Electronic Journal of Mathematical Analysis and Applications
Vol. 5(1) Jan. 2017, pp. 42-49.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

SOME REMARKS ON ALMOST PERIODIC TIME SCALES AND ALMOST PERIODIC FUNCTIONS ON TIME SCALES

YONGKUN LI

Abstract. In this note we communicate some important remarks about the concepts of almost periodic time scales and almost periodic functions on time scales that are proposed by Wang and Agarwal in their recent papers (see [7]-[10]).

1. Introduction

In order to study almost periodic functions, pseudo almost periodic functions, almost automorphic functions, pseudo almost automorphic functions and so on, on time scales, the following concept of almost periodic time scales was proposed in [1, 2]:

Definition 1 [1] A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi = \left\{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\right\} \neq \{0\}. \tag{1}$$

Based on Definition 1, almost periodic functions [1, 2], pseudo almost periodic functions [3], almost automorphic functions [4], almost automorphic functions of order n [5], weighted pseudo almost automorphic functions [6] and weighted piecewise pseudo almost automorphic functions [7] on time scales were defined successfully.

Since Definition 1 requires that time scale $\mathbb{T}$ has a global property, that is, there exists at least one $\tau \in \mathbb{R}$ such that $t \pm \tau \in \mathbb{T}$ for all $t \in \mathbb{T}$, it is very restrictive. This may exclude many interesting time scales. Therefore, it is a challenging and important problem in theory and applications to find new concepts of almost periodic time scales. Recently, Wang and Agarwal in [8, 9, 10] have made some efforts to introduce some new types of almost periodic time scales and almost periodic functions on time scales. However, unfortunately, there are many flaws and mistakes in [8, 9, 10].

Our main purpose of this note is to point out some flaws and mistakes in [8, 9, 10] and give another correction of the concept of almost periodic functions on time scales in [1], which is overcorrected in [8].

2010 *Mathematics Subject Classification.* 26E70; 42A75.
Key words and phrases. Almost periodic time scales, almost periodic functions, time scales.
Submitted Nov. 2, 2015.

2. Preliminaries

In this section, we shall first recall some definitions and state some results which are used in what follows.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \text{ and } \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$. The time scale interval $[a, b]_{\mathbb{T}}$ will be defined by $[a, b]_{\mathbb{T}} = \{t : t \in [a, b] \cap \mathbb{T}\}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be continuous function on $\mathbb{T}$.

For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^{\Delta}(t)$, to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|[y(\sigma(t)) - y(s)] - y^{\Delta}(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|$$

for all $s \in U$.

If y is continuous, then y is right-dense continuous, and if y is delta differentiable at t, then y is continuous at t.

Let y be right-dense continuous. If $Y^{\Delta}(t) = y(t)$, then we define the delta integral by $\int_a^t y(s)\Delta s = Y(t) - Y(a)$ if $a, t \in \mathbb{T}$; $\int_a^t y(s)\Delta s = Y(\rho(t)) - Y(a)$ if $a \in \mathbb{T}, t \notin \mathbb{T}, \rho(t) \geq a$ and $\int_a^t y(s)\Delta s = Y(\sigma(t)) - Y(a)$ if $a \in \mathbb{T}, t \notin \mathbb{T}, \sigma(t) \leq a$; and so on.

Definition 2 [11] A subset S of $\mathbb{R}$ is called relatively dense or relatively dense in $\mathbb{R}$ if there exists a positive number L such that $[a, a + L] \cap S \neq \emptyset$ for all $a \in \mathbb{R}$; a subset $S \subset A \subset \mathbb{R}$ is called relatively dense in A if there exists a positive number L such that $[a, a + L] \cap A \cap S \neq \emptyset$ for all $a \in A$.The number L is called the inclusion length.

Let $\mathbb{T}$ be a time scale and $\tau \in \mathbb{R}$, we denote $\mathbb{T}^{\tau} = \{t + \tau : t \in \mathbb{T}\}$.

Definition 3 [1, 4, 12] A time scale $\mathbb{T}$ is called an invariant under a translation time scale or a periodic time scale if

$$\Pi = \left\{\tau \in \mathbb{R} : \mathbb{T} \cap \mathbb{T}^{\pm\tau} = \mathbb{T}\right\} \neq \{0\}. \tag{2}$$

Remark 1 Since the set defined by (1) is essentially the same set defined by (2), the almost periodic time scale under Definition 1 is the periodic time scale under Definition 3. If $\mathbb{T}$ is invariant under a translation time scale, then $\sup \mathbb{T} = +\infty$ and $\inf \mathbb{T} = -\infty$.

Definition 4 [13] A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi := \left\{\tau \in \mathbb{R} : \mathbb{T}_{\tau} \neq \emptyset\right\}$$

is relatively dense in $\mathbb{R}$, where $\mathbb{T}_{\tau} = \mathbb{T} \cap \{\mathbb{T} - \tau\}$ or $\mathbb{T}_{\tau} = \mathbb{T} \cap \{\mathbb{T} \pm \tau\}$.

Definition 5 [14] A time scale $\mathbb{T}$ is called an almost periodic time scale if the set

$$\Pi := \{\tau \in \mathbb{R} : \mathbb{T}_\tau \neq \emptyset\},$$

where $\mathbb{T}_\tau = \mathbb{T} \cap \{\mathbb{T} - \tau\} = \mathbb{T} \cap \{t - \tau : t \in \mathbb{T}\}$, satisfies

(i) $\Pi \neq \{0\}$,

(ii) if $\tau_1, \tau_2 \in \Pi$, then $\tau_1 \pm \tau_2 \in \Pi$.

Remark 2 If $\mathbb{T}$ is an almost periodic time scale under Definition 3, then $\mathbb{T}$ is also an almost periodic time scale under Definitions 4 and 5. If $\mathbb{T}$ is an almost periodic time scale under Definition 5, then $\mathbb{T}$ is also an almost periodic time scale under Definition 4.

As a slightly modified version of Definition 5, the following definition of almost periodic time scales is given in [15]:

Definition 6 [15] A time scale $\mathbb{T}$ is called an almost periodic time scale if the set

$$\Pi := \{\tau \in \mathbb{R} : \mathbb{T}_\tau \neq \emptyset\},$$

where $\mathbb{T}_\tau = \mathbb{T} \cap \{\mathbb{T} - \tau\} = \mathbb{T} \cap \{t - \tau : t \in \mathbb{T}\}$, satisfies

(i) $\Pi \neq \{0\}$,

(ii) if $\tau_1, \tau_2 \in \Pi$, then $\tau_1 \pm \tau_2 \in \Pi$,

(iii) $\widetilde{\mathbb{T}} := \mathbb{T}(\Pi) = \bigcap_{\tau \in \Pi} \mathbb{T}_\tau \neq \emptyset$.

Definition 7 [1] Let $\mathbb{T}$ be an invariant under a translation time scale. A function $f \in C(\mathbb{T} \times D, \mathbb{E}^n)$ is called an almost periodic function in $t \in \mathbb{T}$ uniformly for $x \in D$ if the ε-translation set of f

$$E\{\varepsilon, f, S\} = \{\tau \in \Pi : |f(t+\tau, x) - f(t, x)| < \varepsilon,\ \forall (t, x) \in \mathbb{T} \times S\}$$

is a relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$ and for each compact subset S of D; that is, for any given $\varepsilon > 0$ and each compact subset S of D, there exists a constant $l(\varepsilon, S) > 0$ such that each interval of length $l(\varepsilon, S)$ contains a $\tau(\varepsilon, S) \in E\{\varepsilon, f, S\}$ such that

$$|f(t+\tau, x) - f(t, x)| < \varepsilon,\ \forall t \in \mathbb{T} \times S.$$

This τ is called the ε-translation number of f.

Lemma 1 [5] Let $\mathbb{T}$ be an invariant under a translation time scale. Then one has

(i) $\Pi \subset \mathbb{T} \Leftrightarrow 0 \in \mathbb{T}$;

(ii) $\Pi \cap \mathbb{T} = \emptyset \Leftrightarrow 0 \notin \mathbb{T}$.

3. Some remarks

By Definition 7, we see that $E\{\varepsilon, f, S\} \subset \Pi$. So, according to Lemma 1 and also, as pointed out in [8], $\mathbb{T} \cap \Pi = \emptyset$ if $0 \notin \mathbb{T}$, and so, in this case, $E\{\varepsilon, f, S\}$ can not be a relatively dense set in $\mathbb{T}$. To fix this flaw, the authors of [8] use "$E\{\varepsilon, f, S\}$ is a relatively dense set in Π" to replace "$E\{\varepsilon, f, S\}$ is a relatively dense set in $\mathbb{T}$" in Definition 7 to give a correction of Definition 7, that is, Definition 1.8 in [8]. Here, we will use "$E\{\varepsilon, f, S\}$ is a relatively dense set in $\mathbb{R}$" to replace "$E\{\varepsilon, f, S\}$ is a relatively dense set in $\mathbb{T}$" in Definition 7 to give another correction of Definition 7 as follows.

Definition 8 Let $\mathbb{T}$ be an invariant under a translation time scale. A function $f \in C(\mathbb{T} \times D, \mathbb{E}^n)$ is called an almost periodic function in $t \in \mathbb{T}$ uniformly for $x \in D$ if the ε-translation set of f

$$E\{\varepsilon, f, S\} = \{\tau \in \Pi : |f(t+\tau, x) - f(t, x)| < \varepsilon,\ \forall (t, x) \in \mathbb{T} \times S\}$$

is a relatively dense set in $\mathbb{R}$ for all $\varepsilon > 0$ and for each compact subset S of D; that is, for any given $\varepsilon > 0$ and each compact subset S of D, there exists a constant $l(\varepsilon, S) > 0$ such that each interval of length $l(\varepsilon, S)$ contains a $\tau(\varepsilon, S) \in E\{\varepsilon, f, S\}$ such that

$$|f(t+\tau, x) - f(t, x)| < \varepsilon, \ \forall t \in \mathbb{T} \times S.$$

τ is called the ε-translation number of f.

The authors of [8] also pointed out that there exist many integrals in [1] such as the following:

$$\int_t^{t+l} f(s)\Delta s,$$

but $t \in \mathbb{T}$ cannot guarantee that $t + l \in \mathbb{T}$, where $l \in \mathbb{R}$ is an inclusion length in Definition 2. However, after we have thoroughly and carefully checked the whole paper [1], there is no such integral in [1] at all. Besides, even if there is such integral in paper [1], according to our convention

$$\int_t^{t+l} f(s)\Delta s := \int_t^{\rho(t+l)} f(s)\Delta s, \quad \text{if} \quad t \in \mathbb{T}, \ t + l \notin \mathbb{T}.$$

Hence, the integral $\int_t^{t+l} f(s)\Delta s$ is well defined.

Remark 3 From the above, we see that if we adopt Definition 8 as the definition of almost periodic functions on time scales, all the results of [1] remain true.

In order to extend the concept of almost periodic time scales, authors of [8] give the following result:

Theorem 1 (Theorem 2.4 in [8]) Let $\mathbb{T}$ be an arbitrary time scale with $\sup \mathbb{T} = +\infty, \inf \mathbb{T} = -\infty$. If $\mu : \mathbb{T} \to \mathbb{R}^+$ is bounded, then $\mathbb{T}$ contains at least one invariant under the translation unit (that is, $\mathbb{T}$ contains at least one sub time scale that is an invariant under a translation time scale).

Unfortunately, the following example shows that Theorem 1 is incorrect.

Example 1 Take $\mathbb{T} = \{-2k, 2k+1 : k \in \mathbb{N}\}$, then $\sup\limits_{t \in \mathbb{T}} \mu(t) = 5$. According to Remark 1, if $\mathbb{T}_*$ is a sub time scale of $\mathbb{T}$ that is an invariant under a translation time scale, then $\sup \mathbb{T}_* = +\infty$ and $\inf \mathbb{T}_* = -\infty$.

Let $\hat{\mathbb{T}}$ be any subset of $\mathbb{T}$ with $\sup \hat{\mathbb{T}} = +\infty$ and $\inf \hat{\mathbb{T}} = -\infty$, then $\hat{\mathbb{T}}$ can be expressed as

$$\hat{\mathbb{T}} = \{-2k_n, 2l_m + 1 : n, m \in \mathbb{N}\},$$

where $\{k_n\}_{n=1}^{\infty}, \{l_m\}_{m=1}^{\infty} \subset \mathbb{N}, k_n < k_{n+1}, l_m < l_{m+1}$ and $\lim\limits_{n\to\infty} k_n = \lim\limits_{m\to\infty} l_m = +\infty$. Then for every $\tau \in \mathbb{R}$,

$$\hat{\mathbb{T}}^{\tau} = \{-2k_n + \tau, 2l_m + 1 + \tau : n, m \in \mathbb{N}\}.$$

In the following, we divide five cases to show that for every $\tau \in \mathbb{R}\backslash\{0\}$, $\hat{\mathbb{T}} \cap \hat{\mathbb{T}}^{\tau} \neq \hat{\mathbb{T}}$, that is, $\mathbb{T}$ contains no any sub time scale that is an invariant under a translation time scale.

Case 1: It is easy to see that $\mathbb{T} \cap \hat{\mathbb{T}}^{\tau} = \emptyset$ for $\tau \in \mathbb{R} \setminus \mathbb{Z}$.

Case 2: If τ is a positive even number, then $2l_1 + 1 \in \hat{\mathbb{T}}$ and $2l_1 + 1 \notin \hat{\mathbb{T}}^{\tau}$, hence $\hat{\mathbb{T}} \cap \hat{\mathbb{T}}^{\tau} \neq \hat{\mathbb{T}}$.

Case 3: If τ is a negative even number, then $-2k_1 \in \hat{\mathbb{T}}$ and $-2k_1 \notin \hat{\mathbb{T}}^{\tau}$, hence $\hat{\mathbb{T}} \cap \hat{\mathbb{T}}^{\tau} \neq \hat{\mathbb{T}}$.

Case 4: If τ is a positive odd number, then there are at last only finite many positive odd numbers in $\{-2k_n + \tau\}_{n=1}^{\infty}$, that is, $\hat{\mathbb{T}}^{\tau}$ only contains at last finite many positive odd numbers. Since there are infinite many positive odd numbers in $\hat{\mathbb{T}}$, hence $\hat{\mathbb{T}} \cap \hat{\mathbb{T}}^{\tau} \neq \hat{\mathbb{T}}$.

Case 5: If τ is a negative odd number, then there are at last only finite many negative even numbers in $\{2l_m + 1 + \tau\}_{n=1}^{\infty}$, that is, $\hat{\mathbb{T}}^{\tau}$ only contains at last finite many negative even numbers. Since there are infinite many negative even numbers in $\hat{\mathbb{T}}$, hence $\hat{\mathbb{T}} \cap \hat{\mathbb{T}}^{\tau} \neq \hat{\mathbb{T}}$.

Remark 4 Example 1 shows that there exists a time scale that satisfies all the conditions of Theorem 1, but it contains no sub time scale that is an invariant under a translation time scale. Therefore, Theorem 1 is incorrect.

Remark 5 The set $\widetilde{\mathbb{T}}$ of Definition 6 is the sub-invariant under the translation unit in $\mathbb{T}$ of Definitions 2.2 and 2.3 in [8].

In [9], the following definitions are given:

Definition 9 (Definition 2.4 in [9]) Let $\mathbb{T}$ be a time scale, we say $\mathbb{T}$ is a zero-periodic time scale if and only if there exists no nonzero real number ω such that $t + \omega \in \mathbb{T}$ for all $t \in \mathbb{T}$.

Definition 10 (Definition 2.5 in [9]) A time scale sequence $\{\mathbb{T}_i\}_{i\in\mathbb{Z}^+}$ is called well-connected if and only if for $i \neq j$, we have $\mathbb{T}_i \cap \mathbb{T}_j = \{t_{ij}^k\}_{k\in\mathbb{Z}}$, where $\{t_{ij}^k\}$ is a countable points set or an empty set, and t_{ij}^k is called the connected point between $\mathbb{T}_i$ and $\mathbb{T}_j$ for each $k \in \mathbb{Z}$, the set $\{t_{ij}^k\}$ is called the connected points set of this well-connected sequence.

Definition 11 (Definition 2.6 in [9]) Let $\mathbb{T}$ be an infinite time scale. We say $\mathbb{T}$ is a changing-periodic or a piecewise-periodic time scale if the following conditions are fulfilled:

- (a) $\mathbb{T} = (\bigcup_{i=1}^{\infty} \mathbb{T}_i) \cup \mathbb{T}_r$ and $\{\mathbb{T}_i\}_{i\in\mathbb{Z}^+}$ is a well-connected time scale sequence, where $\mathbb{T}_r = \bigcup_{i=1}^{k} [\alpha_i, \beta_i]$ and k is some finite number, and $[\alpha_i, \beta_i]$ are closed intervals for $i = 1, 2, \ldots, k$ or $\mathbb{T}_r = \emptyset$;
- (b) S_i is a nonempty subset of $\mathbb{R}$ with $0 \notin S_i$ for each $i \in \mathbb{Z}^+$ and $\Pi = (\bigcup_{i=1}^{\infty} S_i) \cup R_0$, where $R_0 = \{0\}$ or $R_0 = \emptyset$;
- (c) for all $t \in \mathbb{T}_i$ and all $\omega \in S_i$, we have $t + \omega \in \mathbb{T}_i$, i.e., $\mathbb{T}_i$ is an ω-periodic time scale;
- (d) for $i \neq j$, for all $t \in \mathbb{T}_i \setminus \{t_{ij}^k\}$ and all $\omega \in S_j$, we have $t + \omega \notin \mathbb{T}$, where $\{t_{ij}^k\}$ is the connected points set of the timescale sequence $\{\mathbb{T}_i\}_{i\in\mathbb{Z}^+}$;
- (e) $R_0 = \{0\}$ if and only if $\mathbb{T}_r$ is a zero-periodic time scale and $R_0 = \emptyset$ if and only if $\mathbb{T}_r = \emptyset$;

and the set Π is called a changing-periods set of $\mathbb{T}$, $\mathbb{T}_i$ is called the periodic sub-timescale of $\mathbb{T}$ and S_i is called the periods subset of $\mathbb{T}$ or the periods set of $\mathbb{T}_i$, $\mathbb{T}_r$ is called the remain timescale of $\mathbb{T}$ and R_0 the remain periods set of $\mathbb{T}$.

In [9], the following results are given:

Theorem 2 (Theorem 2.11 in [9]) If $\mathbb{T}$ is an infinite time scale and the graininess function $\mu : t \to \mathbb{R}^+$ is bounded, then $\mathbb{T}$ is a changing-periodic time scale.

Theorem 3 (Theorem 2.21 in [9]) (Decomposition theorem of time scales) Let $\mathbb{T}$ be an infinite time scale and the graininess function $\mu : \mathbb{T} \to \mathbb{R}^+$ be bounded, then $\mathbb{T}$ is a changing-periodic time scale, i.e., there exists a countable periodic decomposition such that $\mathbb{T} = (\bigcup_{i=1}^{\infty} \mathbb{T}_i) \cup \mathbb{T}_r$ and $\mathbb{T}_i$ is an ω-periodic sub-timescale, $\omega \in S_i, i \in \mathbb{Z}^+$, where $\mathbb{T}_i, S_i, \mathbb{T}_r$ satisfy the conditions in Definition 11.

Theorem 4 (Theorem 2.23 in [9]) (Periodic coverage theorem of time scales) Let

$\mathbb{T}$ be an infinite time scale and the graininess function $\mu : \mathbb{T} \to \mathbb{R}^+$ be bounded, then $\mathbb{T}$ can be covered by countable periodic time scales.

Unfortunately, we have the following remark:

Remark 6 Theorems 2 and 3 are incorrect. The reasons are as follows:

According to Definition 11, if $\mathbb{T}$ is a changing-periodic time scale, then $\mathbb{T}$ can be expressed as $\mathbb{T} = (\bigcup_{i=1}^{\infty} \mathbb{T}_i) \cup \mathbb{T}_r$, where $\mathbb{T}_i \subset \mathbb{T}$ is a periodic time scale. Hence, Theorem 2 is essentially equivalent to Theorem 1. Therefore, in view of Remark 4, Theorem 2 is incorrect. For the same reason, Theorem 3 is also incorrect.

As to Theorem 4, since its proof is based on Theorem 2, we cannot judge the correctness of it.

The following concept of an index function for changing-periodic time scales in [9] has flaws.

Definition 12 (Definition 2.8 in [9]) Let $\mathbb{T}$ be a changing-periodic time scale, then the function τ

$$\begin{aligned} \tau \quad : \quad & \mathbb{T} \mapsto \mathbb{Z}^+ \cup \{0\}, \\ & \left(\bigcup_{i=1}^{\infty} \mathbb{T}_i \right) \mapsto i, \ where \ t \in \mathbb{T}_i, \ i \in \mathbb{Z}^+, \\ & \mathbb{T}_r \mapsto 0, \ where \ t \in \mathbb{T}_r, \\ & t \mapsto \tau_t \end{aligned}$$

is called an index function for $\mathbb{T}$, where the corresponding periods set of $\mathbb{T}_{\tau_t}$ is denoted as S_{τ_t}. In what follows we shall call S_{τ_t} the adaption set generated by t, and all the elements in S_{τ_t} will be called the adaption factors for t.

Remark 7 According to Definition 10 and Definition 11, for $i \neq j$, we have $\mathbb{T}_i \cap \mathbb{T}_j = \{t_{ij}^k\}_{k\in\mathbb{Z}}$, where $\{t_{ij}^k\}$ is a countable points set or an empty set, and $\mathbb{T}_r \cap \mathbb{T}_i$ may not be an empty set. Therefore, the concept of an index function for changing-periodic time scales is not well defined.

Definition 13 (Definition 3.1 in [9]) Let $\mathbb{T}$ be a changing-periodic time scale, i.e., $\mathbb{T}$ satisfies Definition 11. A function $f \in C(\mathbb{T} \times D, E^n)$ is called a local-almost periodic function in $t \in \mathbb{T}$ uniformly for $x \in D$ if the ε-translation numbers set of f,

$$E\{\varepsilon, f, S\} = \{\tilde{\tau} \in S_{\tau_t} : |f(t+\tilde{\tau}, x) - f(t,x)| < \varepsilon \ for \ all \ (t,x) \in \mathbb{T} \times S\}$$

is a relatively dense set for all $\varepsilon > 0$ and for each compact subset S of D; that is, for any given $\varepsilon > 0$ and each compact subset S of D, there exists a constant $l(\varepsilon, S) > 0$ such that each interval of length $l(\varepsilon, S)$ contains $\tilde{\tau}(\varepsilon, S) \in E\{\varepsilon, f, S\}$ such that

$$|f(t+\tilde{\tau}, x) - f(t,x)| < \varepsilon \ for \ all \ (t,x) \in \mathbb{T} \times S;$$

here, $\tilde{\tau}$ is called the ε-local translation number of f and $l(\varepsilon, S)$ is called the local inclusion length of $E\{\varepsilon, f, S\}$.

Definition 14 (Definition 3.2 in [9]) Assume that $\mathbb{T}$ is a changing-periodic time scale. Let $f(t,x) \in C(\mathbb{T} \times D, E^n)$ if for any given adaption factors sequence $(\alpha^\tau)' \subset S_{\tau_t}$, there exists a subsequence $\alpha^\tau \subset (\alpha^\tau)'$ such that $T_{\alpha^\tau} f(t,x)$ exists uniformly on $\mathbb{T} \times S$, then $f(t,x)$ is called a local-almost periodic function in t uniformly for $x \in D$.

Remark 8 Since Definition 13 and Definition 14 are based on the concept of an index function for changing-periodic time scales, they are not well defined.

In [10, 16], the following definitions of almost periodic time scales and almost periodic functions on time scales are given:

Definition 15 [10, 16] We say that $\mathbb{T}$ is an almost periodic time scale if for any given $\varepsilon_1 > 0$, there exists a constant $l(\varepsilon_1) > 0$ such that each interval of length $l(\varepsilon_1)$ contains a $\tau(\varepsilon_1)$ such that

$$d(\mathbb{T}, \mathbb{T}^\tau) < \varepsilon_1;$$

that is, for any $\varepsilon_1 > 0$, the following set

$$E\{\mathbb{T}, \varepsilon_1\} = \{\tau \in \mathbb{R} : d(\mathbb{T}^\tau, \mathbb{T}) < \varepsilon_1\}$$

is relatively dense. This τ is called the ε_1-translation number of $\mathbb{T}$, $l(\varepsilon_1)$ is called the inclusion length of $E\{\mathbb{T}, \varepsilon_1\}$, and $E\{\mathbb{T}, \varepsilon_1\}$ is called the ε_1-translation set of $\mathbb{T}$.

In the following, we denote $\Pi_\varepsilon = E\{\mathbb{T}, \varepsilon\}$ and $\mathbb{T}^{\Pi_\varepsilon} = \{\mathbb{T}^{-\tau} : -\tau \in E\{\mathbb{T}, \varepsilon\}\}$.

Definition 16 (Definition 5.5 in [10]) Let $\mathbb{T}$ be an almost periodic time scale under Definition 15. A function $f \in C(\mathbb{T} \times D, E^n)$ is called an almost periodic function in $t \in \mathbb{T}$ uniformly for $x \in D$ if the ε_2-translation set of f

$$\begin{aligned} E\{\varepsilon_2, f, S\} = \{\tau \in \Pi_{\varepsilon_1} : &|f(t+\tau, x) - f(t, x)| < \varepsilon_2, \\ &for\ all\ (t, x) \in (\mathbb{T} \cap (\cup_{-\tau} \mathbb{T}^{\Pi_{\varepsilon_1}})) \times S\} \end{aligned}$$

is a relatively dense set for all $\varepsilon_2 > \varepsilon_1 > 0$ and for each compact subset S of D; that is, for any given $\varepsilon_2 > \varepsilon_1 > 0$ and each compact subset S of D, there exists a constant $l(\varepsilon_2, S) > 0$ such that each interval of length $l(\varepsilon_2, S)$ contains $\tau(\varepsilon_2, S) \in E\{\varepsilon_2, f, S\}$ such that

$$|f(t+\tau, x) - f(t, x)| < \varepsilon_2,\ for\ all\ (t, x) \in (\mathbb{T} \cap (\cup_{-\tau} \mathbb{T}^{\Pi_{\varepsilon_1}})) \times S.$$

This τ is called the ε_2-translation number of f and $l(\varepsilon_2, S)$ is called the inclusion length of $E\{\varepsilon_2, f, S\}$.

Remark 9 Since the fact that $\mathbb{T}$ is an almost periodic time scale under Definition 15 may do not guarantee that the set $\{\tau \in \Pi_{\varepsilon_1} = E\{\mathbb{T}, \varepsilon_1\} : \mathbb{T} \cap \mathbb{T}^\tau \neq \emptyset\}$ is relatively dense. Therefore, Definition 16 is not well defined. An correction for this, we refer to [13].

Definition 17 (Definition 6.2 in [10]) Let $\mathbb{T}$ be an almost periodic time scale. A function $f \in C(\mathbb{T} \times D, E^n)$ is called an ε^*-local almost periodic function in $t \in \mathbb{T}$ uniformly for $x \in D$ if there exists some fixed $\varepsilon^* > 0$, such that for all $\varepsilon_1 > \varepsilon^*$ and $\varepsilon^* < d(\mathbb{T}, \mathbb{T}^\tau) < \varepsilon_1$, the ε_2-translation set of f

$$E\{\varepsilon_2, f, S\} = \{\tau \in \Pi_{\varepsilon_1} : |f(t+\tau, x) - f(t, x)| < \varepsilon_2,\ for\ all\ (t, x) \in (\mathbb{T} \cap \mathbb{T}^{-\tau}) \times S\}$$

is a relatively dense set for all $\varepsilon_2 > \varepsilon_1 > 0$ and for each compact subset S of D; that is, for any given $\varepsilon_2 > 0$ and each compact subset S of D, there exists a constant $l(\varepsilon_2, S) > 0$ such that each interval of length $l(\varepsilon_2, S)$ contains a $\tau(\varepsilon_2, S) \in E\{\varepsilon_2, f, S\}$ such that

$$|f(t+\tau, x) - f(t, x)| < \varepsilon_2,\ for\ all\ (t, x) \in (\mathbb{T} \cap \mathbb{T}^{-\tau}) \times S.$$

This τ is called the ε_2-translation number of f and $l(\varepsilon_2, S)$ is called the inclusion length of $E\{\varepsilon_2, f, S\}$.

Remark 10 If we take $\mathbb{T} = \mathbb{Z}$, then $\mathbb{T} \cap \mathbb{T}^\tau = \emptyset$ for $\tau \in \{\tau \in \mathbb{R} : \varepsilon^* < d(\mathbb{T}, \mathbb{T}^\tau) < \varepsilon_1 < 1\}$. Therefore, Definition 17 is not well defined.

As a closing of this note, we shall point out that, in order to define local-almost periodic functions, local-almost automorphic functions and so on, on time scales, one needs a proper definition of almost periodic time scales, which can support these classes of functions on time scales, the almost periodic time scale under Definition 4 may be the most general almost periodic time scale. However, since a time scale under Definition 5 has the concrete property (ii) of Definition 5, for conveniens, the almost periodic time scale under Definition 5 may be the best one on which one can define and investigate local-almost periodic functions, local-almost automorphic functions and so on.

Acknowledgement

This work is supported by the National Natural Sciences Foundation of People's Republic of China under Grant 11361072.

References

[1] Y. K. Li and C. Wang, Uniformly almost periodic functions and almost periodic solutions to dynamic equations on time scales, Abstr. Appl. Anal. 2011 (2011), Article ID 341520, 22 pages.
[2] Y. K. Li and C. Wang, Almost periodic functions on time scales and applications, Discrete Dyn. Nat. Soc. 2011 (2011), Article ID 727068, 20 pages.
[3] Y. K. Li and C. Wang, Pseudo almost periodic functions and pseudo almost periodic solutions to dynamic equations on time scales, Adv. Difference Equ. (2012) 2012:77.
[4] C. Lizama, J.G. Mesquita, Almost automorphic solutions of dynamic equations on time scales, J. Funct. Anal. 265 (2013) 2267-2311.
[5] G. Mophou, G.M. N'Guérékata and A. Milce, Almost automorphic functions of order n and applications to dynamic equations on time scales, Discrete Dyn. Nat. Soc. 2014 (2014), Article ID 410210, 13 pages.
[6] C. Wang and Y.K. Li, Weighted pseudo almost automorphic functions with applications to abstract dynamic equations on time scales, Ann. Polon. Math. 108 (2013) 225-240.
[7] C. Wang and R. P. Agarwal, Weighted piecewise pseudo almost automorphic functions with applications to abstract impulsive ∇-dynamic equations on time scales, Adv. Difference Equ. (2014) 2014:153.
[8] C. Wang and R. P Agarwa, Relatively dense sets, corrected uniformly almost periodic functions on time scales, and generalizations, Adv. Difference Equ. (2015) 2015:312.
[9] C. Wang and R. P Agarwal, Changing-periodic time scales and decomposition theorems of time scales with applications to functions with local almost periodicity and automorphy, Adv. Difference Equ. (2015) 2015:296.
[10] C. Wang and R. P. Agarwal, A classification of time scales and analysis of the general delays on time scales with applications, Math. Meth. Appl. Sci. 39 (2016) 1568-1590.
[11] A. M. Fink, Almost Periodic Differential Equations, Springer-Verlag, Berlin, 1974.
[12] E. R. Kaufmann and Y. N. Raffoul, Periodic solutions for a neutral nonlinear dynamical equation on a time scale, J. Math. Anal. Appl. 319 (2006) 315-325.
[13] Y. K. Li and B. Li, Almost periodic time scales and almost periodic functions on time scales, J. Appl. Math. 2015 (2015), Article ID 730672, 8 pages.
[14] Y. K. Li, L. L. Zhao and L. Yang, C^1-Almost periodic solutions of BAM neural networks with time-varying delays on time scales, The Scientific World J. 2015 (2015), Article ID 727329, 15 pages.
[15] Y. K. Li, B. Li, X.F. Meng, Almost automorphic functions on time scales and almost automorphic solutions to shunting inhibitory cellular neural networks on time scales, J. Nonlinear Sci. Appl. 8 (2015) 1190-1211.
[16] C. Wang and R. P. Agarwal, A further study of almost periodic time scales with some notes and applications, Abstr. Appl. Anal. 2014 (2014), Article ID 267384, 11 pages.

Yongkun Li
Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, People's Republic of China
E-mail address: yklie@ynu.edu.cn