Electronic Journal of Mathematical Analysis and Applications
Vol. 5(1) Jan. 2017, pp. 81-87.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

RADII PROBLEMS OF THE CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS WITH FIXED SECOND COEFFICIENTS

SERAP BULUT AND SAURABH PORWAL

Abstract. The purpose of the present paper is to study certain radii problems for the function

$$f(z)=\left\{\frac{z^{1-\gamma}}{\gamma+\beta}\left(z^{\gamma}\left[R_{\delta}^{m}F(z)\right]^{\beta}\right)'\right\}^{1/\beta},$$

where β be a positive real number, γ be a complex number such that $\gamma+\beta\neq 0$ and the function $F(z)$ varies various subclasses of analytic functions with fixed second coefficients. Relevant connections of the results presented herewith various well-known results are briefly indicated.

1. Introduction

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}:=\{z\in\mathbb{C}:|z|<1\}$.

For $f_j\in\mathcal{A}$ given by

$$f_j(z)=z+\sum_{k=2}^{\infty}a_{k,j}z^k \qquad (j=1,2), \tag{1.2}$$

the Hadamard product (or convolution) f_1*f_2 of f_1 and f_2 is defined by

$$(f_1*f_2)(z)=z+\sum_{k=2}^{\infty}a_{k,1}a_{k,2}z^k. \tag{1.3}$$

Using the convolution (1.3), we introduce the generalization of the Ruscheweyh derivative as follows:

For $f\in\mathcal{A}$, $\delta\geq 0$ and $u>-1$, we consider

$$R_{\delta}^{u}f(z)=\frac{z}{(1-z)^{u+1}}*R_{\delta}f(z) \qquad (z\in\mathbb{U}), \tag{1.4}$$

2000 *Mathematics Subject Classification.* Primary 30C45.
Key words and phrases. Analytic function, Starlike function, Convex function, Close-to-convex function, Spiral-like function, Ruscheweyh derivative.
Submitted July 6, 2015.

where

$$R_\delta f(z) = (1-\delta) f(z) + \delta z f'(z) \qquad (z \in \mathbb{U}).$$

If $f \in \mathcal{A}$ is of the form (1.1), then we obtain the power series expansion of the form

$$R_\delta^u f(z) = z + \sum_{k=2}^{\infty} \left[1 + (k-1)\delta\right] C(u,k) a_k z^k, \tag{1.5}$$

where

$$C(u,k) = \frac{(1+u)_{k-1}}{(k-1)!} \qquad (k \in \mathbb{N} := \{1,2,3,\ldots\}), \tag{1.6}$$

and where $(a)_n$ is the Pochhammer symbol (or shifted factorial) defined (in terms of the Gamma function) by

$$(a)_k := \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1 & \text{if} \quad k = 0,\ a \in \mathbb{C}\backslash\{0\} \\ a(a+1)\cdots(a+k-1) & \text{if} \quad k \in \mathbb{N},\ a \in \mathbb{C}. \end{cases} \tag{1.7}$$

In the case $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we have

$$R_\delta^m f(z) = \frac{z\left[z^{m-1} R_\delta f(z)\right]^{(m)}}{m!}, \tag{1.8}$$

and for $\delta = 0$, we obtain m-th Ruscheweyh derivative introduced in [13], $R_0^m = R^m$.

Remark 1.1. We have

$$\begin{aligned} R_\delta^0 f(z) &= R_\delta f(z), \\ R_\delta^1 f(z) &= z\left[R_\delta^0 f(z)\right]', \\ 2R_\delta^2 f(z) &= z\left[R_\delta^1 f(z)\right]' + R_\delta^1 f(z) \\ (m+1) R_\delta^{m+1} f(z) &= z\left[R_\delta^m f(z)\right]' + m R_\delta^m f(z) \end{aligned}$$

for all $z \in \mathbb{U}$.

Using the generalized Ruscheweyh derivative operator R_δ^m, we consider the following classes.

Definition 1.1. Let $\mathcal{R}_\delta(m,\alpha)$ be the class of functions $f \in \mathcal{A}$ satisfying

$$\Re\left\{\frac{z\left[R_\delta^m f(z)\right]'}{R_\delta^m f(z)}\right\} > \alpha \tag{1.9}$$

for some $0 \le \alpha < 1$, $\delta \ge 0$, $m \in \mathbb{N}_0$, and all $z \in \mathbb{U}$.

Definition 1.2. Let $\mathcal{R}_\delta^\lambda(m,\alpha)$ be the class of functions $f \in \mathcal{A}$ satisfying

$$\Re\left\{e^{i\lambda}\frac{z\left[R_\delta^m f(z)\right]'}{R_\delta^m f(z)}\right\} > \alpha\cos\lambda \tag{1.10}$$

for some $0 \le \alpha < 1$, $\delta \ge 0$, $m \in \mathbb{N}_0$, $\lambda \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and all $z \in \mathbb{U}$.

Definition 1.3. Let $\mathcal{W}_\delta(m,\alpha)$ be the class of functions $f \in \mathcal{S}$ satisfying

$$\Re\left\{\left[R_\delta^m f(z)\right]'\right\} > \alpha \tag{1.11}$$

for some $0 \le \alpha < 1$, $\delta \ge 0$, $m \in \mathbb{N}_0$, and all $z \in \mathbb{U}$.

The class $\mathcal{R}_\delta(m,\alpha)$ is studied by Shaqsi and Darus [11]. Also the class $\mathcal{W}_0(m,\alpha)$ is studied by Oros [9], and Esa and Darus [2].

Remark 1.2. For the special chosing of the parameters, we have the following classes:
(**i**) $\mathcal{R}_0(0,\alpha) = \mathcal{S}^*(\alpha)$ of starlike functions of order α,
(**ii**) $\mathcal{R}_0(1,\alpha) = \mathcal{C}(\alpha)$ of convex functions of order α,
(**iii**) $\mathcal{R}_0^\lambda(0,\alpha) = \mathcal{R}^\lambda(\alpha)$ of spiral-like functions of order α,
(**iv**) $\mathcal{W}_0(0,\alpha) = \mathcal{W}(\alpha)$ of close-to-convex functions of order α.

Libera [5] proved that if $f(z)$ is in $S^*(0)$ or $K(0)$, then the function

$$F(z) = \frac{2}{z}\int_0^z f(t)dt$$

is also an element of $S^*(0)$ or $K(0)$, respectively. Subsequently Livingston [7] considered the converse problem and proved that if $F(z)$ is in $S^*(0)$ or $K(0)$, then the function

$$f(z) = \frac{1}{2}[zF(z)]'$$

belongs to $S^*(0)$ or $K(0)$ in $|z| < \frac{1}{2}$, respectively.

Al-Amiri [1], Gupta et al. [3] and Kumar [4], (see also [14]) generalized these results. In the present paper, we extend these results and establish some interesting results. To prove our main result, we shall require the following definition and lemma.

Let $\mathcal{P}$ denote the class of functions of the form $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ which are analytic in $\mathbb{U}$ and satisfy $\Re\{p(z)\} > 0,\ z \in \mathbb{U}$. Then, it is well-known that $|p_k| \le 2$ for $k = 1, 2, ...,$ (see Nehari [8]). Therefore, without loss of generality, we may assume $p_1 = 2b$, where $0 \le b \le 1$. Let $\mathcal{P}_b$ denote the class of those functions of p for which $p'(0) = 2b$. Tuan and Anh [15] showed that if $p(z) \in \mathcal{P}_b$, then $p(z)$ maps $|z| \le r$ into the disc

$$\left| p(z) - \left(\frac{1+B^2}{1-B^2}\right)\right| \le \frac{2B}{1-B^2}, \quad B = \frac{r(r+b)}{1+br}.$$

Lemma 1.1. [4] *Let β be a non-negative real number and γ be a complex number such that $\gamma + \beta \neq 0$. If $p(z) \in \mathcal{P}_b$, then*

$$\Re\left\{p(z) + \frac{zp'(z)}{\gamma + \beta p(z)}\right\} > 0 \tag{1.12}$$

in $|z| < R[\gamma,\beta,b]$, where $R[\gamma,\beta,b]$ is the smallest positive root of the equation

$$\begin{aligned}&\left|\gamma(1-r^2)(1+2br+r^2) + \beta\left\{(1+br)^2 + r^2(r+b)^2\right\}\right| \\ &-2r\left\{\beta(1+br)(r+b) + (b+2r+br^2)\right\} = 0\ .\end{aligned} \tag{1.13}$$

The result is sharp when γ is a non-negative real number.

2. Main Results

Theorem 2.1. *Let β be a positive real number and γ be a complex number such that $\gamma + \beta \neq 0$. If*

$$F(z) = z + 2az^2 + \cdots \quad \left(0 \le a \le \frac{1-\alpha}{(1+\delta)(1+m)}\right)$$

belongs to $\mathcal{R}_\delta(m,\alpha)$, then the function $f(z)$ defined by

$$f(z) = \left\{\frac{z^{1-\gamma}}{\gamma+\beta}\left(z^\gamma\left[R_\delta^m F(z)\right]^\beta\right)'\right\}^{1/\beta} \tag{2.1}$$

belongs to $\mathcal{S}^*(\alpha)$ *in* $|z| < R\left[\gamma+\alpha\beta, \beta(1-\alpha), \frac{(1+\delta)(1+m)a}{1-\alpha}\right]$.
The result is sharp when γ *is a real number such that* $\gamma \geq -\beta\alpha$.

Proof. From (2.1), we have

$$(\gamma+\beta)\frac{(f(z))^{\beta}}{[R_{\delta}^{m}F(z)]^{\beta}} = \gamma + \beta\frac{z\left[R_{\delta}^{m}F(z)\right]'}{R_{\delta}^{m}F(z)}. \tag{2.2}$$

Substituting

$$q(z) = \frac{z\left[R_{\delta}^{m}F(z)\right]'}{R_{\delta}^{m}F(z)}$$

and differentiating logarithmically, the relation (2.2) yields

$$\frac{zf'(z)}{f(z)} = q(z) + \frac{zq'(z)}{\gamma+\beta q(z)}. \tag{2.3}$$

Now $F(z) \in \mathcal{R}_{\delta}(m,\alpha)$ if and only if there exists a function $p(z)$ in $\mathcal{P}_{\frac{(1+\delta)(1+m)a}{1-\alpha}}$ such that

$$q(z) = (1-\alpha)p(z) + \alpha. \tag{2.4}$$

From (2.3) and (2.4), we obtain

$$\frac{\frac{zf'(z)}{f(z)} - \alpha}{1-\alpha} = p(z) + \frac{zp'(z)}{(\gamma+\alpha\beta) + \beta(1-\alpha)p(z)}. \tag{2.5}$$

Since the real part of right hand side of the equation (2.5) is, by Lemma 1.1, greater than zero in $|z| < R\left[\gamma+\alpha\beta, \beta(1-\alpha), \frac{(1+\delta)(1+m)a}{1-\alpha}\right]$, it follows that

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha$$

in $|z| < R\left[\gamma+\alpha\beta, \beta(1-\alpha), \frac{(1+\delta)(1+m)a}{1-\alpha}\right]$.

To show the sharpness when γ is a real number such that $\gamma \geq -\beta\alpha$, we take

$$R_{\delta}^{m}F(z) = z\left[1 - 2\left(\frac{(1+\delta)(1+m)a}{1-\alpha}\right)z + z^2\right]^{-(1-\alpha)}.$$

Clearly $F(z) = z + 2az^2 + \cdots$, and for this function it is easy to compute from (2.4) that

$$p(z) = \frac{1-z^2}{1-2\left(\frac{(1+\delta)(1+m)a}{1-\alpha}\right)z + z^2}.$$

Hence the sharpness of the result follows from that of Lemma 1.1. This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let* γ *be a complex number such that* $\gamma + 1 \neq 0$. *If*

$$F(z) = z + az^2 + \cdots \quad \left(0 \leq a \leq \frac{1-\alpha}{(1+\delta)(1+m)}\right)$$

belongs to $\mathcal{W}_{\delta}(m,\alpha)$, *then the function* $f(z)$ *defined by*

$$f(z) = \frac{z^{1-\gamma}}{\gamma+1}\left(z^{\gamma}\left[R_{\delta}^{m}F(z)\right]\right)' \tag{2.6}$$

belongs to $\mathcal{W}(\alpha)$ *in* $|z| < R\left[\gamma+1, 0, \frac{(1+\delta)(1+m)a}{1-\alpha}\right]$.
The result is sharp when γ *is a real number such that* $\gamma > -1$.

Proof. The relation (2.6) can be written as

$$(\gamma+1) f(z) = \gamma \left[R_\delta^m F(z)\right] + z \left[R_\delta^m F(z)\right]'.$$

Substituting $q(z) = \left[R_\delta^m F(z)\right]'$ and differentiating the above relation, we have

$$f'(z) = q(z) + \frac{zq'(z)}{\gamma+1}. \tag{2.7}$$

Now $F(z) \in \mathcal{W}_\delta(m, \alpha)$ if and only if there exists a function $p(z)$ in $\mathcal{P}_{\frac{(1+\delta)(1+m)a}{1-\alpha}}$ such that

$$q(z) = (1-\alpha) p(z) + \alpha. \tag{2.8}$$

From (2.7) and (2.8), we obtain

$$\frac{f'(z)-\alpha}{1-\alpha} = p(z) + \frac{zp'(z)}{\gamma+1},$$

and hence by Lemma 1.1,

$$\Re\{f'(z)\} > \alpha$$

holds in $|z| < R\left[\gamma+1, 0, \frac{(1+\delta)(1+m)a}{1-\alpha}\right]$.

To show the sharpness we take

$$R_\delta^m F(z) = \int_0^z \frac{1 - 2\frac{(1+\delta)(1+m)a}{1-\alpha}\alpha t - (1-2\alpha)t^2}{1 - 2\frac{(1+\delta)(1+m)a}{1-\alpha}t + t^2} dt.$$

□

Finally we consider the function $f(z)$ defined by (2.1) is a limiting case for spiral-like functions.

If $\beta \to 0$, the relation (2.1) reduces to

$$f(z) = R_\delta^m F(z) \exp\left[\frac{1}{\gamma}\left\{\frac{z\left[R_\delta^m F(z)\right]'}{R_\delta^m F(z)} - 1\right\}\right], \tag{2.9}$$

where $\gamma \neq 0$. We now prove the following:

Theorem 2.3. *If*

$$F(z) = z + 2ae^{-i\lambda}z^2 + \cdots \quad \left(0 \le a \le \frac{1-\alpha}{(1+\delta)(1+m)\sec\lambda}\right)$$

belongs to $\mathcal{R}_\delta^\lambda(m, \alpha)$ *and* γ *is a complex number with* $\gamma \neq 0$, *then the function* $f(z)$ *defined by (2.9) belongs to* $\mathcal{R}^\lambda(\alpha)$ *in* $|z| < R\left[\gamma, 0, \frac{(1+\delta)(1+m)a\sec\lambda}{1-\alpha}\right]$.
The result is sharp when γ *is a positive real number.*

Proof. Letting

$$q(z) = e^{i\lambda}\frac{z\left[R_\delta^m F(z)\right]'}{R_\delta^m F(z)},$$

we have from (2.9) that

$$e^{i\lambda}\frac{zf'(z)}{f(z)} = q(z) + \frac{1}{\gamma}zq'(z). \tag{2.10}$$

Now $F(z)\in\mathcal{R}_{\delta}^{\lambda}(m,\alpha)$ if and only if there exists a function $p(z)$ in $\mathcal{P}_{\frac{(1+\delta)(1+m)a\sec\lambda}{1-\alpha}}$ such that

$$q(z)=(1-\alpha)\cos\lambda\, p(z)+\alpha\cos\lambda+i\sin\lambda. \tag{2.11}$$

From (2.10) and (2.11), we get

$$\frac{e^{i\lambda}\frac{zf'(z)}{f(z)}-\alpha\cos\lambda-i\sin\lambda}{(1-\alpha)\cos\lambda}=p(z)+\frac{1}{\gamma}zp'(z).$$

Therefore, by Lemma 1.1, the real part of right hand side of this last relation is positive in $|z|<R\left[\gamma,0,\frac{(1+\delta)(1+m)a\sec\lambda}{1-\alpha}\right]$ and hence $\Re\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\}>\alpha\cos\lambda$ in $|z|<R\left[\gamma,0,\frac{(1+\delta)(1+m)a\sec\lambda}{1-\alpha}\right]$.

Sharpness follows by taking

$$R_{\delta}^{m}F(z)=z\left[1-2\frac{(1+\delta)(1+m)a\sec\lambda}{1-\alpha}z+z^{2}\right]^{-(1-\alpha)e^{-i\lambda}\cos\lambda}.$$

□

Remark 2.1. It would be interesting to establish the sharpness of present results for complex γ.

Remark 2.2. If we put $m=\delta=0$ in Theorem $2.1-2.3$, then we obtain corresponding results of Kumar [4].

Acknowledgement

The authors are thankful to the referee for his/her valuable comments and observations which helped in improving the paper.

References

[1] H.S. Al-Amiri, On the radius of univalence of certain analytic functions, Colloq. Math. **28** (1973), 133–139.
[2] G.H. Esa and M. Darus, Application of fractional calculus operators to a certain class of univalent functions with negative coefficient, Int. Math. Forum **2** (2007), no. 57, 2807–2814.
[3] V.P. Gupta, P.K. Jain and I. Ahmad, On the radius of univalence of certain classes of analytic functions with fixed second coefficients, Rend. Math. **12** (1979), 423–430.
[4] V. Kumar, On univalent functions with fixed second coefficient, Indian J. Pure Appl. Math. **14** (1983), no. 11, 1424–1430.
[5] R.J. Libera, Some classes of regular univalent functions, Proc. Amer. Math. Soc. **16** (1965), 755–758.
[6] J.E. Littlewood, On inequalities in the theory of functions, Proc. London Math. Soc. **23** (1925), no. 1, 481–519.
[7] A.E. Livingston, On the radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. **17** (1966), 352–357.
[8] Z. Nehari, Conformal Mapping, Mc-Graw Hill, New York, (1953).
[9] G.I. Oros, On a class of holomorphic functions defined by the Ruscheweyh derivative, Int. J. Math. Math. Sci. **2003**, no. 65, 4139–4144.
[10] S. Owa, On the distortion theorems. I, Kyungpook Math. J. **18** (1978), no. 1, 53–59.
[11] K.A. Shaqsi and M. Darus, On certain subclass of analytic univalent functions with negative coefficients, Appl. Math. Sci., **1** (2007), no. 23, 1121–1128.
[12] H.M. Srivastava and S. Owa, (Eds.), Univalent Functions, Fractional Calculus, and Their Applications, Ellis Horwood Series: Mathematics and Its Applications, Ellis Horwood, Chichester, UK; JohnWiley & Sons, New York, NY, USA, 1989.

[13] S. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc. **49** (1975), 109–115.

[14] P.D. Tuan and V.V. Anh, Radii of starlikeness and convexity of certain classes of analytic functions, J. Math. Anal. Appl. **64** (1978), 146–158.

[15] P.D. Tuan and V.V. Anh, Radii of convexity of two classes of regular functions, Bull. Aust. Math. Soc. **21** (1980), 29–41.

Serap Bulut

Kocaeli University, Civil Aviation College, Arslanbey Campus,41285 Izmit-Kocaeli, TURKEY

E-mail address: serap.bulut@kocaeli.edu.tr

Saurabh Porwal

Department of Mathematics, UIET, CSJM University, Kanpur-208024, (U.P.), India

E-mail address: saurabhjcb@rediffmail.com