theorem by using (4) for positive weights.

Theorem 7 Let $n \in \mathbb{N}, n \geq 4$, $\phi : I \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous, $I \subset \mathbb{R}$ an open interval, $\alpha, \beta \in I$, $\alpha < \beta$. Let $\mathbf{x} = (x_1, x_2, ..., x_l) \in [\alpha, \beta]^l$, $\mathbf{y} = (y_1, y_2, ..., y_m) \in [\alpha, \beta]^m$, $\mathbf{u} = (u_1, u_2, ..., u_l) \in [0, \infty]^l$ and $\mathbf{v} = (v_1, v_2, ..., v_m) \in [0, \infty]^m$ be such that (4) holds for some row stochastic matrix $\mathbf{A} = (a_{ij}) \in \mathcal{M}_{ml}(\mathbb{R})$. If n is even and ϕ is n-convex function. Then the inequality (14) holds. Moreover, if (14) holds and the function

$$L(.) = G(., t) \left(\frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} \phi(t)\, dt + \sum_{k=0}^{n-4} \frac{\phi^{(k+1)}(\alpha)}{k!\,(k+2)} \frac{(t-\alpha)^{k+2}}{\beta-\alpha} - \sum_{k=0}^{n-4} \frac{\phi^{(k+1)}(\beta)}{k!\,(k+2)} \frac{(t-\beta)^{k+2}}{\beta-\alpha} \right). \tag{15}$$

is convex on $[\alpha, \beta]$, then (5) holds.

Proof. Since the function $G(., t)$, $t \in [\alpha, \beta]$, is convex, so by Sherman's theorem it holds that

$$\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{m} v_q G(y_q, t) \geq 0, \quad t \in [\alpha, \beta].$$

Applying Theorem 6 we get (14).

Since (14) holds, the right hand side of (14) can be rewritten in the form

$$\sum_{p=1}^{l} u_p L(x_p) - \sum_{q=1}^{m} v_q L(y_q),$$

where L is defined by (15). If L is convex, then by Sherman's theorem we have

$$\sum_{p=1}^{l} u_p L(x_p) - \sum_{q=1}^{m} v_q L(y_q) \geq 0,$$

i.e. the right hand side of (14) is nonnegative, so the inequality (5) immediately follows.

3. Grüss and Ostrowski type inequalities related to generalized Sherman's Inequality

P. Cerone and S. S. Dragomir [8], considering Čebyšev functional

$$T(f,g) := \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} f(t)g(t)dt - \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} f(t)dt \cdot \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} g(t)dt$$

for Lebesgue integrable functions $f, g : [\alpha, \beta] \to \mathbb{R}$, proved the following two results which contain the Grüss and Ostrowski type inequalities.

Theorem 8. Let $f : [\alpha, \beta] \to \mathbb{R}$ be Lebesgue integrable and $g : [\alpha, \beta] \to \mathbb{R}$ be absolutely continuous with $(\cdot - \alpha)(\beta - \cdot)(g')^2 \in L[\alpha, \beta]$. Then

$$|T(f,g)| \leq \frac{1}{\sqrt{2}}[T(f,f)]^{\frac{1}{2}} \frac{1}{\sqrt{\beta-\alpha}} \left(\int_{\alpha}^{\beta} (x-\alpha)(\beta-x)[g'(x)]^2 dx\right)^{\frac{1}{2}}. \tag{16}$$

The constant $\frac{1}{\sqrt{2}}$ in (16) is the best possible.

Theorem 9 Let $g : [\alpha, \beta] \to \mathbb{R}$ be monotonic nondecreasing and $f : [\alpha, \beta] \to \mathbb{R}$ be absolutely continuous with $f' \in L_\infty[\alpha, \beta]$. Then

$$|T(f,g)| \leq \frac{1}{2(\beta-\alpha)} \|f'\|_\infty \int_{\alpha}^{\beta} (x-\alpha)(\beta-x)dg(x). \tag{17}$$

The constant $\frac{1}{2}$ in (17) is the best possible.

Using previous two theorems we obtain upper bounds for the identity (11) related to generalizations of Sherman's inequality.

To avoid many notations, under the assumptions of Theorem 5 , we define the function $\mathcal{P} : [\alpha, \beta] \to \mathbb{R}$ by

$$\mathcal{P}(s) = \int_{\alpha}^{\beta} \left[\sum_{p=1}^{l} u_p G(x_p, t) - \sum_{q=1}^{m} v_q G(y_q, t)\right] T_{n-2}(t, s)dt. \tag{18}$$

Then $T(\mathcal{P},\mathcal{P})$ denotes Čebyšev functional

$$T(\mathcal{P},\mathcal{P}) = \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathcal{P}^2(s)ds - \left(\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathcal{P}(s)ds\right)^2.$$

Theorem 10 Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n)}$ is absolutely continuous with $(\cdot-\alpha)(\beta-\cdot)(\phi^{(n+1)})^2 \in L[\alpha,\beta]$ and $\mathcal{P}$ be defined as in (18). Then

$$\begin{aligned}
\kappa(\phi;\alpha,\beta) &= \sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{m} v_q\phi(y_q) - \int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G(x_p,t) - \sum_{q=1}^{m} v_q G(y_q,t)\right] \times \\
&\left[\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi(t)\,dt + \sum_{k=0}^{n-4}\frac{\phi^{(k+1)}(\alpha)}{k!\,(k+2)}\frac{(t-\alpha)^{k+2}}{\beta-\alpha} - \sum_{k=0}^{n-4}\frac{\phi^{(k+1)}(\beta)}{k!\,(k+2)}\frac{(t-\beta)^{k+2}}{\beta-\alpha}\right] dt \\
&- \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{\beta-\alpha}\int_\alpha^\beta \mathcal{P}(s)ds \qquad (19)
\end{aligned}$$

and $\kappa(\phi;\alpha,\beta)$ satisfies the estimation

$$|\kappa(\phi;\alpha,\beta)| \le \frac{\sqrt{\beta-\alpha}}{\sqrt{2}(n-1)!}\,[T(\mathcal{P},\mathcal{P})]^{\frac{1}{2}}\left|\int_\alpha^\beta (s-\alpha)(\beta-s)[\phi^{(n+1)}(s)]^2 ds\right|^{\frac{1}{2}}. \qquad (20)$$

Proof. Applying Theorem 8 for $f \to \mathcal{P}$ and $g \to \phi^{(n)}$, we get

$$\begin{aligned}
&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathcal{P}(s)\phi^{(n)}(s)ds - \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathcal{P}(s)ds\cdot\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi^{(n)}(s)ds\right| \\
&\le \frac{1}{\sqrt{2}}[T(\mathcal{P},\mathcal{P})]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left(\int_\alpha^\beta (s-\alpha)(\beta-s)[\phi^{(n+1)}(s)]^2 ds\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore, we have

$$\frac{1}{(n-1)!}\int_\alpha^\beta \mathcal{P}(s)\phi^{(n)}(s)ds = \frac{\left(\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)\right)}{(n-1)!(\beta-\alpha)}\int_\alpha^\beta \mathcal{P}(s)ds + \kappa(\phi;\alpha,\beta),$$

where the remainder $\kappa(\phi;\alpha,\beta)$ satisfies the estimation (20). Now from the identity (11) we obtain (19).

Theorem 11 Let $\phi : [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n)}$ is monotonic nondecreasing on $[\alpha,\beta]$ and $\mathcal{P}$ be defined as in (18). Then (19) holds and $\kappa(\phi;\alpha,\beta)$ satisfies the estimation

$$|\kappa(\phi;\alpha,\beta)| \le \frac{\|\mathcal{P}'\|_\infty}{(n-1)!}\left\{\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right\}. \qquad (21)$$

Proof. Applying Theorem 9 for $f \to \mathcal{P}$ and $g \to \phi^{(n)}$, we get

$$\begin{aligned}
&\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta \mathcal{P}(s)\phi^{(n)}(s)ds - \frac{1}{\beta-\alpha}\int_\alpha^\beta \mathcal{P}(s)ds\cdot\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi^{(n)}(s)ds\right| \\
&\le \frac{1}{2(\beta-\alpha)}\,\|\mathcal{P}'\|_\infty \int_\alpha^\beta (s-\alpha)(\beta-s)\phi^{(n+1)}(s)ds. \qquad (22)
\end{aligned}$$

Since

$$\int_\alpha^\beta (s-\alpha)(\beta-s)\phi^{(n+1)}(s)ds = \int_\alpha^\beta \left[2s-(\alpha+\beta)\right]\phi^{(n)}(s)ds$$
$$= (\beta-\alpha)\left[\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)\right] - 2\left[\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)\right],$$

using identity (11) and the inequality (22) we deduce (21).

In the following theorem we present Ostrowski type inequality related to generalizations of Sherman's inequality.

Theorem 12 Let (p,q) be a pair of conjugate exponents, i.e. $1 \le p,q \le \infty$ and $1/p+1/q=1$. Let $\phi:[\alpha,\beta]\to\mathbb{R}$ be such that $\left|\phi^{(n)}\right|^p \in L[\alpha,\beta]$ and $\mathcal{P}$ be defined as in (18). Then

$$\Bigg|\sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{m} v_q\phi(y_q)$$
$$-\int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G(x_p,t) - \sum_{q=1}^{m} v_q G(y_q,t)\right]\times$$
$$\left[\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi(t)\,dt + \sum_{k=0}^{n-4}\frac{\phi^{(k+1)}(\alpha)}{k!\,(k+2)}\frac{(t-\alpha)^{k+2}}{\beta-\alpha} - \sum_{k=0}^{n-4}\frac{\phi^{(k+1)}(\beta)}{k!\,(k+2)}\frac{(t-\beta)^{k+2}}{\beta-\alpha}\right]dt\Bigg|$$
$$\le \frac{1}{(n-1)!}\left\|\phi^{(n)}\right\|_p \left\|\mathcal{P}\right\|_q.$$

The constant $\left\|\mathcal{P}\right\|_q$ is sharp for $1<p\le\infty$ and the best possible for $p=1$.

Proof. Using the identity (11) and applying Hölder's inequality we obtain

$$\Bigg|\sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{m} v_q\phi(y_q)$$
$$-\int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G(x_p,t) - \sum_{q=1}^{m} v_q G(y_q,t)\right]\times$$
$$\left[\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi(t)\,dt + \sum_{k=0}^{n-4}\frac{\phi^{(k+1)}(\alpha)}{k!\,(k+2)}\frac{(t-\alpha)^{k+2}}{\beta-\alpha} - \sum_{k=0}^{n-4}\frac{\phi^{(k+1)}(\beta)}{k!\,(k+2)}\frac{(t-\beta)^{k+2}}{\beta-\alpha}\right]dt\Bigg|$$
$$\le \frac{1}{(n-1)!}\left\|\phi^{(n)}\right\|_p \left\|\mathcal{P}\right\|_q. \tag{23}$$

For the proof of the sharpness of the constant $\left\|\mathcal{P}\right\|_q$ let us find a function ϕ for which the equality in (23) is obtained.

For $1<p<\infty$ take ϕ to be such that

$$\phi^{(n)}(s) = sgn\mathcal{P}(s)\left|\mathcal{P}(s)\right|^{\frac{1}{1-p}}$$

For $p=\infty$ take $\phi^{(n)}(s) = sgn\mathcal{P}(s)$.

For $p=1$ we prove that

$$\left|\int_\alpha^\beta \mathcal{P}(s)\phi^{(n)}(s)ds\right| \le \max_{s\in[\alpha,\beta]}\left|\mathcal{P}(s)\right|\left(\int_\alpha^\beta \left|\phi^{(n)}(s)\right|ds\right) \tag{24}$$

is the best possible inequality. Assume that $|\mathcal{P}(s)|$ attains its maximum at $s_0 \in [\alpha,\beta]$. First we assume that $\mathcal{P}(s_0) > 0$. For ε small enough we define $\phi_\varepsilon(s)$ by

$$\phi_\varepsilon(s) = \begin{cases} 0, & \alpha \le s \le s_0, \\ \frac{1}{\varepsilon\, n!}(s-s_0)^n, & s_0 \le s \le s_0+\varepsilon, \\ \frac{1}{n!}(s-s_0)^{n-1}, & s_0+\varepsilon \le s \le \beta. \end{cases}$$

Then for ε small enough

$$\left|\int_\alpha^\beta \mathcal{P}(s)\phi^{(n)}(s)ds\right| = \left|\int_{s_0}^{s_0+\varepsilon} \mathcal{P}(s)\frac{1}{\varepsilon}ds\right| = \frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} \mathcal{P}(s)ds.$$

Now from the inequality (24) we have

$$\frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} \mathcal{P}(s)ds \le \mathcal{P}(s_0)\int_{s_0}^{s_0+\varepsilon}\frac{1}{\varepsilon}ds = \mathcal{P}(s_0).$$

Since,

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{s_0}^{s_0+\varepsilon} \mathcal{P}(s)ds = \mathcal{P}(s_0)$$

the statement follows. In the case $\mathcal{P}(s_0) < 0$, we define $\phi_\varepsilon(s)$ by

$$\phi_\varepsilon(s) = \begin{cases} \frac{1}{n!}(s-s_0-\varepsilon)^{n-1},, & \alpha \le s \le s_0, \\ -\frac{1}{\varepsilon n!}(s-s_0-\varepsilon)^n, & s_0 \le s \le s_0+\varepsilon, \\ 0, & s_0+\varepsilon \le s \le \beta, \end{cases}$$

and the rest of the proof is the same as above.

4. Mean value theorems and Exponential convexity with applications

Motivated by the inequality (14), under the assumptions of Theorems 6, we define the linear functional $\Lambda : C^n([\alpha,\beta]) \to \mathbb{R}$ by

$$\begin{aligned}\Lambda(\phi) &= \sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{m} v_q\phi(y_q) \\ &- \int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G(x_p,t) - \sum_{q=1}^{m} v_q G(y_q,t)\right] \times \\ &\left[\frac{1}{\beta-\alpha}\int_\alpha^\beta \phi(t)\,dt + \sum_{k=0}^{n-4}\frac{\phi^{(k+1)}(\alpha)}{k!\,(k+2)}\frac{(t-\alpha)^{k+2}}{\beta-\alpha} - \sum_{k=0}^{n-4}\frac{\phi^{(k+1)}(\beta)}{k!\,(k+2)}\frac{(t-\beta)^{k+2}}{\beta-\alpha}\right]dt. \end{aligned} \tag{25}$$

By Theorem 6, for every n-convex functions $\phi : [\alpha,\beta] \to \mathbb{R}$ we have

$$\Lambda(\phi) \ge 0.$$

Using the linearity of this functional we derive mean-value theorems of Lagrange and Cauchy type.

Theorem 13 Let $\phi \in C^n([\alpha,\beta])$ and $\Lambda : C^n([\alpha,\beta]) \to \mathbb{R}$ be the linear functional defined by (25). Then there exist $\xi \in [\alpha,\beta]$ such that

$$\Lambda(\phi) = \phi^{(n)}(\xi)\Lambda(\varphi),$$

where $\varphi(x) = \frac{x^n}{n!}$.

Proof. Similar to the proof of Theorem 4.1 in [11].

Theorem 14 Let $\phi, \psi \in C^n([\alpha, \beta])$ and $\Lambda : C^n([\alpha, \beta]) \to \mathbb{R}$ be the linear functional defined by (25). Then there exists $\xi \in [\alpha, \beta]$ such that

$$\frac{\Lambda(\phi)}{\Lambda(\psi)} = \frac{\phi^{(n)}(\xi)}{\psi^{(n)}(\xi)},$$

provided that the denominators are non-zero.

Proof. Similar to the proof of Corollary 4.2 in [11].

Remark 2. With assumption that $\frac{\phi^{(n)}}{\psi^{(n)}}$ is an invertible function, as a consequence of the previous theorem we get the single number

$$\xi = \left(\frac{\phi^{(n)}}{\psi^{(n)}}\right)^{-1} \left(\frac{\Lambda(\phi)}{\Lambda(\psi)}\right)$$

which is exactly mean of Chauchy type of the segment $[\alpha, \beta]$.
Applying Exponential convexity method [11], we construct some new families of exponentially convex functions or in the special case logarithmically convex functions. The outcome are some new classes of two-parameter Cauchy-type means. In order that, we introduce the some basic definitions and results on exponential convexity.

Through the rest of paper, I denotes an open interval in $\mathbb{R}$.

Definition 3 For fixed $n \in \mathbb{N}$, a function $f : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} p_i p_j f\left(\frac{x_i + x_j}{2}\right) \geq 0$$

holds for all choices $p_i \in \mathbb{R}$ and $x_i \in I$, $i = 1, ..., n$.
A function $f : I \to \mathbb{R}$ is n-exponentially convex on I if it is n-exponentially convex in the Jensen sense and continuous on I.

The notation of n-exponential convexity is introduced in [14].

Remark 3 From Definition 3 it follows that 1-exponentially convex functions in the Jensen sense are exactly nonnegative functions. Moreover, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leq n$.

Definition 4 A function $f : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

One of the most important properties of exponentially convex functions is their integral representation (see [6, p. 211]).

Theorem 15 The function $f : I \to \mathbb{R}$ is exponentially convex on I if and only if

$$f(x) = \int_{-\infty}^{\infty} e^{tx} d\sigma(s), \tag{26}$$

for some non-decreasing function $\sigma : \mathbb{R} \to \mathbb{R}$.
The next example is deduced using integral representation (26) and some results of the Laplace transform (see [16, p. 214]).

Example 1 The function $f : (0, \infty) \to (0, \infty)$ defined by $f(x) = e^{-k\sqrt{x}}$ is exponentially convex on $(0, \infty)$ for every $k > 0$ since $e^{-k\sqrt{x}} = \int_0^{\infty} e^{-xt} e^{-k^2/4t} \frac{k}{2\sqrt{\pi t^3}} dt$.

Definition 5 A function $f : I \to (0, \infty)$ is said to be logarithmically convex in the Jensen sense if

$$f\left(\frac{x+y}{2}\right) \leq \sqrt{f(x)f(y)}$$

holds for all $x, y \in I$.
Definition 6. A function $f : I \to (0, \infty)$ is said to be logarithmically convex or log-convex if

$$f\left((1-\lambda)s + \lambda t\right) \leq f(s)^{1-\lambda} f(t)^{\lambda}$$

holds for all $s, t \in I, \lambda \in [0, 1]$.
Remark 4 If a function is continuous and log-convex in the Jensen sense then it is also log-convex. We can also easily see that for positive functions exponential convexity implies log-convexity (consider the Definition 4 for $n = 2$).

The following lemmas are equivalent to definition of convexity (see [15]).
Lemma 1 Let $f : I \to \mathbb{R}$ be a convex function. Then for any $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$ the following is valid

$$(x_3 - x_2) f(x_1) + (x_1 - x_3) f(x_2) + (x_2 - x_1) f(x_3) \geq 0.$$

Lemma 2 *Let $f : I \to \mathbb{R}$ be a convex function. Then for any $x_1, x_2, y_1, y_2, \in I$ such that $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$ the following is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}.$$

In order to obtain results regarding the exponential convexity, we define the families of functions as follows.

For every choice of $l+1$ mutually different points $x_0, x_1, ..., x_l \in [\alpha, \beta]$ we define

- $\mathcal{F}_1 = \{\phi_t : [\alpha, \beta] \to \mathbb{R} : t \in I$ and $t \mapsto [x_0, x_1, ..., x_l; \phi_t]$ is n-exponentially convex in the Jensen sense on $I\}$
- $\mathcal{F}_2 = \{\phi_t : [\alpha, \beta] \to \mathbb{R} : t \in I$ and $t \mapsto [x_0, x_1, ..., x_l; \phi_t]$ is exponentially convex in the Jensen sense on $I\}$
- $\mathcal{F}_3 = \{\phi_t : [\alpha, \beta] \to \mathbb{R} : t \in I$ and $t \mapsto [x_0, x_1, ..., x_l; \phi_t]$ is 2-exponentially convex in the Jensen sense on $I\}$

Theorem 16 Let Λ be the linear functional defined as in (25) associated with family $\mathcal{F}_1$. Then the following statements hold:

(i) The function $t \mapsto \Lambda(\phi_t)$ is n-exponentially convex in the Jensen sense on I.
(ii) If the function $t \mapsto \Lambda(\phi_t)$ is continuous on I, then it is n-exponentially convex on I.

Proof. (i) Let $p_j, s_j \in \mathbb{R}$, $j = 1, ..., n$, and $s_{jk} = \frac{s_j + s_k}{2}$, $1 \leq j, k \leq n$. We consider the function $h : [\alpha, \beta] \to \mathbb{R}$ defined by

$$h(x) = \sum_{j,k=1}^{n} p_j p_k \phi_{s_{jk}}(x),$$

where $\phi_{s_{jk}} \in \mathcal{F}_1$.
Since the function $t \mapsto [x_0, x_1, ..., x_l; \phi_t]$ is n-exponentially convex in the Jensen sense on I, we have

$$[x_0, x_1, ..., x_l; h] = \sum_{j,k=1}^{n} p_j p_k \left[x_0, x_1, ..., x_l; \phi_{s_{jk}}\right] \geq 0.$$

Hence, the function h is l-convex. Therefore, we have

$$\Lambda(h) = \sum_{j,k=1}^{n} p_j p_k \Lambda\left(\phi_{s_{jk}}\right) \geq 0.$$

Now we conclude that the function $t \mapsto \Lambda(\phi_t)$ is n-exponentially convex in the Jensen sense on I what we need to prove.
(ii) Follows from (i) and Definition 4.

The following corollary is an easy consequence of the previous theorem.

Corollary 1 Let Λ be the linear functional defined as in (25) associated with family $\mathcal{F}_2$. Then the following statements hold:

(i) The function $t \mapsto \Lambda(\phi_t)$ is exponentially convex in the Jensen sense on I.
(ii) If the function $t \mapsto \Lambda(\phi_t)$ is continuous on I, then it is exponentially convex on I.

Corollary 2 Let Λ be the linear functional defined as in (25) associated with family $\mathcal{F}_3$. Then the following statements hold:

(i) If the function $t \mapsto \Lambda(\phi_t)$ is continuous on I, then it is 2-exponentially convex on I. If $t \mapsto \Lambda(\phi_t)$ is additionally positive, then it is also log-convex on I. Furthermore, for every choice $r, s, t \in I$, such that $r < s < t$, it holds

$$[\Lambda(\phi_s)]^{t-r} \leq [\Lambda(\phi_r)]^{t-s} [\Lambda(\phi_t)]^{s-r}.$$

(ii) If the function $t \mapsto \Lambda(\phi_t)$ is positive and differentiable on I, then for all $r, s, u, v \in I$ such that $r \leq u$, $s \leq v$, we have

$$M_{r,s}(\Lambda, \mathcal{F}_3) \leq M_{u,v}(\Lambda, \mathcal{F}_3), \tag{27}$$

where

$$M_{r,s}(\Lambda, \mathcal{F}_3) = \begin{cases} \left(\frac{\Lambda(\phi_r)}{\Lambda(\phi_s)}\right)^{\frac{1}{r-s}}, & r \neq s, \\ \exp\left(\frac{\frac{d}{dr}(\Lambda(\phi_r))}{\Lambda(\phi_r)}\right), & r = s. \end{cases} \tag{28}$$

Proof (i) The first part of statement is an easy consequence of Theorem 16 and the second one of Remark 4.
Since the function $t \mapsto \Lambda(\phi_t)$ is log-convex on I, i.e. the function $t \mapsto \log \Lambda(\phi_t)$ is convex on I, then applying Lemma 4 we have

$$(t - s) \log \Lambda(\phi_r) + (r - t) \log \Lambda(\phi_s) + (s - r) \log \Lambda(\phi_t) \geq 0$$

for every choice $r, s, t \in I$, such that $r < s < t$. Therefore, we have

$$[\Lambda(\phi_s)]^{t-r} \leq [\Lambda(\phi_r)]^{t-s} [\Lambda(\phi_t)]^{s-r}.$$

(ii) Applying Lemma 4 to the convex function $t \mapsto \log A(\phi_t)$, we get

$$\frac{\log \Lambda(\phi_r) - \log \Lambda(\phi_s)}{r-s} \leq \frac{\log \Lambda(\phi_u) - \log \Lambda(\phi_v)}{u-v} \tag{29}$$

for $r \leq u$, $s \leq v$, $r \neq u$, $s \neq v$. Therefore, we have

$$M_{r,s}(\Lambda, \mathcal{F}_3) \leq M_{u,v}(\Lambda, \mathcal{F}_3).$$

Case $r = s$, $u = v$ follows from (29) as limiting case.

Remark 5 Note that the results from Theorem 4, Corollary 4 and Corollary 4 still hold when two of the points $x_0, ..., x_l \in [a, b]$ coincide, say $x_1 = x_0$, for a family of differentiable functions ϕ_t such that the function $t \mapsto \phi_t[x_0, ..., x_l]$ is an n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all $(l+1)$ points coincide for a family of l differentiable functions with the same property. The proofs are obtained by (10) and suitable characterization of convexity.

As an example of application of the previous results, consider the family of functions

$$\Omega = \{\varphi_t : (0, \infty) \to (0, \infty) : t \in (0, \infty)\}$$

defined by

$$\varphi_t(x) = \frac{e^{-x\sqrt{t}}}{\left(-\sqrt{t}\right)^n}.$$

Since $\frac{d^n \varphi_t}{dx^n}(x) = e^{-x\sqrt{t}} > 0$, the function φ_t is n-convex function for every $t > 0$. Moreover, the function $t \mapsto \frac{d^n \varphi_t}{dx^n}(x)$ is exponentially convex. Therefore, using the same arguments as in proof of Theorem 4, we conclude that the function $t \mapsto [x_0, x_1, ..., x_l; \varphi_t]$ is exponentially convex (and so exponentially convex in the Jensen sense). Then from Corollary 1 it follows that $t \mapsto \Lambda(\varphi_t)$ is exponentially convex in the Jensen sense. It is easy to verify that the function $t \mapsto \Lambda(\varphi_t)$ is continuous, so it is exponentially convex.

For this family of functions, with assumption that $[\alpha, \beta] \subset (0, \infty)$ and $t \mapsto \Lambda(\varphi_t)$ is positive, (28) becomes

$$\mu_{\eta,\zeta} = \left(\frac{\zeta^n}{\eta^n} \cdot \frac{\sum_{p=1}^{l} u_p e^{-x_p\sqrt{\eta}} - \sum_{q=1}^{m} v_q e^{-y_q\sqrt{\eta}} - A_1}{\sum_{p=1}^{l} u_p e^{-x_p\sqrt{\zeta}} - \sum_{q=1}^{m} v_q e^{-y_q\sqrt{\zeta}} - B_1} \right)^{\frac{1}{\eta-\zeta}}, \quad \eta \neq \zeta,$$

$$\mu_{\eta,\eta} = \exp\left(\frac{\sum_{q=1}^{m} v_q y_q e^{-y_q\sqrt{\eta}} - \sum_{p=1}^{l} u_p x_p e^{-x_p\sqrt{\eta}} + A_2}{2\sqrt{\eta}\left(\sum_{p=1}^{l} u_p e^{-x_p\sqrt{\eta}} - \sum_{q=1}^{m} v_q e^{-y_q\sqrt{\eta}} - A_1\right)} - \frac{n}{\eta} \right),$$

where

$$A_1 = \int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G(x_p,t) - \sum_{q=1}^{m} v_q G(y_q,t) \right] \times$$
$$\left[\frac{e^{-\beta\sqrt{\eta}} - e^{-\alpha\sqrt{\eta}}}{\beta-\alpha} + \sum_{k=0}^{n-4} \frac{(-\sqrt{\eta})^{k+1} e^{-\alpha\sqrt{\eta}}}{k!\,(k+2)} \frac{(t-\alpha)^{k+2}}{\beta-\alpha} - \sum_{k=0}^{n-4} \frac{(-\sqrt{\eta})^{k+1} e^{-\beta\sqrt{\eta}}}{k!\,(k+2)} \frac{(t-\beta)^{k+2}}{\beta-\alpha} \right] dt$$

$$B_1 = \int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G(x_p,t) - \sum_{q=1}^{m} v_q G(y_q,t) \right] \times$$
$$\left[\frac{e^{-\beta\sqrt{\zeta}} - e^{-\alpha\sqrt{\zeta}}}{\beta-\alpha} + \sum_{k=0}^{n-4} \frac{(-\sqrt{\zeta})^{k+1} e^{-\alpha\sqrt{\zeta}}}{k!\,(k+2)} \frac{(t-\alpha)^{k+2}}{\beta-\alpha} - \sum_{k=0}^{n-4} \frac{(-\sqrt{\zeta})^{k+1} e^{-\beta\sqrt{\zeta}}}{k!\,(k+2)} \frac{(t-\beta)^{k+2}}{\beta-\alpha} \right] dt$$

$$A_2 = \int_\alpha^\beta \left[\sum_{p=1}^{l} u_p G(x_p,t) - \sum_{q=1}^{m} v_q G(y_q,t) \right] \times$$
$$\left[\frac{e^{-\beta\sqrt{\zeta}} - e^{-\alpha\sqrt{\zeta}}}{\beta-\alpha} + \sum_{k=0}^{n-4} \frac{\frac{d^{k+1}}{dx^{k+1}}(xe^{-x\sqrt{\eta}})|_{x=\alpha}}{k!\,(k+2)} \frac{(t-\alpha)^{k+2}}{\beta-\alpha} - \sum_{k=0}^{n-4} \frac{\frac{d^{k+1}}{dx^{k+1}}(xe^{-x\sqrt{\eta}})|_{x=\beta}}{k!\,(k+2)} \frac{(t-\beta)^{k+2}}{\beta-\alpha} \right] dt$$

Using Theorem 14 it follows that

$$M_{\eta,\zeta}(\Lambda,\Omega) = -\left(\sqrt{\eta}+\sqrt{\zeta}\right) \log \mu_{\eta,\zeta}(\Lambda,\Omega)$$

satisfies

$$\alpha \le M_{\eta,\zeta}(\Lambda,\Omega) \le \beta,$$

i.e. $M_{\eta,\zeta}(\Lambda,\Omega)$ is mean. By Corollary 2, using (27), it follows that this mean is monotonic.

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