Electronic Journal of Mathematical Analysis and Applications
Vol. 5(1) Jan. 2017, pp. 50-63.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

SOME FIXED POINT THEOREMS FOR MAPPINGS SATISFYING A GENERAL MULTIPLICATIVE CONTRACTIVE CONDITION OF INTEGRAL TYPE

BADSHAH-E-ROME AND MUHAMMAD SARWAR

Abstract. The objective of this manuscript is to study the existence and uniqueness of fixed points for general multiplicative contractive condition of integral type. Our work generalizes and extends some well known results of the literature.

1. Introduction and Preliminaries

Michael Grossman and Robert Katz [13] presented the concept of *multiplicative calculus* also termed as *exponential calculus* where the ordinary product and ratio are used as exponential sum and difference respectively on the domain of positive real numbers. Florack and Van Assen [12] employed multiplicative calculus in biomedical image exploration. Bashirov *et al.*[7] signalized versatile problems in various fields where multiplicative calculus is more efficient and effective than the Newtonian calculus for modeling. Using the notion of multiplicative calculus, Bashirov and Bashirova [8] obtained function that exhibits dynamics of literary text. Bashirov *et al*[5] established the fundamental theorem of multiplicative calculus. They also defined multiplicative distance, thus provided basis for multiplicative metric spaces.

Özavsar and Cevikel [16] presented the notion of multiplicative contraction mapping. Along with some other results they proved the well known Banach contraction principle for such contraction in the framework of multiplicative metric spaces. He *et al.* [14] improved the work of [16] in terms of two pairs of self-mappings satisfying certain commutative conditions on multiplicative metric spaces. Mujahid Abbas *et*

2010 *Mathematics Subject Classification.* 47H10, 54H25, 55M20.
Key words and phrases. multiplicative metric, multiplicative convergence, multiplicative open ball, multiplicative Cauchy sequence.
Submitted Oct. 17, 2015.

al.[3] established common fixed point theorems for quasi-weak commutative mappings on a closed ball in the context of multiplicative metric spaces and also solved multiplicative integral and multiplicative differential equations. Branciari [6] was the first to establish an integral version of Banach contraction principle. Rhoades [18] and Liu *at al.*[15] extended and improved the result of Brancairi. The authors in [1, 2, 4, 9, 10, 15, 18, 20, 21] obtained some fixed point results for mappings satisfying more general contractive conditions of this type.

In this paper we establish some fixed point results for mapping satisfying a general multiplicative contraction of integral type. Our results generalise the results of Branciari [6], Rhoades [18]and Liu *at al.*[15] in the setting of multiplicative metric spaces. For further details about multiplicative calculus, multiplicative metric space and related concepts, we refer the reader to [7, 11, 13, 16, 17, 19] The following definitions and results will be needed in sequel.

Definition 1.1. [5] Let M be a nonempty set. A mapping $d : M \times M \to [1, \infty)$ is said to be multiplicative metric if the following conditions are satisfied for all $x, y, z \in M$.

$$\begin{aligned} (1)\ \ d(x,y) &= 1 \Leftrightarrow x = y \\ (2)\ \ d(x,y) &= d(y,x) \\ (3)\ \ d(x,z) &\leq d(x,y).d(y,z).(\text{multiplicative triangular inequality }) \end{aligned}$$

Definition 1.2. [5] For $x \in \mathbb{R}_+ \to \mathbb{R}_+$ multiplicative absolute value of x is defined as follows

$$|x|^* = \begin{cases} x & \text{if } x \geq 1 \\ \frac{1}{x} & \text{if } x < 1 \end{cases}$$

.

Example 1.1. [16] Let R^n_+ be the collection of all $n-$tupples of positive real numbers. And let
$d^* : R^n_+ \times R^n_+ \to R$ be defined as

$$d^*(x, y) = |\frac{x_1}{y_1}|^* \cdot |\frac{x_2}{y_2 1}|^* \cdots |\frac{x_n}{y_n}|^*$$

where $x = (x_1, x_2 \cdots x_n), y = (y_1, y_2 \cdots y_n) \in R^n_+$. Then clearly $d^*(x, y)$ is a multiplicative metric.

Definition 1.3. [16] (Multiplicative reverse triangular inequality)Let (X, d) be a multiplicative metric space. Then we have the following inequality

$$\frac{1}{d(x,y)} \leq \frac{d(x,z)}{d(y,z)} \leq d(x,y) \Leftrightarrow |\frac{d(x,z)}{d(y,z)}|^* \leq d(x,y)$$

This is called *multiplicative reverse triangular inequality.*

Definition 1.4. [16] (Multiplicative open ball) Let (X, d) be a multiplicative metric space. If $a \in X$ and $r > 1$ then subset $B_r(a) = B(a; r) = \{x \in X : d(a, x) < r\}$ of X is called multiplicative open ball centered at a with radius r.

Definition 1.5. [16] (Limit point) Let A be any subset of a multiplicative metric space (X, d). A point $x \in X$ is called limit point of A if and only if $(A \cap B_\epsilon(x)) - \{x\} \neq \phi$ for every $\epsilon > 1$

Definition 1.6. [16] A sequence $\{x_n\}$ in a multiplicative metric space (X,d) is said to be multiplicative convergent to a point $x \in X$ if for a given $\epsilon > 1$ there exits a positive integer n_0 such that
$d(x_n, x) < \epsilon$ for all $n \geq n_0$
or equivalently, if for every multiplicative open ball $B_\epsilon(x)$ there exists a positive integer n_0 such that $n \geq n_0 \Rightarrow x_n \in B_\epsilon(x)$ then the sequence $\{x_n\}$ is said to be multiplicative convergent to a point $x \in X$ denoted by $x_n \to x(n \to \infty)$

Definition 1.7. [16] A sequence $\{x_n\}$ in a multiplicative metric space (X,d) is said to be multiplicative Cauchy sequence if for all $\epsilon > 1$ there exits a positive integer n_0 such that
$d(x_n, x_m) < \epsilon$ for all $n, m \geq n_0$

Definition 1.8. [16] A multiplicative metric space (X, d) is said to be complete if every multiplicative Cauchy sequence in Xconverges in X.

Lemma 1.1. [16] *A sequence $\{x_n\}$ in a multiplicative metric space (X, d) is multiplicative Cauchy sequence if and only if $d(x_n, x_m) \to 1(n, m \to \infty)$*

2. Main Results

Our first theorem in the main result generalizes the result of Rhoades [18] to multiplicative metric space.

Theorem 2.1. *Let (M,d) be a complete multiplicative metric space, $\rho \in [0, 1)$ and $T : M \to M$ be a mapping such that, for each $x, y \in M$,*

$$\int_1^{d(Tx,Ty)} \varphi(s)^{ds} \leq \rho \int_1^{m(x,y)} \varphi(s)^{ds}, \tag{1}$$

Where

$$m(x, y) = max\Big\{d(x, y), d(x, Tx), d(y, Ty), \sqrt{d(x, Ty) \cdot d(y, Tx)}\Big\} \tag{2}$$

and
$\varphi : [1, \infty) \to [1, \infty)$ *is a Lebesgue-integrable mapping which is summable, nonnegative, such that*

$$\int_1^{\delta} \varphi(s)^{ds} > 0 \text{ for each } \delta > 1 \tag{3}$$

Then T has a unique fixed point.

Proof. Let $x \in M$ and, define $x_n = T^n x$. For each integer $n \geq 1$, from (1)

$$\int_1^{d(x_n,x_{n+1})} \varphi(s)^{ds} \leq \rho \int_1^{m(x_{n-1},x_n)} \varphi(s)^{ds}. \tag{4}$$

Using (2)

$$m(x_{n-1},x_n) = \max\left\{d(x_{n-1},x_n), d(x_n,x_{n+1}), \sqrt{d(x_{n-1},x_{n+1})}\right\} \tag{5}$$

Using multiplicative triangular inequality, we have

$$\begin{aligned}\sqrt{d(x_{n-1},x_{n+1})} &\leq \sqrt{d(x_{n-1},x_n)\cdot d(x_n,x_{n+1})} \\ &\leq \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}\end{aligned}$$

Therefore,

$$m(x_{n-1},x_n) \leq \max\{d(x_{n-1},x_n), (x_n,x_{n+1})\}. \tag{6}$$

Substituting into (4), we obtain

$$\begin{aligned}\int_1^{d(x_n,x_{n+1})} \varphi(s)^{ds} &\leq \rho \int_1^{\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}} \varphi(s)^{ds} \\ &= \rho \max\left\{\int_1^{d(x_{n-1},x_n)} \varphi(s)^{ds}, \int_1^{d(x_n,x_{n+1})} \varphi(s)^{ds}\right\} \\ &= \rho \int_1^{d(x_{n-1},x_n)} \varphi(s)^{ds} \quad \because \text{The other case is not possible.} \\ \Rightarrow \int_1^{d(x_n,x_{n+1})} \varphi(s)^{ds} &\leq \rho \int_1^{d(x_{n-1},x_n)} \varphi(s)^{ds} \leq \cdots \leq \rho^n \int_1^{d(x_0,x_1)} \varphi(s)^{ds}.\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} \int_1^{d(x_n,x_{n+1})} \varphi(s)^{ds} = 0 \qquad \because \rho \in [0,1)$$

which implies that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 1 \tag{7}$$

We will prove that $\{x_n\}$ is a multiplicative Cauchy sequence. Suppose by the way of contradiction that it is not. Then there exists some $\epsilon > 1$ such that for an integer k there exist integers $m(k) > n(k) > k$ such that

$$d(x_{m(k)},x_{n(k)}) > \epsilon \qquad \text{and} \qquad d(x_{m(k)},x_{n(k)-1}) \leq \epsilon. \tag{8}$$

Using (2) we have

$$m(x_{m(k)-1},x_{n(k)-1}) = \max\{d(x_{m(k)-1},x_{n(k)-1}),d(x_{m(k)-1},x_{m(k)}),d(x_{n(k)-1},x_{n(k)}) \\ ,\sqrt{d(x_{m(k)-1},x_{n(k)})\cdot d(x_{n(k)-1},x_{m(k)})}\}. \quad (9)$$

Using (7),

$$\lim_{k\to\infty} d(x_{m(k)-1},x_{m(k)}) = \lim_{k\to\infty} d(x_{n(k)-1},x_{n(k)}) = 1.$$

Therefore

$$\lim_{k\to\infty}\int_1^{d(x_{m(k)-1},x_{m(k)})}\varphi(s)^{ds} = \lim_{k\to\infty}\int_1^{d(x_{n(k)-1},x_{n(k)})}\varphi(s)^{ds} = 0. \quad (10)$$

Using multiplicative triangular inequality and (8)

$$\begin{aligned} d(x_{m(k)-1},x_{n(k)-1}) &\le d(x_{m(k)-1},x_{m(k)})\cdot d(x_{m(k)},x_{n(k)-1}) \\ &\le d(x_{m(k)-1},x_{m(k)})\cdot\epsilon \\ \Rightarrow \lim_{k\to\infty} d(x_{m(k)-1},x_{n(k)-1}) &\le \lim_{k\to\infty} d(x_{m(k)-1},x_{m(k)})\cdot\epsilon = 1\cdot\epsilon=\epsilon. \end{aligned}$$

Therefore

$$\lim_{k\to\infty}\int_1^{d(x_{m(k)-1},x_{n(k)-1})}\varphi(s)^{ds} \le \int_1^{\epsilon}\varphi(s)^{ds}. \quad (11)$$

Again using multiplicative triangular inequality and (8), we have

$$\begin{aligned} &\sqrt{d(x_{m(k)-1},x_{n(k)})\cdot d(x_{n(k)-1},x_{m(k)})} \\ \le\ &\sqrt{d(x_{m(k)-1},x_{m(k)})\cdot d(x_{m(k)},x_{n(k)-1})\cdot d(x_{n(k)-1},x_{n(k)})\cdot d(x_{n(k)-1},x_{m(k)})} \\ =\ &\sqrt{d(x_{m(k)-1},x_{m(k)})\cdot d(x_{n(k)-1},x_{n(k)})}\cdot d(x_{n(k)-1},x_{m(k)}) \\ \le\ &\sqrt{d(x_{m(k)-1},x_{m(k)})\cdot d(x_{n(k)-1},x_{n(k)})}\cdot\epsilon. \end{aligned}$$

Letting $k\to\infty$ and using (7),we obtain

$$\lim_{k\to\infty}\sqrt{d(x_{m(k)-1},x_{n(k)})\cdot d(x_{n(k)-1},x_{m(k)})} \le \epsilon.$$

Therefore

$$\lim_{k\to\infty}\int_1^{\sqrt{d(x_{m(k)-1},x_{n(k)})\cdot d(x_{n(k)-1},x_{m(k)})}}\varphi(s)^{ds} \le \int_1^{\epsilon}\varphi(s)^{ds}. \quad (12)$$

Combining (9), (10) , (11) and (12) we have

$$\int_1^{m(x_{m(k)-1},x_{n(k)-1})}\varphi(s)^{ds} \le \int_1^{\epsilon}\varphi(s)^{ds}. \quad (13)$$

Using (1) , (8) and (13), we have

$$\int_1^{\epsilon} \varphi(s)^{ds} < \int_1^{d(x_{m(k)},x_{n(k)}} \varphi(s)^{ds} \leq \rho \int_1^{m(x_{m(k)-1},x_{n(k)-1})} \varphi(s)^{ds} \leq \rho \int_1^{\epsilon} \varphi(s)^{ds}.$$

Which is a contradiction. Therefore $\{x_n\}$ is multiplicative Cauchy sequence. As (M,d) is a complete multiplicative metric space, therefore $\{x_n\}$ converges to some point w of M.

Using (1) and (2), we have

$$\begin{aligned}\int_1^{d(Tw,x_{n+1})} \varphi(s)^{ds} &\leq \rho \int_1^{m(w,x_n)} \varphi(s)^{ds} \\ &= \rho\max\{^{d(w,x_n)}\varphi(s)^{ds}, \int_1^{d(w,Tw)} \varphi(s)^{ds}, \int_1^{d(x_n,x_{n+1})} \varphi(s)^{ds} \\ &\quad, \int_1^{\sqrt{d(w,x_{n+1})\cdot d(x_n,Tw)}} \varphi(s)^{ds}\}.\end{aligned}$$

Letting $n \to \infty$ we obtain

$$\begin{aligned}\int_1^{d(Tw,w)} \varphi(s)^{ds} &\leq \rho\max\{\int_1^{d(w,w)} \varphi(s)^{ds}, \int_1^{d(w,Tw)} \varphi(s)^{ds}, \int_1^{d(w,w)} \varphi(s)^{ds} \\ &\quad, \int_1^{\sqrt{d(w,w)\cdot d(w,Tw)}} \varphi(s)^{ds}\} \\ &= \rho \max\{0, \int_1^{d(w,Tw)} \varphi(s)^{ds}, \int_1^{\sqrt{d(w,Tw)}} \varphi(s)^{ds}\}. = \rho \int_1^{d(w,Tw)} \varphi(s)^{ds}.\end{aligned}$$

Therefore $\int_1^{d(Tw,w)} \varphi(s)^{ds} = 0 \Rightarrow d(Tw,w) = 1 \Rightarrow Tw = w$. That is w is fixed point of T. Next we are going to show that fixed point of T is unique, suppose on the contrary that w and z are two distinct fixed points of T in M. Using (1) and (2), we have,

$$\begin{aligned}&\int_1^{d(z,w)} \varphi(s)^{ds} = \int_1^{d(Tz,Tw)} \varphi(s)^{ds} \leq \rho \int_1^{m(z,w)} \varphi(s)^{ds} \\ &= \rho \max\left\{\int_1^{d(z,w)} \varphi(s)^{ds}, \int_1^{d(z,Tz)} \varphi(s)^{ds}, \int_1^{d(w,Tw)} \varphi(s)^{ds}, \int_1^{\sqrt{d(z,Tw)\cdot d(w,Tz)}} \varphi(s)^{ds}\right\} \\ &= \rho \max\left\{\int_1^{d(z,w)} \varphi(s)^{ds}, 0\right\} \\ &\Rightarrow \int_1^{d(z,w)} \varphi(s)^{ds} \leq \rho \int_1^{d(z,w)} \varphi(s)^{ds} \Rightarrow \int_1^{d(z,w)} \varphi(s)^{ds} = 0 \Rightarrow d(z,w) = 1 \Rightarrow z = w.\end{aligned}$$

Hence fixed point of T is unique. $\square$

If in Theorem 2.1, we let $m(x,y) = d(x,y)$, then the following corollary is deduced. Which is actually multiplicative version of the result of Branciari.

Corollary 2.1. *Let (M,d) be a complete multiplicative metric space, $\rho \in]0,1[$ and let $T : M \to M$. be a mapping such that, for each $x, y \in M$,*

$$\int_1^{d(Tx,Ty)} \varphi(s)^{ds} \leq \rho \int_1^{d(x,y)} \varphi(s)^{ds},$$

Where $\varphi : [1,\infty) \to [1,\infty)$ is a Lebesgue-integrable mapping which is summable on each compact subset of $[1,\infty)$, nonnegative, and such that

$$\int_1^{\delta} \varphi(s)^{ds} > 0 \text{ for each } \delta > 1$$

Then T has a unique fixed point.

To prove the next result, we need the following lemma.

Lemma 2.1. [15] *Let $\upsilon : R^+ \to R^+$ be a Lebesgue-integrable, summable on each compact subset of R^+, $\int_1^{\delta} \upsilon(s)^{ds} > 0$ for each $\delta > 1$ and $\{x_n\}_{n\in N}$ be a nonnegative sequence with $lim_{n\to\infty} x_n = a$, then*

$$lim_{n\to\infty} \int_1^{x_n} \upsilon(s)^{ds} = \int_1^{a} \upsilon(s)^{ds}$$

Theorem 2.2. *Let T be a mapping from a complete multiplicative metric space (M,d)into itself satisfying*

$$\int_1^{d(Tx,Ty)} \upsilon(s)^{ds} \leq \rho(d(x,y)) \int_1^{d(x,y)} \upsilon(s)^{ds}, \forall x, y \in M \tag{14}$$

Where $\upsilon : [1,\infty) \to [1,\infty)$ is a Lebesgue-integrable mapping which is summable, nonnegative, such that

$$\int_1^{\delta} \upsilon(s)^{ds} > 0 \text{ for each } \delta > 1 \tag{15}$$

and $\rho : [1,\infty) \to [0,1)$ is a mapping with

$$\lim_{k\to r} \rho(k) < 1, \forall r > 1$$

.
Then T has a unique fixed point.

Proof. Let x be an arbitrary point of M. Using (14) we have

$$\begin{aligned}\int_1^{d(T^n x, T^{n+1}x)} \upsilon(s)^{ds} &\leq \rho(d(T^{n-1}x, T^n x)) \int_1^{d(T^{n-1}x, T^n x)} \upsilon(s)^{ds} \\ &\leq \int_1^{d(T^{n-1}x, T^n x)} \upsilon(s)^{ds} \quad \forall n \in N \quad \because 0 \leq \rho(d(x,y)) < 1.\end{aligned}$$

Next we show that $d(T^n x, T^{n+1}x) \leq d(T^{n-1}x, T^n x) \ \ \forall n \in N$. Suppose that it doesn't hold. Then there exists some $m \in N$ for which $d(T^m x, T^{m+1}x) > d(T^{m-1}x, T^m x)$. Using the property of υ we have,

$$\begin{aligned}&\int_1^{d(T^m x, T^{m+1}x)} \upsilon(s)^{ds} \leq \rho(d(T^{m-1}x, T^m x)) \int_1^{d(T^{m-1}x, T^m x)} \upsilon(s)^{ds} \\ \leq\ &\int_1^{d(T^{m-1}x, T^m x)} \upsilon(s)^{ds} \leq \int_1^{d(T^m x, T^{m+1}x)} \upsilon(s)^{ds} \\ \leq\ &\rho(d(T^{m-1}x, T^m x)) \int_1^{d(T^{m-1}x, T^m x)} \upsilon(s)^{ds} < \int_1^{d(T^{m-1}x, T^m x)} \upsilon(s)^{ds} \\ \Rightarrow\ &\int_1^{d(T^{m-1}x, T^m x)} \upsilon(s)^{ds} < \int_1^{d(T^{m-1}x, T^m x)} \upsilon(s)^{ds}.\end{aligned}$$

Which is a contradiction.
Hence $d(T^n x, T^{n+1}x) \leq d(T^{n-1}x, T^n x)$, that is $\{d(T^n x, T^{n+1}x)\}_{n \in N}$ is monotonically nonincreasing bounded below, therefore there exists some constant $\eta \geq 1$ with

$$\lim_{n \to \infty} d(T^n x, T^{n+1}x) = \eta.$$

We claim that $\eta = 1$. Suppose it is not true, rather $\eta > 1$. Using (14), Lemma 2.1 and property of υ, we have

$$\begin{aligned}0 < \int_1^{\eta} \upsilon(s)^{ds} &= \lim_{n \to \infty} \int_1^{d(T^n x, T^{n+1}x)} \upsilon(s)^{ds} \\ &\leq \lim_{n \to \infty} \{\rho(d(T^{n-1}x, T^n x)) \int_1^{d(T^{n-1}x, T^n x)} \upsilon(s)^{ds}\} \\ &= \lim_{n \to \infty} \rho(d(T^{n-1}x, T^n x)) \lim_{n \to \infty} \int_1^{d(T^{n-1}x, T^n x)} \upsilon(s)^{ds} \\ &= \{\lim_{k \to \eta} \rho(k)\} \int_1^{\eta} \upsilon(s)^{ds} \\ &< \int_1^{\eta} \upsilon(s)^{ds} \qquad \because \lim_{k \to \eta} \rho(k) < 1 \text{ for } \eta > 1.\end{aligned}$$

Which is not possible.

$$\text{Hence} \quad \lim_{n\to\infty} d(T^n x, T^{n+1}x) = 1. \tag{16}$$

We are going to show that $\{T^n x\}$ is a Cauchy sequence, that is

$$\forall \varepsilon > 1 \ \ \exists \ \ \alpha_\varepsilon \in N \ \ | \ \ \forall \ \ p, q \in N, p > q > \alpha_\varepsilon \ \ d(T^p x, T^q x) < \varepsilon.$$

Suppose on the contrary that $\{T^n x\}$ is not a Cauchy sequence, that is there exists an $\varepsilon > 1$ such that for each $\mu \in N$ there are $p_\mu, q_\mu \in N$, with $p_\mu > q_\mu > \mu$, and $d(T^{p_\mu}x, T^{q_\mu}x) \geq \varepsilon$. For each μ, let p_μ be the minimal i.e $d(T^{p_\mu}x, T^{q_\mu}x) \geq \varepsilon$ but $d(T^i x, T^{q_\mu}x) < \varepsilon$ where $i \in \{q_\mu + 1, q_\mu + 2, \cdots, p_\mu - 1\}$.
Now using multiplicative triangular inequality and multiplicative reverse triangular inequality, we have $\forall \ \ \mu \in N$

$$d(T^{p_\mu}x, T^{q_\mu}x) \leq d(T^{q_\mu}x, T^{p_\mu - 1}x) \cdot d(T^{p_\mu - 1}x, T^{p_\mu}x); \tag{17}$$

$$\left| \frac{d(T^{p_\mu}x, T^{q_\mu + 1}x)}{d(T^{p_\mu}x, T^{q_\mu}x)} \right| \leq d(T^{q_\mu}x, T^{q_\mu + 1}x); \tag{18}$$

$$\left| \frac{d(T^{p_\mu + 1}x, T^{q_\mu + 1}x)}{d(T^{p_\mu}x, T^{q_\mu + 1}x)} \right| \leq d(T^{p_\mu}x, T^{p_\mu + 1}x) \tag{19}$$

$$\left| \frac{d(T^{p_\mu + 1}x, T^{q_\mu + 2}x)}{d(T^{p_\mu + 1}x, T^{q_\mu + 1}x)} \right| \leq d(T^{p_\mu + 1}x, T^{q_\mu + 2}x). \tag{20}$$

From (17), $\varepsilon \leq d(T^{p_\mu}x, T^{q_\mu}x) \leq d(T^{q_\mu}x, T^{p_\mu - 1}x) \cdot d(T^{p_\mu - 1}x, T^{p_\mu}x)$. Letting $\mu \to \infty$ and using (16), we have

$$\begin{aligned} &\varepsilon \leq \lim_{\mu\to\infty} d(T^{p_\mu}x, T^{q_\mu}x) < \varepsilon \cdot 1 \\ &\lim_{\mu\to\infty} d(T^{p_\mu}x, T^{q_\mu}x) = \varepsilon. \end{aligned} \tag{21}$$

Letting $\mu \to \infty$ and using (16),(18) and (21), we get

$$\begin{aligned} &\left| \frac{\lim_{\mu\to\infty} d(T^{p_\mu}x, T^{q_\mu + 1}x)}{\varepsilon} \right| \leq 1 \\ &\Rightarrow \lim_{\mu\to\infty} d(T^{p_\mu}x, T^{q_\mu + 1}x) = \varepsilon. \end{aligned} \tag{22}$$

Letting $\mu \to \infty$ and using (16),(19) and (22), we get

$$\begin{aligned} &\left| \frac{\lim_{\mu\to\infty} d(T^{p_\mu + 1}x, T^{q_\mu + 1}x)}{\varepsilon} \right| \leq 1 \\ &\Rightarrow \lim_{\mu\to\infty} d(T^{p_\mu + 1}x, T^{q_\mu + 1}x) = \varepsilon. \end{aligned} \tag{23}$$

Similarly letting $\mu \to \infty$ and using (16),(20) and (23), we get

$$\begin{aligned} &\left|\frac{\lim_{\mu\to\infty} d(T^{p_\mu+1}x, T^{q_\mu+2}x)}{\varepsilon}\right| \leq 1 \\ &\Rightarrow \lim_{\mu\to\infty} d(T^{p_\mu+1}x, T^{q_\mu+2}x) = \varepsilon. \end{aligned} \tag{24}$$

In view of (14), we have

$$\int_1^{d(T^{p_\mu+1}x, T^{q_\mu+2}x)} \upsilon(s)^{ds} \leq \rho(d(T^{p_\mu}x, T^{q_\mu+1}x)) \int_1^{d(T^{p_\mu}x, T^{q_\mu+1}x)} \upsilon(s)^{ds}.$$

Taking limit $\mu \to \infty$ and using (22) and (24), we have

$$\begin{aligned} \int_1^{\varepsilon} \upsilon(s)^{ds} &= \lim_{\mu\to\infty} \int_1^{d(T^{p_\mu+1}x, T^{q_\mu+2}x)} \upsilon(s)^{ds} \\ &\leq \lim_{\mu\to\infty} \rho(d(T^{p_\mu}x, T^{q_\mu+1}x)) \lim_{\mu\to\infty} \int_1^{d(T^{p_\mu}x, T^{q_\mu+1}x)} \upsilon(s)^{ds} \\ &= \{\lim_{k\to\varepsilon} \rho(k)\} \int_1^{\varepsilon} \upsilon(s)^{ds} < \int_1^{\varepsilon} \upsilon(s)^{ds} \qquad \because \lim_{k\to\varepsilon} \rho(k) < 1 \text{ for } \varepsilon > 1. \end{aligned}$$

Which is not possible. Hence $\{T^n x\}$ is a Cauchy sequence. As (M, d) is complete multiplicative metric space, therefore there must be some $w \in M$ such that $\lim_{n\to\infty} T^n x = w$. We claim that w is fixed point of T. Using (14), we have

$$\begin{aligned} 0 \leq \int_1^{d(T^{n+1}x, Tw)} \upsilon(s)^{ds} &\leq \rho(d(T^n x, w)) \int_1^{d(T^n x, w)} \upsilon(s)^{ds} \\ &\leq \int_1^{d(T^n x, w)} \upsilon(s)^{ds} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Which implies that

$$\lim_{n\to\infty} \int_1^{d(T^{n+1}x, Tw)} \upsilon(s)^{ds} = 0 \Rightarrow \lim_{n\to\infty} d(T^{n+1}x, Tw) = 1. \tag{25}$$

Using (16),(25) and multiplicative triangular inequality we conclude that

$$\begin{aligned} &d(w, Tw) \leq d(w, T^n x) \cdot d(T^n x, T^{n+1}x) \cdot d(T^{n+1x}, Tw) \to 1 \text{ as } n \to \infty \\ &\Rightarrow d(w, Tw) = 1 \Rightarrow Tw = w. \end{aligned}$$

Next we show that T has unique fixed point. Suppose on the contrary that $z \in M$ is another fixed point of T distinct from w then

$$\int_1^{d(w,z)} \upsilon(s)^{ds} = \int_1^{d(Tw,Tz)} \upsilon(s)^{ds} \leq \rho(d(w, z)) \int_1^{d(w,z)} \upsilon(s)^{ds} < \int_1^{d(w,z)} \upsilon(s)^{ds},$$

which is not possible. Hence fixed point of T is unique. This completes the proof. □

Theorem 2.3. *Let T be a mapping from a complete multiplicative metric space (M,d)into itself satisfying*

$$\int_1^{d(Tx,Ty)} \upsilon(s)^{ds} \leq \rho(d(x,y)) \int_1^{d(x,Tx)} \upsilon(s)^{ds} + \varrho(d(x,y)) \int_1^{d(y,Ty)} \upsilon(s)^{ds}, \forall x, y \in M \tag{26}$$

Where $\upsilon : [1,\infty) \to [1,\infty)$ is a Lebesgue-integrable mapping which is summable, nonnegative, such that

$$\int_1^{\delta} \upsilon(s)^{ds} > 0 \text{ for each } \delta > 1 \tag{27}$$

and $\rho, \varrho : [1,\infty) \to [0,1)$ are mappings with

$$\rho(s) + \varrho(s) < 1, \quad \forall s \in [1,\infty), \quad \lim_{k\to 1} \varrho(k) < 1 \text{ and } \lim_{k\to r} \frac{\rho(k)}{1-\varrho(k)} < 1, \forall r > 1.$$

Then T has a unique fixed point.

Proof. Let x be an arbitrary point of M. Using (26) we have

$$\begin{aligned}
\int_1^{d(T^nx,T^{n+1}x)} \upsilon(s)^{ds} &\leq \rho(d(T^{n-1}x,T^nx)) \int_1^{d(T^{n-1}x,T^nx)} \upsilon(s)^{ds} \\
&+ \varrho(d(T^{n-1}x,T^nx)) \int_1^{d(T^nx,T^{n+1}x)} \upsilon(s)^{ds}, \quad \forall n \in N \\
\int_1^{d(T^nx,T^{n+1}x)} \upsilon(s)^{ds} &\leq \frac{\rho(d(T^{n-1}x,T^nx))}{\{1-\varrho(d(T^{n-1}x,T^nx))\}} \int_1^{d(T^{n-1}x,T^nx)} \upsilon(s)^{ds} \\
&< \int_1^{d(T^{n-1}x,T^nx)} \upsilon(s)^{ds} \quad \because \rho(s)+\varrho(s) < 1 \Rightarrow \frac{\rho(s)}{1-\varrho(s)} < 1.
\end{aligned}$$

Arguing as in the proof of the *Theorem 2.2*, it can be very easily proved that $\{d(T^nx,T^{n+1}x)\}_{n\in N}$ is monotonically nonincreasing sequence converging to 1,

$$\text{i.e } \lim_{n\to\infty} d(T^nx,T^{n+1}x) = 1. \tag{28}$$

Next we are going to show that $\{T^nx\}$ is a Cauchy sequence, that is

$$\forall \varepsilon > 1 \;\; \exists \;\; \alpha_\varepsilon \in N \;\; | \;\; \forall \;\; p, q \in N, p > q > \alpha_\varepsilon \;\; d(T^px,T^qx) < \varepsilon.$$

Suppose on the contrary that $\{T^nx\}$ is not a Cauchy sequence, that is there exists an $\varepsilon > 1$ such that for each $\mu \in N$ there are $p_\mu, q_\mu \in N$, with $p_\mu > q_\mu > \mu$, and $d(T^{p_\mu}x,T^{q_\mu}x) \geq \varepsilon$. For each μ, let p_μ be the minimal i.e $d(T^{p_\mu}x,T^{q_\mu}x) \geq \varepsilon$ but $d(T^ix,T^{q_\mu}x) < \varepsilon$ where $i \in \{q_\mu+1, q_\mu+2, \cdots, p_\mu-1\}$. As in the proof of the

Theorem 2.2, it can be easily verified that in this case (21)-(24) hold. Now using $(26), (24), (28)$ and the property of υ,, we have

$$\begin{aligned}
0 < \int_1^{\varepsilon} \upsilon(s)^{ds} &= \lim_{\mu\to\infty} \int_1^{d(T^{p_\mu+1}x, T^{q_\mu+2}x)} \upsilon(s)^{ds} \\
&\leq \lim_{\mu\to\infty} \rho(d(T^{p_\mu}x, T^{q_{\mu+1}}x)) \lim_{\mu\to\infty} \int_1^{d(T^{p_\mu}x, T^{p_\mu+1}x)} \upsilon(s)^{ds} \\
&+ \lim_{\mu\to\infty} \varrho(d(T^{p_\mu}x, T^{q_{\mu+1}}x)) \lim_{\mu\to\infty} \int_1^{d(T^{q_\mu+1}x, T^{q_\mu+2}x)} \upsilon(s)^{ds} \\
&= 0.
\end{aligned}$$

Which is not possible. Hence $\{T^n x\}$ is a Cauchy sequence. As (M, d) is complete multiplicative metric space, therefore there must be some $w \in M$ such that $\lim_{n\to\infty} T^n x = w$, which implies that $\lim_{n\to\infty} d(T^{n+1}x, Tw) = d(w, Tw)$. We claim that w is fixed point of T, i.e $d(w, Tw) = 1$. Suppose $d(w, Tw) \neq 1$, then using (27)and (28) we have

$$\begin{aligned}
0 &< \int_1^{d(w,Tw)} \upsilon(s)^{ds} = \lim_{n\to\infty} \int_1^{d(T^{n+1}x, Tw)} \upsilon(s)^{ds} \\
&\leq \lim_{n\to\infty} \left(\rho(d(T^n, w)) \int_1^{d(T^n x, T^{n+1}w)} \upsilon(s)^{ds} \right) \\
&+ \lim_{n\to\infty} \left(\varrho(d(T^n, w)) \int_1^{d(w,Tw)} \upsilon(s)^{ds} \right) \\
&= \left(\lim_{k\to 1} \varrho(k) \right) \int_1^{d(w,Tw)} \upsilon(s)^{ds} < \int_1^{d(w,Tw)} \upsilon(s)^{ds}.
\end{aligned}$$

Which is contradiction. Hence $d(w, Tw) = 1$. That is $Tw = w$. Next we show that T has unique fixed point. Suppose on the contrary that $z \in M$ is another fixed point of T distinct from w then

$$\begin{aligned}
0 &< \int_1^{d(w,z)} \upsilon(s)^{ds} = \int_1^{d(Tw,Tz)} \upsilon(s)^{ds} \\
&\leq \rho(d(w, z)) \int_1^{d(w,Tw)} \upsilon(s)^{ds} + \varrho(d(w, z)) \int_1^{d(z,Tz)} \upsilon(s)^{ds} \\
&= 0.
\end{aligned}$$

Which is not possible. Hence T has unique fixed point. This competes our proof. $\square$

Corollary 2.2. *If $\rho(r) = k$ for all $r \in [1, \infty)$ where $k \in]0, 1[$ is a constant, then Theorem 2.2 brings the result of Branciari; moreover if $\upsilon(s) = 1$ for all $s \in [1, \infty)$, then Theorem 2.2 reduces to Banach contraction principle.*

Remark 1. Theorems 2.2 and 2.3 generalize the results of *Liu et al* [15] from metric spaces to multiplicative metric spaces.

.

Acknowledgement
The authors are grateful to the editor and anonymous reviewers for their careful reviews, valuable comments and remarks to improve this manuscript.

References

[1] MA Ahmed, S Chauhan, HA Nafadi, "Unified common fixed point theorems for hybrid mappings in modified intuitionistic fuzzy metric spaces via an implicit retion",*Kochi Journal of Mathematics* 9, 153-168, 2014.
[2] MA Ahmed and HA Nafadi, "Common fixed point theorems for hybrid pairs of maps in fuzzy metric spaces", *Journal of the Egyptian Mathematical Society*, **22**, 3, pp. 453-458, 2014.
[3] M. Abbas, Bashir Ali and Yusuf I. Suleiman, "Common Fixed Points of Locally Contractive Mappings in Multiplicative Metric Spaces with Application", *International Journal of Mathematics and Mathematical Sciences*,Volume 2015, Article ID 218683, 7, pp.
[4] A. Aliouche, A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type, Journal of Mathematical Analysis and Applications, vol. **322**, no. 2, pp. 796-802, 2006.
[5] A.E. Bashirov, E.M. Kurpinar and A. Ozyapici, Multiplicative calculus and its applications, J. Math.Analy. App., **337**(2008) 36-48.
[6] A. Branciari, A fixed point theorem for mappings satisfying a general contractive condition of integral type, Int. J. Math. Math. Sci. **29** (2002), no. 9, 531-536.
[7] A. E. Bashirov, E. Misirli, Y. Tandogdu, and A. Ozyapici, "On modeling with multiplicative differential equations", *A Journal of Chinese Universities,* vol. **26**(2011), pp. 425–438
[8] A Bashirov, G Bashirova, "Dynamics of literary texts and diffusion", *Online Journal of Communication and Media Technologies*, 2011, 1(3): 60-82.425-438, 2011.
[9] A. Djoudi and A. Aliouche, Common fixed point theorems of Gregus type for weakly compatiblemappings satisfying contractive conditions of integral type, *Journal ofMathematical Analysis and Applications*, vol. **329**, no. 1, pp. 31-45, 2007.
[10] M Imdad, MA Ahmed, HA Nafadi, "Common fixed point theorems for hybrid pairs of maps in fuzzy metric spaces" *Thai Journal of Mathematics* 12 (3), 749-760, 2014.
[11] J Englehardt, J Swartout, CLoewenstine. "A new theoretical discrete growth distribution with verification for microbial counts in water", *Risk Analysis*, 2009, 29(6): 841-856.
[12] L. Florack and H. V. Assen, "Multiplicative calculus in biomedical image analysis," *Journal ofMathematical Imaging and Vision*, vol. 42, no. 1, pp. 64-75, 2012.
[13] M Grossman, RKatz. *Non-Newtonian Calculus*, Pigeon Cove, Lee Press, Massachusats, 1972.
[14] X. He, M. Song, and D. Chen, "Common fixed points for weak commutative mappings on a multiplicative metric space", *Fixed PointTheory and Applications*, vol. 2014, article 48, 2014.

[15] Liu, Z, Li, X, Kang, SM, Cho, SY: Fixed point theorems for mappings satisfying contractive conditions of integral type and applications. Fixed Point Theory Appl. 2011, Article ID 64 (2011).
[16] M. Őzavsar and A. C. Cevikel, Fixed point of multiplicative contraction mappings on multiplicative metric space. arXiv:1205.5131v1 [matn.GN] (2012).
[17] M Riza, A Özyapici, E Kurpinar, "Multiplicative finite difference methods", *Quarterly of Applied Mathematics*, 2009, 67(4): 745-754.
[18] B. E. Rhoades, Two fixed-point theorems for mappings satisfying a general contractive condition of integral type, International Journal ofMathematics andMathematical Sciences, vol. **2003**,no. 63, pp. 4007-4013, 2003.
[19] D Stanley. A multiplicative calculus, Primus, 1999, IX(4): 310-326.
[20] D. Turko?glu and I. Altun, A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying an implicit relation, to appear in Matematica Mexicana. Boletin. Tercera Serie.
[21] P. Vijayaraju, B. E. Rhoades, and R. Mohanraj, A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type, International Journal of Mathematics and Mathematical Sciences, vol. **2005**, no. 15, pp. 2359-2364, 2005.

Authors Address
Department of Mathematics University of Malakand, Chakdara Dir(L), Khyber PakhtunKhwa, Pakistan.
E-mail address: baadeshah@yahoo.com; sarwarswati@gmail.com