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MATHEMATICAL STUDY OF THE THREE DIMENSIONAL OSCILLATIONS OF A HEAVY ALMOST HOMOGENEOUS LIQUID PARTIALLY FILLING AN ELASTIC CONTAINER

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ABSTRACT. We present in this article a theoretical work to treat the coupling between the structure elasticity and the heterogeneousness of a liquid. Considering an almost homogeneous, incompressible and inviscid liquid in an elastic container, using functional analysis, we obtain a variational formulation of the small amplitude oscillations of the coupled problem around the equilibrium position, then two operatorial equations in a suitable Hilbert space are analyzed. We show that the spectrum of the system is real and consists of a countable set of eigenvalues, and an essential continuous spectrum filling an interval and corresponding physically to a domain of resonance.

1. INTRODUCTION

The study of the classical problem of the small oscillations of an inviscid or viscous homogeneous liquid in a rigid container has been the subject of many works [6, 7, 8]. The case of an elastic container with homogeneous liquid was studied in details in the book [9] and, more recently in [8]. On the other hand, the general case of a viscous heterogeneous liquid was treated in [4], and the planar case of a heterogeneous inviscid incompressible liquid in a rigid container was studied, first by Rayleigh and then by Capodanno and his collaborators [1, 2, 3].

In this aim, we propose here a theoretical study of the three-dimensional oscillations of an incompressible inviscid liquid taking into account the effects of heterogeneousness which are neglected by the majority of authors.

In this contexte, we consider an elastic container, the external boundary of which is fixed, that is partially filled by an almost-homogeneous heavy liquid. After writing the general equations of motion of the system, we linearize the problem assuming small displacements from an equilibrium position. As a second step, and under the hypothesis that the liquid is almost-homogeneous, we reformulate the equations as a variational problem, and finally, as an operatorial problem involving a bounded linear operators on suitable Hilbert space. Finally, we show that the spectrum of the relevant operator, is composed by a discrete part and an essential part filling an interval and corresponding physically to a domain of resonance: we argue that the presence of the essential part of the spectrum

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is due the hypothesis of almost-homogeneity, in contrast to the classical case in which the fluid is homogeneous and the spectrum is entirely discrete [8].

Our work can be used in various applications : tank filled by liquid; ship, train, truck containing a liquid, where the stability of the system is important, and the knowledge of the natural frequencies is essential in the design process of liquid tanks and implementing active control systems in space vehicles.

2. Problem statement

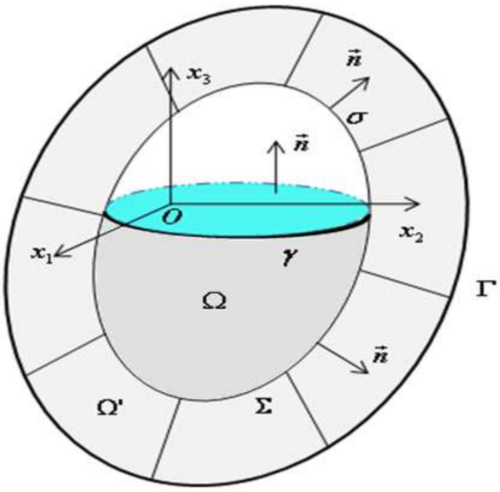

FIGURE 1. Model of the system.

We consider an elastic body that occupies a domain Ω' bounded by a regular closed fixed surface Γ and an regular closed internal surface. The domain bounded by this surface is partially filled by an heavy incompressible inviscid liquid, that occupies in equilibrium position a domain Ω bounded by a surface Σ and the horizontal free surface γ. We denote by σ the part of the internal surface of the body that is above Ω and is wetted by a gas with constant pressure $\mathcal{P}^0$.

We use an orthogonal coordinate system $Ox_1x_2x_3$, Ox_1x_2 being in the plane of γ and Ox_3 directed upwards. The system is supposed at the constant temperature and in a constant gravity field $\vec{g} = -g\vec{x}_3$.

We study the small oscillations of the system elastic body-liquid about its equilibrium position in the framework of the linear theory.

3. Equations of motion

3.1. Equations of motion of the elastic body

We denote by ρ', λ', μ' the density and the Lamé coefficients, wich we suppose constant, of the body.

Let $\vec{u}'_0(x_1,x_2,x_3)$ the displacement of the particle of the body that occupies, in the natural state, the position (x_1,x_2,x_3), from the natural state to the equilibrium state. We denote by

$\vec{n}(x_1,x_2,x_3)$ the unit vector normal to σ and directed to the interior of Ω'. We have:

$$0 = -\rho' g\delta_{i3} + \frac{\partial\sigma'_{ij}(\vec{u}'_0)}{\partial x_j} \qquad \text{in} \qquad \Omega' \ (i,j=1,2,3) \tag{1}$$

$$\vec{u}'_{0|\Gamma} = 0 \tag{2}$$

$$\sigma'_{ij}(\vec{u}'_0)n_j = -\mathcal{P}^0 n_i \qquad \text{on } \sigma \tag{3}$$

where we have set

$$\sigma'_{ij}(\vec{u}'_0) = \lambda'\delta_{ij}\mathrm{div}\vec{u}'_0 + 2\mu'\varepsilon'_{ij}(\vec{u}'_0),$$

$$\varepsilon_{ij}(\vec{u}'_0) = \frac{1}{2}\left(\frac{\partial u'_{0i}}{\partial x_j} + \frac{\partial u'_{0j}}{\partial x_i}\right).$$

The $\varepsilon_{ij}(\vec{u}'_0)$ are the components of the deformation tensor, the $\sigma'_{ij}(\vec{u}'_0)$ are the components of the stress tensor.
Let $\vec{u}'_0(x_1,x_2,x_3)$ the displacement of a particle of the body from its equilibrium position to its position at the instant t.
We have

$$\rho'\frac{\partial^2(u'_i+u'_{0i})}{\partial t^2} = -\rho' g\delta_{i3} + \frac{\partial\sigma'_{ij}(\vec{u}'+\vec{u}'_0)}{\partial x_j} \qquad \text{in} \qquad \Omega' \tag{4}$$

$$(\vec{u}'+\vec{u}'_0)|_\Gamma = 0 \tag{5}$$

$$\sigma'_{ij}(\vec{u}'+\vec{u}'_0)n_j = -\mathcal{P}^0 n_i \qquad \text{on } \sigma \tag{6}$$

Taking into account the equations (1), (2), (3), we obtain

$$\rho'\ddot{u}'_i = \frac{\partial\sigma'_{ij}(\vec{u}')}{\partial x_j} \qquad \text{in} \qquad \Omega' \left(\ddot{u}'_i = \frac{\partial^2(u'_i)}{\partial t^2}\right) \tag{7}$$

$$\vec{u}'|_\Gamma = 0 \tag{8}$$

$$\sigma'_{ij}(\vec{u}')n_j = 0 \qquad \text{on } \sigma \tag{9}$$

We will write in the following the kinematic and dynamic conditions on Σ.

3.2. Equations of motion of the liquid

We suppose that the liquid is heterogeneous. We denote by $\vec{u}(x,t)$ the displacement from the equilibrium state of the particle that occupies the position $x(x_1,x_2,x_3)$ at the instant t, and by $\rho^*(x,t)$, $\mathcal{P}(x,t)$ the density and the pressure in this point.
We have

$$\rho^*\ddot{\vec{u}} = -\overrightarrow{\mathrm{grad}}\mathcal{P} - \rho^* g\vec{x}_3 \qquad \text{(Euler's equation)} \tag{10}$$

$$\mathrm{div}\dot{\vec{u}} = 0 \qquad \text{(incompressibility) in } \Omega \tag{11}$$

$$\frac{\partial\rho^*}{\partial t} + \mathrm{div}(\rho^*\dot{\vec{u}}) = 0 \qquad \text{(continuity equation)} \tag{12}$$

If $\vec{n}$ is the unit vector normal to $\Sigma\cup\gamma$ directed to the exterior of Ω, we have the kinematic condition

$$\vec{u}\cdot\vec{n} = \vec{u}'\cdot\vec{n} \qquad \text{on } \Sigma \tag{13}$$

Let ρ_0, p_0 the density and the pressure in the equilibrium state; we have

$$-\overrightarrow{\text{grad}} p_0 - \rho_0 g \vec{x}_3 = 0$$

so that p_0 and ρ_0 are functions of x_3 only and we have

$$\frac{\mathrm{d}p_0(x_3)}{\mathrm{d}x_3} = -\rho_0(x_3) g$$

We set

$$\rho^*(x,t) = \rho_0(x_3) + \tilde{\rho}(x,t) + \cdots,$$
$$\mathcal{P}(x,t) = p_0(x_3) + p(x,t) + \cdots$$

where $\tilde{\rho}$, p are of the first order with respect to the amplitude of the oscillations and the dots indicate terms of the second order.
The linearized continuity equation is

$$\frac{\partial \tilde{\rho}}{\partial t} + \text{div}(\rho_0 \dot{\vec{u}}) = 0$$

or, since $\text{div}\dot{\vec{u}} = 0$

$$\frac{\partial \tilde{\rho}}{\partial t} + \dot{\vec{u}} \cdot \overrightarrow{\text{grad}} \rho_0 = 0$$

or, integrating from the date of the equilibrium to the instant t

$$\tilde{\rho} = -\rho_0'(x_3) u_3$$

Then the Euler's equation (10) takes the form

$$\rho^* \ddot{\vec{u}} = -\overrightarrow{\text{grad}} p - (\rho^* - \rho_0) g \vec{x}_3$$

and finally the linearized Euler's equation is

$$\rho_0(x_3) \ddot{\vec{u}} = -\overrightarrow{\text{grad}} p + \rho_0'(x_3) g u_3 \vec{x}_3 \qquad \text{in } \Omega \tag{14}$$

After integration, the equation (11) gives

$$\text{div}\vec{u} = 0 \qquad \text{in } \Omega \tag{15}$$

3.3. The dynamic conditions

a) The equation of the moving free surface γ_t is

$$x_3 = u_3(x_1, x_2, 0, t) + \cdots$$

or, writing $u_{n|\gamma}$ for $\vec{u} \cdot \vec{n}$ on γ :

$$x_3 = u_{n|\gamma} + \cdots$$

We must write the the pressure $\mathcal{P}$ of the liquid is equal to $\mathcal{P}^0$ on γ_t.
We have

$$p_0(u_{n|\gamma} + \cdots) + p_{|\gamma} + \cdots = \mathcal{P}^0$$

Then, at the first order

$$p_0(0) + u_{n|\gamma} \frac{\mathrm{d}p_0}{\mathrm{d}x_3}(0) + p_{|\gamma} = \mathcal{P}^0$$

or

$$p_{|\gamma} = \rho_0(0) g u_{n|\gamma} \tag{16}$$

b) Let us write the dynamic conditions on the surface Σ_t, position of Σ at the instant t.
M being a point of Σ, we denote by M_ℓ (resp. M_s) the particle of the liquid (resp. the

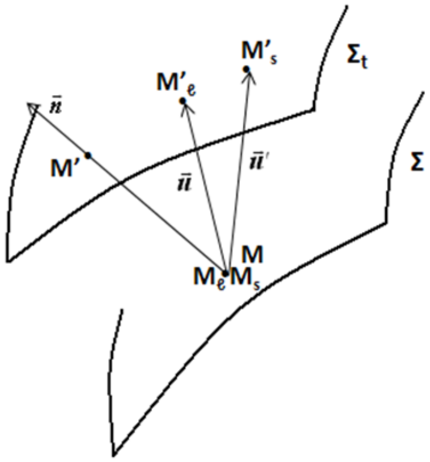

FIGURE 2. Configurations of Σ and Σ_t.

body) wich occupy the position in the equilibrium position.
If M'_ℓ and M'_s are the position of M_ℓ and M_s on Σ_t at the instant t, we have

$$\overrightarrow{MM'_\ell} = \vec{u} \qquad ; \qquad \overrightarrow{MM'_s} = \vec{u}'$$

In linear theory, we admit that the unit vectors normal to Σ_t in M'_ℓ and M'_s are equal to the unit vector $\vec{n}$ normal to Σ in M and that the pressure of the liquid in M'_ℓ is equal to the pressure $\mathcal{P}(M',t)$ in M', intersection of Σ_t with the normal in M to Σ.
Therefore, the dynamic boundary conditions on Σ_t can be written

$$\sigma'_{ij}(\vec{u}'_0 + \vec{u}')n_j = -\mathcal{P}(M',t)n_i$$

Replacing $\sigma'_{ij}(\vec{u}'_0)n_j$ by $-p_0(M)n_i$, we obtain

$$\sigma'_{ij}(\vec{u}')n_j = -[\mathcal{P}(M',t) - p_0(M)]n_i \qquad \text{on } \Sigma$$

But, $u_{n|\Sigma}$ being the normal compnent of $\vec{u}$ on Σ, we have

$$\mathcal{P}(M',t) = \mathcal{P}(M + u_{n|\Sigma}\vec{n},t) = \mathcal{P}(M,t) + \overrightarrow{\text{grad}}\mathcal{P}(M,t)\cdot u_{n|\Sigma}\vec{n} + \cdots$$

$u_{n|\Sigma}$ being of the first order, we can, in linear theory, replace $\overrightarrow{\text{grad}}\mathcal{P}(M,t)$ by

$$\overrightarrow{\text{grad}}p_0(M) = -\rho_0(M)g\vec{x}_3$$

so that we have

$$\mathcal{P}(M',t) = \mathcal{P}(M,t) - \rho_0(M)gu_{n|\Sigma}\, n_{3|\Sigma} + \cdots$$

and finally

$$\sigma'_{ij}(\vec{u}')n_j = -[p(M,t) - \rho_{0|\Sigma}gn_3u_n]n_i \qquad \text{on } \Sigma \tag{17}$$

3.4. The case of the almost homogeneous liquid
Let h the maximum height of the liquid in the equilibrium position.
We suppose that

$$\rho_0(x_3) = \rho(1 - \beta x_3) + o(\beta h),$$

where ρ and β are positive constants, β being sufficiently small so that $(\beta h)^2$, $(\beta h)^3, \cdots$ are negligible with respect to βh.
Then, the liquid is called *"almost-homogeneous in Ω"*.
Like in the Boussinesq approximation for the convective motions of the viscous liquids, substituting in the equation (14)

$$\rho_0 \text{ by } \rho \text{ and } \rho' \text{ by } -\rho\beta,$$

we replace it by the approximated equation

$$\rho\ddot{\vec{u}} = -\overrightarrow{\text{grad}} p - \rho\beta g u_3 \vec{x}_3 \qquad \text{in } \Omega \tag{18}$$

Finally, in the case of an almost-homogeneous liquid, the equations of motion are the equations (7), (8), (9), (18), (15), (13), (16), (17)(in the last two equations, $\rho_0(0)$ and $\rho_{0|\Sigma}$ replaced by ρ).

4. Variational formulation of the problem

4.1. A formal variational formulation

For a formal calculation, we introduce the space of the kinematically admissible displacements:

$$H = \left\{ (\vec{w}', \vec{w})^t \,/\, \vec{w}'_{|\Gamma} = 0, \text{ div}\vec{w} = 0 \text{ on } \Omega, \; w_{n|\Sigma} = w'_{n|\Sigma} \right\}$$

with $\vec{w}'$, $\vec{w}$ sufficiently smooth respectively in Ω' and Ω. This space will be precised later.

Theorem 4.1. *A formal variational equation of the problem is:*

$$\begin{cases} \displaystyle\int_{\Omega'} \rho' \ddot{\vec{u}}' \cdot \bar{\vec{w}}' \, d\Omega' + \int_{\Omega} \rho \ddot{\vec{u}} \cdot \bar{\vec{w}} \, d\Omega + \left[\int_{\Omega'} \sigma'_{ij}(\vec{u}') \varepsilon'_{ij}(\bar{\vec{w}}') \, d\Omega' + \rho g \int_{\Sigma} u'_n n_3 \bar{w}'_n \, d\Sigma \right] \\ \qquad + \rho g \displaystyle\int_{\gamma} u_{n|\gamma} \bar{w}_{n|\gamma} \, d\gamma + \rho\beta g \int_{\Omega} u_3 \bar{w}_3 \, d\Omega = 0 \end{cases}$$

for all admissible $\vec{w}$, $\vec{w}'$.

Proof. i) At first, we have

$$\int_{\Omega'} \rho' \ddot{\vec{u}}' \cdot \bar{\vec{w}}' \, d\Omega' = \int_{\Omega'} \frac{\partial \sigma'_{ij}(\vec{u}')}{\partial x_j} \bar{w}'_i \, d\Omega' = \int_{\Omega'} \left[\frac{\partial}{\partial x_j} [\sigma'_{ij}(\vec{u}') \bar{w}'_i] - \sigma'_{ij}(\vec{u}') \frac{\partial \bar{w}'_i}{\partial x_j} \right] d\Omega'$$

The σ'_{ij} being symmetrical, we have

$$\int_{\Omega'} \sigma'_{ij}(\vec{u}') \frac{\partial \bar{w}'_i}{\partial x_j} \, d\Omega' = \int_{\Omega'} \sigma'_{ij}(\vec{u}') \varepsilon'_{ij}(\bar{\vec{w}}') \, d\Omega'$$

On the other hand, the Green formula gives

$$\int_{\Omega'} \frac{\partial}{\partial x_j} [\sigma'_{ij}(\vec{u}') \bar{w}'_i] \, d\Omega' = \int_{\Gamma \cup \sigma \cup \Sigma} \sigma'_{ij}(\vec{u}') \bar{w}'_i n_{ej} \, d(\partial\Omega'),$$

where, $\vec{n}_e$ is the unit vector that is normal to $\partial\Omega'$ and directed to the exterior of Ω'.
We have

$$\bar{w}'_i = 0 \text{ on } \Gamma, \qquad \sigma'_{ij}(\vec{u}') n_{ej} = 0 \text{ on } \sigma,$$

so that the right hand side of the last integral is reduced to

$$-\int_{\Sigma} \sigma'_{ij}(\vec{u}') n_j \bar{w}'_i \, d\Sigma$$

Taking into account the dynamic conditions (17) on Σ, we obtain

$$\int_{\Omega'} \rho' \ddot{\vec{u}}' \cdot \bar{\vec{w}}' \, d\Omega' = \int_{\Sigma} (p - \rho g n_3 u_n) \bar{w}'_i n_i \, d\Sigma - \int_{\Omega'} \sigma'_{ij}(\vec{u}') \varepsilon'_{ij}(\bar{\vec{w}}') \, d\Omega' \tag{19}$$

For the liquid, we have, from the equation (18):

$$\int_\Omega \rho \ddot{\vec{u}} \cdot \bar{\vec{w}}\, \mathrm{d}\Omega = -\int_\Omega \overrightarrow{\mathrm{grad}} p \cdot \bar{\vec{w}}\, \mathrm{d}\Omega - \rho\beta g \int_\Omega u_3 \bar{w}_3\, \mathrm{d}\Omega$$

We can write

$$\begin{aligned} -\int_\Omega \overrightarrow{\mathrm{grad}} p \cdot \bar{\vec{w}}\, \mathrm{d}\Omega &= -\int_\Omega [\mathrm{div}(p\bar{\vec{w}}) - p\,\mathrm{div}(\bar{\vec{w}})]\, \mathrm{d}\Omega \\ &= -\int_{\gamma\cup\Sigma} p\bar{w}_n\, \mathrm{d}(\partial\Omega) \\ &= -\int_\gamma \rho g u_{n|\gamma} \bar{w}_{n|\gamma} \mathrm{d}\gamma - \int_\Sigma p_{|\Sigma} \bar{w}_{n|\Sigma}\, \mathrm{d}\Sigma, \end{aligned}$$

so that we have

$$\int_\Omega \rho \ddot{\vec{u}} \cdot \bar{\vec{w}}\, \mathrm{d}\Omega = -\int_\gamma \rho g u_{n|\gamma} \bar{w}_{n|\gamma} \mathrm{d}\gamma - \int_\Sigma p_{|\Sigma} \bar{w}_{n|\Sigma}\, \mathrm{d}\Sigma - \rho\beta g \int_\Omega u_3 \bar{w}_3\, \mathrm{d}\Omega \tag{20}$$

Adding (19) and (20), the terms containing $p_{|\Sigma}$ disappear since $w_{n|\Sigma} = w'_{n|\Sigma}$ and we obtain the formal variational equation of the problem

$$\left\{ \begin{array}{l} \displaystyle\int_{\Omega'} \rho' \ddot{\vec{u}}' \cdot \bar{\vec{w}}'\, \mathrm{d}\Omega' + \int_\Omega \rho \ddot{\vec{u}} \cdot \bar{\vec{w}}\, \mathrm{d}\Omega + \left[\int_{\Omega'} \sigma'_{ij}(\vec{u}')\varepsilon'_{ij}(\bar{\vec{w}}')\, \mathrm{d}\Omega' + \rho g \int_\Sigma u'_n n_3 \bar{w}'_n\, \mathrm{d}\Sigma \right] \\ \qquad + \rho g \displaystyle\int_\gamma u_{n|\gamma} \bar{w}_{n|\gamma}\, \mathrm{d}\gamma + \rho\beta g \int_\Omega u_3 \bar{w}_3\, \mathrm{d}\Omega = 0 \end{array} \right. \tag{21}$$

for all admissible $\vec{w}$, $\vec{w}'$.

ii) Conversely we are going to prove that, if $\vec{u}'$ and $\vec{u}$ are functions of t with values in the space of the admissible virtual displacements H and verifying (21),$\vec{u}'$ and $\vec{u}$ are solutions of the problem.
We take $\vec{w}'$ sufficiently smooth in Ω' with $\vec{w}'_{n|\Gamma} = 0$, but $\vec{w}$ sufficiently smooth in Ω, verifying $w_{n|\Sigma} = w'_{n|\Sigma}$, and we introduce a multiplier Λ associated to the condition $\mathrm{div}\vec{w} = 0$.
The equation (21) takes the form

$$\left\{ \begin{array}{l} \displaystyle\int_{\Omega'} \rho' \ddot{\vec{u}}' \cdot \bar{\vec{w}}'\, \mathrm{d}\Omega' + \int_\Omega \rho \ddot{\vec{u}} \cdot \bar{\vec{w}}\, \mathrm{d}\Omega + \left[\int_{\Omega'} \sigma'_{ij}(\vec{u}')\varepsilon'_{ij}(\bar{\vec{w}}')\, \mathrm{d}\Omega' + \rho g \int_\Sigma n_3 u'_n \bar{w}'_n\, \mathrm{d}\Sigma \right] \\ \qquad + \rho g \displaystyle\int_\gamma u_{n|\gamma} \bar{w}_{n|\gamma}\, \mathrm{d}\gamma + \rho\beta g \int_\Omega u_3 \bar{w}_3\, \mathrm{d}\Omega + \int_\Omega \Lambda \mathrm{div}\bar{\vec{w}}\, \mathrm{d}\Omega = 0 \end{array} \right.$$

for all admissible $\vec{w}$, $\vec{w}'$.
We have

$$\begin{aligned} \int_{\Omega'} \sigma'_{ij}(\vec{u}')\varepsilon'_{ij}(\bar{\vec{w}}')\, \mathrm{d}\Omega' &= \int_{\Omega'} \left[\frac{\partial}{\partial x_j}[\sigma'_{ij}(\vec{u}')\bar{w}'_i] - \frac{\partial(\sigma'_{ij}(\vec{u}'))}{\partial x_j} \bar{w}'_i \right] \mathrm{d}\Omega' \\ &= -\int_\Sigma \sigma'_{ij}(\vec{u}') n_j \bar{w}'_i\, \mathrm{d}\Sigma - \int_\sigma \sigma'_{ij}(\vec{u}') n_j \bar{w}'_i\, \mathrm{d}\sigma \\ &\quad - \int_{\Omega'} \frac{\partial(\sigma'_{ij}(\vec{u}'))}{\partial x_j} \bar{w}'_i\, \mathrm{d}\Omega' \\ \int_\Omega \Lambda \mathrm{div}\bar{\vec{w}}\, \mathrm{d}\Omega &= \int_\Omega \left[\mathrm{div}(\Lambda\bar{\vec{w}}) - \overrightarrow{\mathrm{grad}}\Lambda \cdot \bar{\vec{w}} \right] \mathrm{d}\Omega \\ &= \int_\Sigma \Lambda_{|\Sigma} \bar{w}_{n|\Sigma}\, \mathrm{d}\Sigma + \int_\gamma \Lambda_{|\gamma} \bar{w}_{n|\gamma} \mathrm{d}\gamma - \int_\Omega \overrightarrow{\mathrm{grad}}\Lambda \cdot \bar{\vec{w}}\, \mathrm{d}\Omega \end{aligned}$$

Carrying in the previous variational equation, we obtain

$$\left\{\begin{array}{l}\displaystyle\int_{\Omega'}\left(\rho'\ddot{u}'_i-\frac{\partial\sigma'_{ij}(\vec{u}')}{\partial x_j}\right)\bar{w}'_i\,\mathrm{d}\Omega'+\int_{\Omega}\left(\rho\ddot{\vec{u}}-\overrightarrow{\mathrm{grad}}\Lambda+\rho\beta g u_3\vec{x}_3\right)\cdot\bar{\vec{w}}\,\mathrm{d}\Omega\\ \displaystyle+\int_{\gamma}\left(\rho g u_{n|\gamma}+\Lambda_{|\gamma}\right)\bar{w}_{n|\gamma}\mathrm{d}\gamma-\int_{\Sigma}\left(\sigma'_{ij}(\vec{u}')n_j-\rho g n_{3|\Sigma}u'_{n|\Sigma}n_i-\Lambda_{|\Sigma}n_i\right)\bar{w}'_{i|\Sigma}\,\mathrm{d}\Sigma\\ \displaystyle-\int_{\sigma}\sigma'_{ij}(\vec{u}')n_j\bar{w}'_{i|\sigma}\,\mathrm{d}\sigma=0\end{array}\right.$$

Choosing $\vec{w}'\in[\mathcal{D}(\Omega')]^3$. So, we have $w_{n|\Sigma}=w'_{n|\Sigma}=0$. Then we obtain

$$\left\{\begin{array}{l}\displaystyle\int_{\Omega'}\left(\rho'\ddot{u}'_i-\frac{\partial\sigma'_{ij}(\vec{u}')}{\partial x_j}\right)\bar{w}'_i\,\mathrm{d}\Omega'+\int_{\Omega}\left(\rho\ddot{\vec{u}}-\overrightarrow{\mathrm{grad}}\Lambda+\rho\beta g u_3\vec{x}_3\right)\cdot\bar{\vec{w}}\,\mathrm{d}\Omega\\ \displaystyle+\int_{\gamma}\left(\rho g u_{n|\gamma}+\Lambda_{|\gamma}\right)\bar{w}_{n|\gamma}\mathrm{d}\gamma=0\end{array}\right.$$

Taking $\vec{w}=0$, compatible with $w_{n|\Sigma}=0$, we obtain

$$\int_{\Omega'}\left(\rho'\ddot{u}'_i-\frac{\partial\sigma'_{ij}(\vec{u}')}{\partial x_j}\right)\bar{w}'_i\,\mathrm{d}\Omega'=0\qquad\forall\vec{w}'\in\left[\mathcal{D}(\Omega')\right]^3$$

and then

$$\rho'\ddot{u}'_i-\frac{\partial\sigma'_{ij}(\vec{u}')}{\partial x_j}=0\qquad\text{in}\ \left(\left[\mathcal{D}(\Omega')\right]^3\right)'$$

Afterwards, we have

$$\int_{\Omega}\left(\rho\ddot{\vec{u}}-\overrightarrow{\mathrm{grad}}\Lambda+\rho\beta g u_3\vec{x}_3\right)\cdot\bar{\vec{w}}\,\mathrm{d}\Omega+\int_{\gamma}\left(\rho g u_{n|\gamma}+\Lambda_{|\gamma}\right)\bar{w}_{n|\gamma}\mathrm{d}\gamma=0$$

for each $\vec{w}$ verifying $w_{n|\Sigma}=0$.
Taking $\vec{w}\in[\mathcal{D}(\Omega)]^3$, compatible with $w_{n|\Sigma}=0$, we have

$$\rho\ddot{\vec{u}}-\overrightarrow{\mathrm{grad}}\Lambda+\rho\beta g u_3\vec{x}_3=0\qquad\text{in}\ \left[\mathcal{D}(\Omega)'\right]^3$$

Remain

$$\int_{\gamma}\left(\rho g u_{n|\gamma}+\Lambda_{|\gamma}\right)\bar{w}_{n|\gamma}\mathrm{d}\gamma=0$$

for each $\vec{w}$ verifying $w_{n|\Sigma}=0$.
$w_{n|\gamma}$ being arbitrary, we have

$$\rho g u_{n|\gamma}+\Lambda_{|\gamma}=0$$

Then, we have

$$-\int_{\Sigma}\left\{\sigma'_{ij}(\vec{u}')n_j-\left[\rho g n_3u'_n-\Lambda\right]n_i\right\}_{|\Sigma}\bar{w}'_{n|\Sigma}\,\mathrm{d}\Sigma-\int_{\sigma}\sigma'_{ij}(\vec{u}')n_j\bar{w}'_{n|\sigma}\,\mathrm{d}\sigma=0$$

for each $\vec{w}'$ such that $w'_{n|\Gamma}=0$.
$w'_{n|\Sigma}$ and $w'_{n|\sigma}$ being arbitrary, we have

$$\sigma'_{ij}(\vec{u}')n_j=\left(\rho g n_3u'_n-\Lambda\right)n_i\qquad\text{on}\ \Sigma$$

$$\sigma'_{ij}(\vec{u}')n_j=0\qquad\text{on}\ \sigma$$

Setting $p=-\Lambda$ [11], we obtain the Euler's equation and the dynamic boundary conditions, and also, the mechanical interpretation of the multiplier Λ.□

4.2. A precise variational formulation

We suppose that $\vec{u}'$ and $\vec{u}$ belong to the spaces

$$\vec{u}' \in \hat{\Xi}^1(\Omega') = \left\{ \vec{u}' \in \Xi^1(\Omega') \overset{\text{def}}{=} \left[H^1(\Omega')\right]^3;\ \vec{u}'_{|\Gamma} = 0 \right\}\ ;$$

$$\vec{u} \in J(\Omega) = \left\{ \vec{u} \in \mathscr{L}^2(\Omega) \overset{\text{def}}{=} \left[L^2(\Omega)\right]^3;\ \text{div}\vec{u} = 0 \right\}$$

with

$$u'_{n|\Sigma} = u_{n|\Sigma}$$

Obviously, $u'_{n|\Sigma}$ must belong to $H^{1/2}(\Sigma) \subset L^2(\Sigma)$.
Since $\vec{u}$ must belong to $J(\Omega)$, we seek it in the form

$$\vec{u} = \vec{v} + \vec{U}$$

with

$$\vec{v} \in J_0(\Omega) = \left\{ \vec{v} \in \mathscr{L}^2(\Omega);\ \text{div}\vec{v} = 0;\ v_{n|\Sigma\cup\gamma} = 0 \right\}$$

$$\vec{U} \in G_h(\Omega) = \left\{ \vec{U} = \overrightarrow{\text{grad}}\Phi;\ \Phi \in H^1(\Omega);\ \int_\Omega \Phi \, \text{d}\Omega = 0;\ \text{div}\vec{U} = \Delta\Phi = 0 \right\}$$

In accordance to the orthogonal decomposition in $\mathscr{L}^2(\Omega)$ [6]

$$J(\Omega) = J_0(\Omega) \oplus G_h(\Omega)$$

Let us recall [6] that

$$\mathscr{L}^2(\Omega) = J_0(\Omega) \oplus G(\Omega),$$

where $G(\Omega)$ is the space of the potential fields and that

$$G(\Omega) = G_h(\Omega) \oplus G_0(\Omega),$$

where

$$G_0(\Omega) = \left\{ \overrightarrow{\text{grad}}q,\ q \in H_0^1(\Omega) \right\}$$

The Euler's equation (18) can be written

$$\ddot{\vec{v}} + \ddot{\vec{U}} = -\frac{1}{\rho}\overrightarrow{\text{grad}}p - \beta g v_3 \vec{x}_3 - \beta g U_3 \vec{x}_3$$

Consequently, if P_0 is the orthogonal projector from $\mathscr{L}^2(\Omega)$ into $J_0(\Omega)$, we have

$$\ddot{\vec{v}} = -\beta g P_0(v_3 \vec{x}_3) - \beta g P_0(U_3 \vec{x}_3) \tag{22}$$

Let us set

$$\vec{w} = \bar{\vec{v}} + \bar{\vec{U}}$$

$J_0(\Omega)$ and $G_h(\Omega)$ being orthogonal, we have

$$\int_\Omega \rho \ddot{\vec{u}} \cdot \bar{\vec{w}} \, \text{d}\Omega = \int_\Omega \rho \left(\ddot{\vec{v}} \cdot \bar{\vec{v}} + \ddot{\vec{U}} \cdot \bar{\vec{U}} \right) \text{d}\Omega$$

On the other hand, since $w_{n|\gamma} = 0$, we have $u_{n|\gamma} = U_{n|\gamma}$.
Therefore, the variational equation (21) takes the form

$$\left\{\begin{array}{l} \displaystyle\int_{\Omega'} \rho' \ddot{\vec{u}}' \cdot \bar{\vec{w}}'\, d\Omega' + \int_{\Omega} \rho \left(\ddot{\vec{v}} \cdot \bar{\vec{v}} + \ddot{\vec{U}} \cdot \bar{\vec{U}} \right) d\Omega \\ \qquad + \left[\displaystyle\int_{\Omega'} \sigma'_{ij}(\vec{u}')\varepsilon'_{ij}(\bar{\vec{w}}')\, d\Omega' + \rho g \int_{\Sigma} n_{3|\Sigma} u'_{n|\Sigma} \bar{w}'_{n|\Sigma}\, d\Sigma \right] \\ \qquad + \rho g \displaystyle\int_{\gamma} U_{n|\gamma} \bar{U}_{n|\gamma}\, d\gamma + \rho\beta g \int_{\Omega} (v_3 + U_3) \left(\bar{v}_3 + \bar{U}_3 \right) d\Omega = 0 \end{array}\right.$$

But we have

$$\beta g \int_{\Omega} v_3 \bar{v}_3\, d\Omega = \int_{\Omega} \beta g P_0(v_3 \vec{x}_3) \cdot \bar{\vec{v}}\, d\Omega;$$
$$\beta g \int_{\Omega} U_3 \bar{v}_3\, d\Omega = \int_{\Omega} \beta g P_0(U_3 \vec{x}_3) \cdot \bar{\vec{v}}\, d\Omega,$$

so that, in the precedent variational equation appears

$$\int_{\Omega} \rho \left[\ddot{\vec{v}} + \beta g P_0(v_3 \vec{x}_3) + \beta g P_0(U_3 \vec{x}_3) \right] \cdot \bar{\vec{v}}\, d\Omega = 0$$

Then, we can deduce from the last variational equation the following:

Theorem 4.2. *The variational equation of the problem is: to find* $(\vec{v}, \vec{U}, \vec{u}') \in J_0(\Omega) \times G_h(\Omega) \times \hat{\Xi}^1(\Omega')$ *such that*

$$\left\{\begin{array}{l} \displaystyle\int_{\Omega'} \rho' \ddot{\vec{u}}' \cdot \bar{\vec{w}}'\, d\Omega' + \int_{\Omega} \rho \ddot{\vec{U}} \cdot \bar{\vec{U}}\, d\Omega + \left[\int_{\Omega'} \sigma'_{ij}(\vec{u}')\varepsilon'_{ij}(\bar{\vec{w}}')\, d\Omega' + \rho g \int_{\Sigma} n_{3|\Sigma} u'_{n|\Sigma} \bar{w}'_{n|\Sigma}\, d\Sigma \right] \\ \qquad + \rho g \displaystyle\int_{\gamma} U_{n|\gamma} \bar{U}_{n|\gamma}\, d\gamma + \rho\beta g \int_{\Omega} (v_3 + U_3) \bar{U}_3\, d\Omega = 0 \end{array}\right. \tag{23}$$

for all $(\bar{\vec{v}}, \bar{\vec{U}}, \vec{w}') \in J_0(\Omega) \times G_h(\Omega) \times \hat{\Xi}^1(\Omega')$.

4.3. Transformation of the equation (23)

In this subsection we are giving another formulation of the problem (23).
i) At first, it is well-known that in $\hat{\Xi}^1(\Omega')$, $\left(\int_{\Omega'} \sigma'_{ij}(\vec{u}')\varepsilon'_{ij}(\bar{\vec{u}}')\, d\Omega' \right)^{1/2}$ defines a norm which is equivalent to the classical norm $\|\vec{u}'\|_1$ of $\Xi^1(\Omega')$.
On the other hand, by virtue of a trace theorem, we have

$$\left| \rho g \int_{\Sigma} \left| n_{3|\Sigma} \right| \left| u'_{n|\Sigma} \right|^2 d\Sigma \right| \le c\rho g \left\| \vec{u}' \right\|_1^2,$$

c being a positive constant depending on Σ.
Therefore, if ρ or the amount of the liquid is sufficiently small,

$$\left(\int_{\Omega'} \sigma'_{ij}(\vec{u}')\varepsilon'_{ij}(\bar{\vec{u}}')\, d\Omega' + \rho g \int_{\Sigma} \left| n_{3|\Sigma} \right| \left| u'_{n|\Sigma} \right|^2 d\Sigma \right)^{1/2}$$

defines in $\hat{\Xi}^1(\Omega')$ a norm equivalent to $\|\vec{u}'\|_1$.
We suppose it in the following. This norm is denoted by $|||\vec{u}'|||_1$, the associated product $[\vec{u}', \vec{w}']_1$.
On the other hand, we remark that, if $\vec{U} \in G_h(\Omega) \subset E(\Omega) = \left\{ \vec{U} \in \mathscr{L}^2(\Omega);\ \mathrm{div}\vec{U} = 0 \right\}$,

$U_{n|\Sigma\cup\gamma}$ has sens as element of $H^{-1/2}(\Sigma\cup\gamma)$ [10].
Let us introduce the space

$$V=\left\{\begin{array}{l} W=\left(\begin{array}{c}\vec{u}'\\ \vec{U}\end{array}\right);\ \vec{u}'\in\hat{\Xi}^1(\Omega')=\left\{\vec{u}'\in\Xi^1(\Omega'),\ \vec{u}'_{|\Gamma}=0\right\}\\ \vec{U}=\overrightarrow{\text{grad}}\Phi,\ \Phi\in\tilde{H}^1(\Omega)=\left\{\Phi\in H^1(\Omega);\ \displaystyle\int_{\Sigma\cup\gamma}\Phi\mathrm{d}(\partial\Omega)=0\right\};\\ \text{div}\vec{U}=0;\ U_{n|\gamma}\in L^2(\gamma);\ U_{n|\Sigma}=u'_{n|\Sigma}\in H^{1/2}(\Sigma)\end{array}\right\}$$

equipped with the hilbertian norm defined by

$$\|W\|_V^2=\left|\left\|\vec{u}'\right\|\right|_1^2+\int_\Omega\left|\vec{U}\right|^2d\Omega+\left\|U_{n|\gamma}\right\|^2_{L^2(\gamma)}+\left\|U_{n|\Sigma}\right\|^2_{H^{1/2}(\Sigma)}$$

and setting

$$\tilde{W}=\left(\begin{array}{c}\vec{w}'\\ \vec{\tilde{U}}\end{array}\right),$$

the space χ, completion of V for the norm associated to the scalar product

$$(W,\tilde{W})_\chi=\int_{\Omega'}\rho'\vec{u}'\cdot\bar{\vec{w}}'\,\mathrm{d}\Omega'+\int_\Omega\rho\vec{U}\cdot\bar{\vec{\tilde{U}}}\,\mathrm{d}\Omega$$

Then, the variational equation (23) takes the form

$$\left(\ddot{W},\tilde{W}\right)_\chi+\left[\vec{u}',\vec{w}'\right]_1+\rho g\int_\gamma U_{n|\gamma}\bar{\tilde{U}}_{n|\gamma}\,\mathrm{d}\gamma+\rho\beta g\int_\Omega(v_3+U_3)\bar{\tilde{U}}_3\,\mathrm{d}\Omega=0;\ \forall\ \tilde{W}\in V \quad (24)$$

ii) In order to obtain a definitive form of this equation, we introduce a few operators.
We set

$$\beta g P_0\left(v_3\vec{x}_3\right)=A_{11}\vec{v}\qquad;\qquad\beta g P_0\left(U_3\vec{x}_3\right)=A_{12}W$$

A_{11} (resp A_{12}) is bounded from $J_0(\Omega)$ (resp χ) into $J_0(\Omega)$.
Then, the equation (22) can be written

$$\ddot{\vec{v}}+A_{11}\vec{v}+A_{12}W=0 \quad (25)$$

and we have

$$\int_\Omega\beta g v_3\bar{\vec{v}}_3\,\mathrm{d}\Omega=\left(A_{11}\vec{v},\vec{v}\right)_{J_0(\Omega)};\qquad\int_\Omega\beta g U_3\bar{\vec{v}}_3\,\mathrm{d}\Omega=\left(A_{12}W,\vec{v}\right)_{J_0(\Omega)}$$

A_{11} is self-adjoint and not negative. Its spectrum will be studied in the following parafraph.
On the other hand, we have for $\vec{v}\in J_0(\Omega)$, $\tilde{W}\in\chi$:

$$\left|\int_\Omega\beta g v_3\bar{\tilde{U}}_3\,\mathrm{d}\Omega\right|\le c_0\left\|\vec{v}\right\|_{J_0(\Omega)}\left\|\vec{\tilde{U}}\right\|_{\mathscr{L}^2(\Omega)}\le c_0'\left\|\vec{v}\right\|_{J_0(\Omega)}\left\|\tilde{W}\right\|_\chi$$

where c_0 and c_0' are suitable positive constants.
Therefore, we can write

$$\int_\Omega\beta g v_3\bar{\tilde{U}}_3\,\mathrm{d}\Omega=\left(A_{21}\vec{v},\tilde{W}\right)_\chi$$

A_{21} being bounded from $J_0(\Omega)$ into χ.
It is easy to see that A_{21} and A_{12} are mutually adjoint.
Indeed, we have

$$\left(A_{21}\vec{v},\tilde{W}\right)_\chi=\overline{\int_\Omega\beta g\tilde{U}_3\bar{v_3}\,\mathrm{d}\Omega}=\overline{\left(A_{12}\tilde{W},\vec{v}\right)}_{J_0(\Omega)}=\left(\vec{v},A_{12}\tilde{W}\right)_{J_0(\Omega)}$$

In the same manner, we can write

$$\beta g \int_{\Omega} U_3 \bar{\tilde{U}}_3 \mathrm{d}\Omega = \left(A_{22} W, \tilde{W}\right)_{\chi}$$

A_{22} being bounded from χ into χ, self-adjoint, not negative.
Finally, from the variational equation (24) we have the

Theorem 4.3. *The final variational formulation of the problem is: to, find $W \in V$ such that*

$$\left(\ddot{W}, \tilde{W}\right)_{\chi} + \left[\vec{u}', \vec{w}'\right]_1 + \rho g \int_{\gamma} U_{n|\gamma} \bar{\tilde{U}}_{n|\gamma} \,\mathrm{d}\gamma + \rho \left(A_{21} \vec{v} + A_{22} W, \tilde{W}\right)_{\chi} = 0 \ \ \forall \, \tilde{W} \in V \qquad (26)$$

5. The spectrum of the operator A_{11}

In order to study the spectrum of the problem, it is necessary to study the spectrum of the self-adjoint operator A_{11}. This operator was widely studied in [5], and we have the following

Theorem 5.1. *Let $\sigma(A_{11})$ the spectrum of the operator A_{11} and $\sigma_e(A_{11})$ its essential spectrum. We have*

$$\sigma(A_{11}) = \sigma_e(A_{11}) = [0, \beta g]$$

6. Operatorial equations of the problem

In this paragraph we want to deduce an operatorial equation from the variational equation (26), by studying the hermitian sesquilinear form on $V \times V$ defined by:

$$a\left(W, \tilde{W}\right) = \left[\vec{u}', \vec{w}'\right]_1 + \rho g \int_{\gamma} U_{n|\gamma} \bar{\tilde{U}}_{n|\gamma} \,\mathrm{d}\gamma$$

Lemma 6.1. *The hermitian sesquilinear form $a\left(W, \tilde{W}\right)$ is continuous and coercive on $V \times V$, and the embedding $V \subset \chi$, obviously dense and continuous, is compact.*

Proof. We use a method that can be found in the book [10].
1) It is sufficient to prove that $[a(W,W)]^{1/2}$ defines on V a norm equivalent to $\|W\|_V$, i.e. there exist $C > 0$ such that

$$a(W,W) \geq C \|W\|_V^2 \quad \forall \ W \in V$$

or

$$|||\vec{u}'|||_1^2 + \int_{\Omega} \left|\vec{U}\right|^2 \mathrm{d}\Omega + \left\|U_{n|\gamma}\right\|_{L^2(\gamma)}^2 + \left\|U_{n|\Sigma}\right\|_{H^{1/2}(\Sigma)}^2 \leq C^{-1} \left[|||\vec{u}'|||_1^2 + \rho g \int_{\gamma} \left|U_{n|\gamma}\right|^2 \mathrm{d}\gamma\right]$$

It is sufficient to prove the ineqality

$$\int_{\Omega} \left|\vec{U}\right|^2 \mathrm{d}\Omega \leq C' \left(\int_{\gamma} \left|U_{n|\gamma}\right|^2 \mathrm{d}\gamma + \int_{\Sigma} \left|U_{n|\Sigma}\right|^2 \mathrm{d}\Sigma\right) \qquad (C' > 0) \ \text{ for all admissible } \vec{U} \quad (27)$$

We have

$$U_{n|\Sigma} = u'_{n|\Sigma}$$

$$\left\|u'_{n|\Sigma}\right\|_{H^{1/2}(\Sigma)} \leq d \, |||\vec{u}'|||_1 \qquad (d > 0)$$

by virtue of a trace theorem,

$$\left\|u'_{n|\Sigma}\right\|_{L^2(\Sigma)} \leq d' \left\|u'_{n|\Sigma}\right\|_{H^{1/2}(\Sigma)} \leq dd' \, |||\vec{u}'|||_1 \qquad (d' > 0)$$

Then, we have

$$\begin{cases} |||\vec{u}'|||_1^2 + \int_\Omega \left|\vec{U}\right|^2 d\Omega + \|U_{n|\gamma}\|^2_{L^2(\gamma)} + \|U_{n|\Sigma}\|^2_{H^{1/2}(\Sigma)} \leq \left(1 + C'd^2d'^2 + d^2\right) |||\vec{u}'|||_1^2 \\ \quad + \left(C'+1\right) \|U_{n|\gamma}\|_{L^2(\gamma)} \qquad \text{(c.q.f.d)} \end{cases}$$

We must prove the ineqality (27), i.e.

$$\int_\Omega \left|\overrightarrow{\text{grad}}\Phi\right|^2 d\Omega \leq C' \left[\int_\Gamma \left|\frac{\partial\Phi}{\partial n}|_\gamma\right|^2 d\gamma + \int_\Sigma \left|\frac{\partial\Phi}{\partial n}|_\Sigma\right|^2 d\Sigma\right] \qquad \text{for all admissible } \Phi.$$

We consider the Neumann problem

$$\Delta\Phi = 0 \text{ in } \Omega; \ \frac{\partial\Phi}{\partial n}|_\gamma = \delta \in L^2(\gamma); \ \frac{\partial\Phi}{\partial n}|_\Sigma = \tau \in L^2(\Sigma)$$

Let $\Psi \in H^1(\Omega)$. From $\int_\Omega \Delta\Phi\bar{\Psi}d\Omega = 0$ and Green formula, we have

$$\int_\Omega \overrightarrow{\text{grad}}\Phi \cdot \overrightarrow{\text{grad}}\bar{\Psi}d\Omega = \int_\gamma \delta\bar{\Psi}_{|\gamma}d\gamma + \int_\Sigma \tau\bar{\Psi}_{|\Sigma}d\Sigma$$

$\Psi = 1$ gives the classical compatibility condition

$$\int_\gamma \delta d\gamma + \int_\Sigma \tau d\Sigma = 0$$

Thus, $\Phi \in \tilde{H}^1(\Omega)$ is solution of the problem

$$\int_\Omega \overrightarrow{\text{grad}}\Phi \cdot \overrightarrow{\text{grad}}\bar{\Psi}d\Omega = \int_\gamma \delta\bar{\Psi}_{|\gamma}d\gamma + \int_\Sigma \tau\bar{\Psi}_{|\Sigma}d\Sigma \qquad \forall\Psi \in \tilde{H}^1(\Omega) \tag{28}$$

Classically, the left-hand side is a scalar product in $\tilde{H}^1(\Omega)$ and by virtue of a trace theorem, the right-hand side is a continuous linear functional on $\tilde{H}^1(\Omega)$. The equation (28) has one and only one solution by Lax-Milgram theorem.
Setting $\Psi = \Phi$ and using a trace theorem, we have

$$\int_\Omega \left|\overrightarrow{\text{grad}}\Phi\right|^2 d\Omega \leq k\left(\|\delta\|_{L^2(\gamma)} + \|\tau\|_{L^2(\Sigma)}\right) \cdot \|\Phi\|_{\tilde{H}^1(\Omega)} \qquad (k > 0)$$

By the Poincaré inequality, $\|\Phi\|_{\tilde{H}^1(\Omega)}$ and $\left\|\overrightarrow{\text{grad}}\Phi\right\|_{\mathscr{L}^2(\Omega)}$ define equivalent norms, so that we have

$$\left\|\overrightarrow{\text{grad}}\Phi\right\|_{\mathscr{L}^2(\Omega)} \leq k'\left(\|\delta\|_{L^2(\gamma)} + \|\tau\|_{L^2(\Sigma)}\right) \qquad (k' > 0)$$

Which prove the inequality (27).

2) Let $W^p = \begin{pmatrix} \vec{u}'^p \\ \vec{U}^p \end{pmatrix} \in V$ a sequence weakly convergent in V to $W^* = \begin{pmatrix} \vec{u}'^* \\ \vec{U}^* \end{pmatrix} \in V \subset \chi$
.
By Rellich theorem, $\vec{u}'^p$ converges strongly in $\mathscr{L}^2(\Omega')$.
Then, we must prove that

$$\int_\Omega \left|\vec{U}^p - \vec{U}^*\right|^2 d\Omega \to 0 \text{ when } p \to +\infty$$

Setting

$$\vec{U}^p = \overrightarrow{\text{grad}}\Phi^p \qquad ; \qquad \vec{U}^* = \overrightarrow{\text{grad}}\Phi^*$$

we have

$$\overrightarrow{\text{grad}}(\Phi^p - \Phi^*) \to 0 \text{ weakly in } \mathscr{L}^2(\Omega)$$

and then

$$(\Phi^p - \Phi^*) \to 0 \text{ weakly in } \tilde{H}^1(\Omega)$$

Using a trace theorem, we have

$$\begin{cases} (\Phi^p - \Phi^*)\left|_\gamma \to 0 \text{ strongly in } L^2(\gamma)\right. \\ (\Phi^p - \Phi^*)|_\Sigma \to 0 \text{ strongly in } L^2(\Sigma) \end{cases}$$

The mapping

$$W \in V \to \begin{pmatrix} \frac{\partial \Phi}{\partial n}|_\gamma \\ \frac{\partial \Phi}{\partial n}|_\Sigma \end{pmatrix} \in L^2(\gamma) \times H^{1/2}(\Sigma)$$

being continuous, if we set

$$\begin{cases} \frac{\partial \Phi^p}{\partial n}|_\gamma = \delta^p \\ \frac{\partial \Phi^*}{\partial n}|_\gamma = \delta^* \end{cases} ; \qquad \begin{cases} \frac{\partial \Phi^p}{\partial n}|_\Sigma = \tau^p \\ \frac{\partial \Phi^*}{\partial n}|_\Sigma = \delta^* \end{cases}$$

we have

$$\begin{cases} \delta^p - \delta^* \to 0 \text{ weakly in } L^2(\gamma) \\ \tau^p - \tau^* \to 0 \text{ weakly in } H^{1/2}(\Sigma) \text{ and } \text{ strongly in } L^2(\Sigma) \end{cases}$$

Now, we write (28) for Φ^p and Φ^* and take the difference; we obtain

$$\int_\Omega \left|\overrightarrow{\text{grad}}(\Phi^p - \Phi^*)\right|^2 d\Omega = \int_\gamma (\delta^p - \delta^*)\overline{(\Phi^p - \Phi^*)}\,|_\gamma\, d\gamma + \int_\Sigma (\tau^p - \tau^*)\overline{(\Phi^p - \Phi^*)}\,|_\Sigma\, d\Sigma$$

By virtue of the precedents results, the integrals of the right-hand side tend to zero and we have

$$\int_\Omega \left|\overrightarrow{\text{grad}}(\Phi^p - \Phi^*)\right|^2 d\Omega \to 0 \qquad \text{(c.q.f.d)}$$

The variational equation (26) can be written

$$\left(\ddot{W}, \tilde{W}\right)_\chi + a\left(W, \tilde{W}\right) + \rho\left(A_{21}\vec{v} + A_{22}W, \tilde{W}\right)_\chi = 0 \qquad \forall\, \tilde{W} \in V$$

Let us call A the unbounded operator of χ, that is associated to the form $a(.,.)$ and the pair (V, χ). By virtue of a Lemma 6.1 and refrence [6], it is well known that this equation is equivalent to the operatorial equation

$$\ddot{W} + AW + \rho\,(A_{21}\vec{v} + A_{22}W) = 0 \qquad W \in V \tag{29}$$

Consequently we have

Theorem 6.2. *The operatorial equations of the problem are*

$$\begin{cases} \ddot{\vec{v}} + A_{11}\vec{v} + A_{12}W = 0 \\ \ddot{W} + AW + \rho\,(A_{21}\vec{v} + A_{22}W) = 0 \\ \vec{v} \in J_0(\Omega),\ W \in V \end{cases}$$

Remark 6.1: *Operatorial equations with bounded operators*
We can eliminate the unbounded operator A by setting

$$A^{1/2}W = W_0 \in \chi$$

We obtain the equations with bounded coefficients

$$\ddot{\vec{v}} + A_{11}\vec{v} + A_{12}A^{-1/2}W_0 = 0 \tag{30}$$

$$A^{-1}\ddot{W}_0 + \rho A^{-1/2}A_{21}\vec{v} + \left(I_\chi + \rho A^{-1/2}A_{22}A^{-1/2}\right)W_0 = 0 \tag{31}$$

$$\vec{v} \in J_0(\Omega),\ W_0 \in \chi.$$

The operators A^{-1}, $A_{12}A^{-1/2}$, $A^{-1/2}A_{21}$, $A^{-1/2}A_{22}A^{-1/2}$ are compact.□

7. The spectrum of the problem

Let us seek the solutions that depend on time according to the exponential law $e^{i\omega t}$, ω real.
We obtain

$$\omega^2\vec{v} = A_{11}\vec{v} + A_{12}A^{-1/2}W_0 \tag{32}$$

$$\omega^2 A^{-1}W_0 = \rho A^{-1/2}A_{21}\vec{v} + \left(I_\chi + \rho A^{-1/2}A_{22}A^{-1/2}\right)W_0 \tag{33}$$

or, setting $\nu = \omega^{-2}$

$$\vec{v} = \nu A_{11}\vec{v} + \nu A_{12}A^{-1/2}W_0 \tag{34}$$

$$A^{-1}W_0 = \nu\rho A^{-1/2}A_{21}\vec{v} + \nu\left(I_\chi + \rho A^{-1/2}A_{22}A^{-1/2}\right)W_0 \tag{35}$$

Theorem 7.1. *The spectrum of the problem is composed by an essential part, which fills the closed interval* $[0,\beta g]$*, and a discrete part that lies outside this interval and is comprised of a countable set of positive real eigenvalues, whose accumulation point is the infinity. Physically, the interval* $[0,\beta g]$ *is a domain of resonance.*

Proof. *i) The spectrum in the domain:* $\omega^2 > \beta g$
We have

$$|\nu| < (\beta g)^{-1}$$

Since $\|A_{11}\| = \beta g$, $I_{J_0(\Omega)} - \nu A_{11}$ is invertible and the operatorial function $R(\nu) = (I_{J_0(\Omega)} - \nu A_{11})^{-1}$ is holomorphic in the domain $|\nu| < (\beta g)^{-1}$.
The equation (34)gives

$$\vec{v} = \nu R(\nu)A_{12}A^{-1/2}W_0$$

Carrying in the equation (35), we obtain

$$Q(\nu)W_0 \overset{\text{def}}{=} \left[\nu^2\rho A^{-1/2}A_{21}R(\nu)A_{12}A^{-1/2} + \nu\left(I_\chi + \rho A^{-1/2}A_{22}A^{-1/2}\right) - A^{-1}\right]W_0 = 0$$

$Q(\nu)$ is a self-adjoint operatorial function, which is holomorphic in the domain $|\nu| < (\beta g)^{-1}$.
We have

$$Q(0) = -A^{-1} \text{ compact and definite positive,}$$

$$Q'(0) = I_\chi + \rho A^{-1/2}A_{22}A^{-1/2} \text{ strongly positive.}$$

Consequently, by virtue of a theorem of theory of the operator pencils [6], for each ε, $0 < \varepsilon < (\beta g)^{-1}$, in the interval $]0,\varepsilon[$, there is a denumerable infinity of eigenvalues ν_k that tend to zero when k tends to infinity. The corresponding eigenelements $\{W_{0k}\}$ form a Riesz basis in a subspace of χ with finite defect.
For our problem, there is a denumerable infinity of eigenvalues $\omega_k^2 = \nu_k^{-1}$, real positive and that tend to $+\infty$ when $k \to +\infty$.

ii) The spectrum in the domain: $0 \le \omega^2 \le \beta g$
The equation (33) can be written

$$\left(I_\chi + \rho A^{-1/2}A_{22}A^{-1/2} - \omega^2 A^{-1}\right)W_0 = -\rho A^{-1/2}A_{21}\vec{v}$$

Since $\omega^2 \leq \beta g$ and A_{22} contains the factor βg, the coefficient of W_0 is a self-adjoint bounded operator that is strongly positive if βg is sufficiently small.
Under this condition we can write

$$W_0 = -\rho \left(I_\chi + \rho A^{-1/2} A_{22} A^{-1/2} - \omega^2 A^{-1} \right)^{-1} A^{-1/2} A_{21} \vec{v}$$

Carrying in the equation (32), we obtain

$$A_{11}\vec{v} - N(\omega^2)\vec{v} = \omega^2 \vec{v}, \qquad \vec{v} \in J_0(\Omega)$$

with

$$N(\omega^2) = \rho A_{12} A^{-1/2} \left(I_\chi + \rho A^{-1/2} A_{22} A^{-1/2} - \omega^2 A^{-1} \right)^{-1} A^{-1/2} A_{21}$$

$N(\omega^2)$ is an analytical function of ω^2 in the domain $\omega^2 \leq \beta g$; for each ω^2 in this domain, it is compact operator.
Setting

$$Z(\omega^2) = A_{11} - N(\omega^2)$$

we obtain the equation

$$\left(Z(\omega^2) - \omega^2 I_{J_0(\Omega)} \right) \vec{v} = 0, \qquad \vec{v} \in J_0(\Omega)$$

Let us fix ω_1^2 in $[0, \beta g]$. Since $N(\omega^2)$ is compact, we can apply to the operator $Z(\omega_1^2)$ a classical Weyl theorem [6]; we have

$$\sigma_e(Z(\omega_1^2)) = \sigma_e(A_{11}) = [0, \beta g].$$

Let $\omega_2^2 \in [0, \beta g]$; there exists for the operator $Z(\omega_1^2)$ a Weyl sequence $\{\vec{v}_n\}$ depending on ω_1^2 and ω_2^2 [6] such that $\vec{v}_n \to 0$ weakly in $J_0(\Omega)$; $\inf_{J_0(\Omega)} \|\vec{v}_n\|_{J_0(\Omega)} > 0$; $\left(Z(\omega_1^2) - \omega_2^2 I_{J_0(\Omega)} \right) \vec{v}_n \to 0$ in $J_0(\Omega)$.
Choosing $\omega_2^2 = \omega_1^2$, we have for the corresponding Weyl sequence $\left\{ \vec{\tilde{v}}_n \right\}$, depending on ω_1^2 only :

$$\left(Z(\omega_1^2) - \omega_1^2 I_{J_0(\Omega)} \right) \vec{\tilde{v}}_n \to 0 \text{ in } J_0(\Omega).$$

Therefore, ω_1^2 belongs to the essential spectrum of the problem

$$\left(Z(\omega^2) - \omega^2 I_{J_0(\Omega)} \right) \vec{v} = 0$$

ω_1^2 being arbitrary in $[0, \beta g]$, this essential spectrum is $[0, \beta g]$.□

8. Conclusion

Like in papers quoted in references, we study the influence of a small heterogeneousness of the liquid on the oscillations, the important point being the presence of an essential spectum.
In works in the process of publication and in progress, we study analogous problems concerning viscous and viscoelastic liquids, the oscillations of a rigid body in a bounded tank containing a almosthomogeneous liquid (for instance a lake), and the influence of surface tensions.

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