Electronic Journal of Mathematical Analysis and Applications
Vol. 5(1) Jan. 2017, pp. 106-111.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

THE DOUBLE χ^2 WITH TWO INNER PRODUCT DEFINED BY MUSIELAK-ORLICZ FUNCTIONS

DEEPMALA, LAKSHMI NARAYAN MISHRA AND N. SUBRAMANIAN

Abstract. In this paper, we introduce the χ^2 inner product space and also discuss general properties in an inner product of χ^2.

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex double sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Tripathy [1] and Mursaleen [2] and Mursaleen and Edely [3, 4], Subramanian and Misra [5], Pringsheim [6], Moricz and Rhoades [7], Robison [8], Savas et al. [9], Raj et al. [10], Francesco Tulone [11] and many others [12-17].

Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ give one space is said to be convergent if and only if the double sequence (S_{mn}) is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n = 1, 2, 3, ...) .$$

A double sequence $x = (x_{mn})$ is said to be double analytic if

$$sup_{m,n} |x_{mn}|^{\frac{1}{m+n}} < \infty.$$

The vector space of all double analytic sequences are usually denoted by Λ^2. A sequence $x = (x_{mn})$ is called double entire sequence if

$$|x_{mn}|^{\frac{1}{m+n}} \to 0 \text{ as } m, n \to \infty.$$

The vector space of all double entire sequences are usually denoted by Γ^2. Let the set of sequences with this property be denoted by Λ^2 and Γ^2 is a metric space with the metric

$$d(x, y) = sup_{m,n} \left\{ |x_{mn} - y_{mn}|^{\frac{1}{m+n}} : m, n : 1, 2, 3, ... \right\}, \tag{1}$$

2010 *Mathematics Subject Classification.* 40A05, 40C05, 40D05.

Key words and phrases. Inner product; analytic sequence; χ^2 space; Musielak-Orlicz function; $p-$ metric space; weights.

Submitted: November 09, 2015.

forall $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ $in\, \Gamma^2$. Let $\phi = \{finite\ sequences\}$.
Consider a double sequence $x = (x_{mn})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\delta_{ij}$ for all $m,n \in \mathbb{N}$,

$$\delta_{mn} = \begin{pmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & ...1 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \end{pmatrix}$$

with 1 in the $(m,n)^{th}$ position and zero otherwise.
A double sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)!\,|x_{mn}|)^{\frac{1}{m+n}} \to 0$ as $m,n \to \infty$. The double gai sequences will be denoted by χ^2.
Let M and Φ are mutually complementary modulus functions. Then, we have:
(i) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y), (Young's\, inequality) \tag{2}$$

(ii) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \tag{3}$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u) \tag{4}$$

Lindenstrauss and Tzafriri [12] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{x \in w : \textstyle\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,\ for\, some\, \rho > 0\right\},$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{\rho > 0 : \textstyle\sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p\,(1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.
A sequence $f = (f_{mn})$ of modulus function is called a Musielak-modulus function [see [13]]. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|\,u - (f_{mn})(u) : u \geq 0\}, m,n = 1,2,\cdots$$

is called the complementary function of a Musielak-modulus function f. For a given Musielak modulus function f, the Musielak-modulus sequence space t_f and its subspace h_f are defined as follows

$$t_f = \left\{x \in w^2 : M_f(|x_{mn}|)^{1/m+n} \to 0\, as\, m,n \to \infty\right\},$$

$$h_f = \left\{x \in w^2 : M_f(|x_{mn}|)^{1/m+n} \to 0\, as\, m,n \to \infty\right\},$$

where M_f is a convex modular defined by

$$M_f(x) = \textstyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}(|x_{mn}|)^{1/m+n}, x = (x_{mn}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x,y) = \left\{\textstyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{|x_{mn}|^{1/m+n}}{mn}\right)\right\}.$$

2. Definition and Preliminaries

We first the notion of two inner product spaces and two metric spaces, which have been introduced. Let X be a real vector space of dimension $w \geq 2$. A real valued function $\langle ., .|.\rangle$ which satisfies the following properties on X^2 is called a two inner product of Musielak on X, and the pair $(X, \langle ., .|.\rangle)$ is called a two inner product space of Musielak. (i.e) $d_p(\langle x_1, x_1|z\rangle, \ldots, \langle x_n, x_n|z\rangle) = \|(d_1 \langle x_1, x_1|z\rangle, \ldots, d_n \langle x_n, x_n|z\rangle)\|_p$ on X satisfying the following four conditions:
(i) $\|(d_1 \langle x_1, x_1|z\rangle, \ldots, d_n \langle x_n, x_n|z\rangle)\|_p = 0$ if and and only if $d_1 \langle x_1, x_1|z\rangle, \ldots, d_n \langle x_n, x_n|z\rangle$ are linearly dependent,
(ii) $\|(d_1 \langle x_1, x_1|z\rangle, \ldots, d_n \langle x_n x_n|z\rangle)\|_p$ is invariant under permutation,
(iii) $\|(\alpha d_1 \langle x_1, x_1|z\rangle, \ldots, d_n \langle x_n, x_n|z\rangle)\|_p = |\alpha| \|(d_1 \langle x_1, x_1|z\rangle, \ldots, d_n \langle x_n, x_n|z\rangle)\|_p, \alpha \in \mathbb{R}$
(iv) $d_p (\langle x_1, y_1|z\rangle, \langle x_2, y_2|z\rangle \cdots \langle x_n, y_n|z\rangle) =$
$\left(d_X \langle x_1, x_2, \cdots x_n, x_1, x_2, \cdots x_n|z\rangle^p + d_Y \langle y_1, y_2, \cdots y_n, y_1, y_2, \cdots y_n|z\rangle^p\right)^{1/p} for 1 \leq p < \infty$; (or)
(v) $d (\langle x_1, y_1|z\rangle, \langle x_2, y_2|z\rangle, \cdots \langle x_n, y_n|z\rangle) :=$
$\sup \{d_X \langle x_1, x_2, \cdots x_n, x_1, x_2, \cdots x_n|z\rangle, d_Y \langle y_1, y_2, \cdots y_n, y_1, y_2, \cdots y_n|z\rangle\},$
for $x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y$ is called the $p-$ two inner product metric of the Cartesian inner product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.
A trivial example of p two inner product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the two inner product space is the p norm:

$$\|(d_1 \langle x_1, x_1|z\rangle, \ldots, d_n \langle x_n, x_n|z\rangle)\|_E = sup\left(|det(d_{mn} \langle x_{mn}, x_{mn}|z\rangle)|\right) =$$
$$sup \left(\begin{vmatrix} d_{11} \langle x_{11}, x_{11}|z\rangle & d_{12} \langle x_{12}, x_{12}|z\rangle & ... & d_{1n} \langle x_{1n}, x_{1n}|z\rangle \\ d_{21} \langle x_{21}, x_{21}|z\rangle & d_{22} \langle x_{22}, x_{22}|z\rangle & ... & d_{2n} \langle x_{2n}, x_{2n}|z\rangle \\ . & & & \\ . & & & \\ . & & & \\ d_{n1} \langle x_{n1}, x_{n1}|z\rangle & d_{n2} \langle x_{n2}, x_{n2}|z\rangle & ... & d_{nn} \langle x_{mn}, x_{mn}|z\rangle \end{vmatrix} \right)$$

where $x_i = (x_{i1}, \cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots n$.
If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.
The function $d(A, B) = \|A - B\|$, which satisfies the following four properties, is called a two metric and the pair $(X, d(A, B))$ is called 2 metric space:
$(N_1)\, d\langle x, z\rangle \geq 0$, for $x, z \in X, d\langle x, z\rangle = 0$ if and only if x and z are linearly independent,
$(N_2)\, d\langle x, z\rangle = d\langle z, x\rangle$, for $x, z \in X$,
$(N_3)\, d\langle \alpha x, z\rangle = |\alpha|\, d\langle x, z\rangle$, for $x, z \in X$ and $\alpha \in \mathbb{R}$,
$(N_4)\, d\langle x + y, z\rangle = d\langle x, z\rangle + d\langle y, z\rangle$, for $x, y, z \in X$.
A sequence $x = (x_{mn})$ in two metric space $(X, d(A, B))$ is convergent if there is an $\xi \in X$ such that $d(x, y) = lim_{ij\to\infty} \left\|x^{ij} - \xi - y\right\| = 0$ and $d(x, y) = lim_{ij\to\infty} \left\|x^{ij} - \xi - z\right\| = 0$ for some linearly independent vectors $y, z \in X$. Moreover, a sequence $x = (x_{mn})$ in $(X, d(A, B))$ is a called Cauchy sequence if $lim_{i,j,r,s\to\infty} \left\|x^{ij} - x^{rs} - y\right\| = 0$ and $lim_{i,j,r,s\to\infty} \left\|x^{ij} - x^{rs} - z\right\| = 0$ for some linearly independent vectors $y, z \in X$. A two metric space is complete if every Cauchy

sequence there is convergent. A complete two metric space is called a two Banach metric space.
The space χ^2 can be equipped with the following two metric $\|. - .\|$:

$$d(x_1,x_2) = \|x_1 - x_2\| = (x_{11}k_{1i} - x_{12}k_{1i}) - (x_{21}k_{2j} - x_{22}k_{2j}) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left\|\begin{vmatrix} x_{11}k_{1i} & x_{12}k_{1i} \\ x_{21}k_{2j} & x_{22}k_{2j}\end{vmatrix}\right\| \quad (5)$$

The two metric is given by the equation (2.1)does not satisfy the parallelogram law. To show this, we can take for example

$$x = x_{mn} = \begin{pmatrix} 1/2 & 1/12 & 0, & 0 & ... \\ 0 & 0 & 0 & 0 & ... \\ . & & & & \\ . & & & & \\ . & & & & \end{pmatrix} \text{and } y = y_{mn} = \begin{pmatrix} 1/2 & -1/12 & 0 & 0 & ... \\ 0 & 0 & 0 & 0 & ... \\ . & & & & \\ . & & & & \\ . & & & & \end{pmatrix} \in$$

χ^2,
for all $m,n = 1,2,\cdots$, then we have $d(x_1,x_2) = 1 = d(y_1,y_2)$,
$d(x_1+y_1,x_2+y_2) = 1.414, d(x_1-y_1,x_2-y_2) = 1$. Hence, $[d(x_1+y_1,x_2+y_2)]^2 + d(x_1-y_1,x_2-y_2)^2 \neq 2\left[(d(x_1,x_2))^2 + (d(y_1,y_2))^2\right]$. Hence it is not satisfied by the law. Therefore χ^2 is not an inner product space.

2.1. **Proposition.** There is no constant $C>0$ such that $\|x-z\|^2 \geq C\|x_1-x_2\|^{r,s}$ for every $x,z \in \chi^2$.

Proof Let $z = z_{mn} = \begin{pmatrix} 1 & 0 & 0, & 0 & ... \\ 0 & 0 & 0 & 0 & ... \\ . & & & & \\ . & & & & \\ . & & & & \end{pmatrix}$, for each $r,s \in \mathbb{N}$, take $x^{[r,s]} = \left(x_{mn}^{[r,s]}\right) = \left(\frac{1}{(m+n)!m^{1/r}n^{1/s}}\right)$. Then we have $\left\|x^{[r,s]} - z\right\|^2 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left\|\begin{vmatrix} x_{11}^{[r,s]}k_{1i} & x_{12}k_{1i} \\ z_{21}^{[r,s]}k_{2j} & z_{22}k_{2j}\end{vmatrix}\right\|^2 = \sum_{m\neq1}\sum_{n\neq1}\left(\frac{1}{(m+n)!m^{1/r}n^{1/s}}\right)^2 = \sum_{m\neq1}\sum_{n\neq1}\left(\frac{1}{(m+n)!m^{2/r}n^{2/s}}\right)$
$\leq \sum_{m\neq1}\sum_{n\neq1}\left(\frac{1}{(m+n)!m^{2/r}n^{2/s}}\right) < \infty$
while $\left\|x^{[r,s]} - z\right\|^{r,s} \leq \sum_{m\neq1}\sum_{n\neq1}\left(\frac{1}{(m+n)!m^1n^1}\right) < \infty$ which tends to ∞ as $r,s\to\infty$. Hence $\frac{\left\|x^{[r,s]}-z\right\|^2}{\left\|x^{[r,s]}-z\right\|^{r,s}} \to 0$ as $r,s\to\infty$. So, there is no constant $C>0$ such that $\|x-z\|^2 \geq C\|x_1-x_2\|^{r,s}$ for every $x,z\in\chi^2$.

2.2. **Definition.** Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\Im_{ij}$ for all $m,n\in\mathbb{N}$; where $\Im_{ij}$ denotes the double sequence whose only non zero term is $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i,j\in\mathbb{N}$.
An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]}\to x$.

2.3. **Proposition:** χ^2 has AK.
Proof: Let $x = (x_{mn}) \in \chi^2$ and take $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\Im_{ij}$ for all $m,n\in\mathbb{N}$. Hence

$d\left(x, x^{[r,s]}\right) = inf\left\{sup_{mn}\left((m+n)!\left|x_{mn}\right|\right)^{1/m+n} : m \geq r+1, n \geq s+1\right\} \to 0$ as $m, n \to \infty$. Therefore, $x^{[r,s]} \to x$ as $r, s \to \infty$ in χ^2 Thus χ^2 has AK. This completes the proof.

2.4. **Proposition.** There is no constant $C = C_{uv} > 0$ such that $\|x - z\|^2 \leq C\|x_1 - x_2\|^{u,v}$ for every $x, z \in \chi^2$.

Proof: Let $z = z_{mn} = \begin{pmatrix} 1 & 0 & 0, & 0 & ... \\ 0 & 0 & 0 & 0 & ... \\ . & & & & \\ . & & & & \\ . & & & & \end{pmatrix}$. Suppose that such a constant exists.

Then, for $\Im_{ij}$ denotes the double sequence whose only non zero term is $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place $(i, j \geq 2)$ for each $i, j \in \mathbb{N}$, . we have

$$1 \leq C\left|u_1 v_1 u_m v_n\right|^2$$

for each $m, n \geq 2$. But this cannot be true, since $u_m v_n \to 0$ as $m, n \to \infty$.

Competing Interests: The authors declare that there is no conflict of interests regarding the publication of this research paper.

Acknowledgement

The authors are extremely grateful to the anonymous learned referee(s) for their keen reading, valuable suggestion and constructive comments for the improvement of the manuscript. The first author Deepmala is thankful to Science and Engineering Research Board (SERB), India for funding under National Post-Doctoral Fellowship Scheme reference no. PDF/2015/000799.

References

[1] B.C. Tripathy, On statistically convergent double sequences, Tamkang J. Math., 34(3), (2003), 231-237.
[2] M. Mursaleen, Almost strongly regular matrices and a core theorem for double sequences, J. Math. Anal. Appl., 293(2), (2004), 523-531.
[3] M. Mursaleen and O.H.H. Edely, Statistical convergence of double sequences, J. Math. Anal. Appl., 288(1), (2003), 223-231.
[4] M. Mursaleen and O.H.H. Edely, Almost convergence and a core theorem for double sequences, J. Math. Anal. Appl., 293(2), (2004), 532-540.
[5] N. Subramanian and U.K. Misra, The semi normed space defined by a double gai sequence of modulus function, Fasciculi Math., 46, (2010).
[6] F. Moricz and B.E. Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, Math. Proc. Camb. Phil. Soc., 104, (1988), 283-294.
[7] A. Pringsheim, Zurtheorie derzweifach unendlichen zahlenfolgen, Math. Ann., 53, (1900), 289-321.
[8] G.M. Robison, Divergent double sequences and series, Amer. Math. Soc. Trans., 28, (1926), 50-73.
[9] E. Savas and Richard F. Patterson, On some double almost lacunary sequence spaces defined by Orlicz functions, Filomat (Niš), 19, (2005), 35-44.
[10] Kuldip Raj and Sunil K. Sharma, Lacunary sequence spaces defined by a Musielak Orlicz function, Le Matematiche, Vol. LXVIII-Fasc. I, (2013), 33-51.
[11] Francesco Tulone, Regularity of some method of summation for double sequences , Le Matematiche, Vol. LXV-Fasc. II, (2010), 45-48.
[12] J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, *Israel J. Math.*, **10** (1971), 379-390.

[13] J. Musielak, Orlicz Spaces,Lectures Notes in Math., 1034, Springer-Verlag, (1983).
[14] Deepmala, N. Subramanian, L.N. Mishra; The Topological groups of Triple Almost Lacunary χ^3 sequence spaces defined by a Orlicz function, Electronic Journal of Mathematical Analysis and Applications, Vol. 4(2) July 2016, pp. 272- 280.
[15] Deepmala, N. Subramanian, V.N. Mishra, The Generalized difference of $\int \chi^{2I}$ of fuzzy real numbers over $p-$ metric spaces defined by Musielak Orlicz function, New Trends in Math. Sci., (2016), in press.
[16] V.N. Mishra, N. Subramanian, I.A. Khan, Riesz Triple Almost $\left(\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}\right)$ Lacunary $\chi^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}$ sequence spaces defined by a Orlicz function-III, Int. J. Math. Math. Sci. (2016), in press.
[17] Deepmala, N. Subramanian, V.N. Mishra, Double Almost $(\lambda_m\mu_n)$ in χ^2-Riesz space, Southeast Asian Bulletin of Mathematics, (2016), in press.

Deepmala,
SQC and OR Unit, Indian Statistical Institute, 203 B. T. Road, Kolkata-700 108, West Bengal, India
E-mail address: dmrai23@gmail.com, deepmaladm23@gmail.com

Lakshmi Narayan Mishra,
Department of Mathematics, National Institute of Technology, Silchar-788 010, Assam, India
E-mail address: lakshminarayanmishra04@gmail.com

N. Subramanian,
Department of Mathematics, SASTRA University,Thanjavur-613 401, India
E-mail address: nsmaths@yahoo.com