Electronic Journal of Mathematical Analysis and Applications
Vol. 5(1) Jan. 2017, pp. 122-134.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

A NEW SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY DERIVATIVE OPERATOR

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Abstract. In the present paper, we introduce a new subclass of harmonic univalent functions in the unit disc U by using Derivative operator. Also, we obtain coefficient conditions, distortion bounds, convolution conditions, convex combinations, extreme points and discuss a class preserving integral operator. Relevant connections of the results presented here with various known results are briefly indicated.

1. Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a complex domain D if both u and v are real harmonic in D. In any simply-connected domain $D \subset C$, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D, see [4].

Denote by $S_H(j)$, the class of functions $f = h + \overline{g}$ that are harmonic, univalent and sense-preserving in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$ with normalization $f(0) = h(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H(j)$, we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=j+1}^{\infty} a_k z^k, \ g(z) = \sum_{k=1}^{\infty} b_k z^k, \ |b_1| < 1. \tag{1.1}$$

For $j = 1$ the class $S_H(j)$ reduce to the class S_H of harmonic univalent functions in U and for $j = 1, g \equiv 0$ it reduce to the class S of normalized analytic univalent functions.

Al-Shaqsi and Darus [3] introduced the derivative operator for functions f of the form (1.1) as:

$$D_\lambda^n f(z) = D_\lambda^n h(z) + (-1)^n \overline{D_\lambda^n g(z)}, \ n, \lambda \in N_0 = N \cup \{0\}, \ z \in U, \tag{1.2}$$

where

2000 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Harmonic, univalent functions, Derivative operator.
Submitted April 10, 2015.

$D_\lambda^n h(z) = z + \sum_{k=j+1}^{\infty} k^n C(\lambda,k) a_k z^k$, $D_\lambda^n g(z) = \sum_{k=1}^{\infty} k^n C(\lambda,k) b_k z^k$ and $C(\lambda,k) = \binom{k+\lambda-1}{\lambda}$.

It is easy to see that for $\lambda = 0$ the operator D_λ^n reduce to the modified Salagean derivative operator introduced by Jahangiri *et al.* [6].

Now we introduce the class $G_H(n,\lambda,j,\alpha,\rho,t)$ of functions of the form (1.1) that satisfy the following condition

$$\Re\left\{(1+\rho e^{i\eta})\frac{D_\lambda^{n+1} f(z)}{D_\lambda^n f_t(z)} - \rho e^{i\eta}\right\} > \alpha, \tag{1.3}$$

where $0 \le \alpha < 1, \eta \in R, \rho \ge 0, j \in N, n, \lambda \in N_0, 0 \le t \le 1$, $f_t(z) = (1-t)z + tf(z)$ and $D_\lambda^n f(z)$ is defined by (1.2).

Let $\overline{G}_H(n,\lambda,j,\alpha,\rho,t)$ denote the subclass of $G_H(n,\lambda,j,\alpha,\rho,t)$ consisting of harmonic functions $f_n = h + \overline{g}_n$ such that h and g_n are of the form

$$h(z) = z - \sum_{k=j+1}^{\infty} |a_k| z^k,\ g_n(z) = (-1)^n \sum_{k=1}^{\infty} |b_k| z^k. \tag{1.4}$$

Assigning specific values to $n, \lambda, j, \alpha, \rho$ and t in the subclass $G_H(n,\lambda,j,\alpha,\rho,t)$, we obtain the following known subclasses studied earlier by various researchers.

(i) $G_H(n,\lambda,1,\alpha,\rho,1) \equiv G_H(n,\lambda,\alpha,\rho)$ studied by Pathak *et al.* [9].
(ii) $G_H(n,\lambda,1,\alpha,0,1) \equiv G_H(n,\lambda,\alpha)$ studied by Al-Shaqsi and Darus [3].
(iii) $G_H(n,0,1,\alpha,1,1) \equiv G_H(n,\alpha)$ studied by Yalcin *et al.*[13].
(iv) $G_H(0,\lambda,1,\alpha,0,1) \equiv G_H(\lambda,\alpha)$ studied by Murugusundaramoorthy and Vijya [8].
(v) $G_H(0,1,1,\alpha,1,1) \equiv G_H(\alpha)$ studied by Rosy *et al.* [10].
(vi) $G_H(0,0,1,\alpha,0,1) \equiv S_H^*(\alpha)$ studied by Jahangiri [5],(see also [11], [12]).
(vii) $G_H(1,0,1,\alpha,0,1) \equiv HK(\alpha)$ studied by Jahangiri [5].
(viii) $G_H(n,0,1,\alpha,0,1) \equiv M_H(n,\alpha)$ studied by Jahangiri *et al.* [6].
(ix) $G_H(1,0,1,\alpha,\rho,1) \equiv HUC(\rho,\alpha)$ studied by Kim *et al.* [7].
(x) $G_H(0,0,1,\alpha,\rho,1) \equiv HUS^*(\rho,\alpha)$ studied by Ahuja *et al.* [2].
(xi) $G_H(1,0,1,\alpha,0,0) \equiv N_H(\alpha)$ studied by Ahuja and Jahangiri [1].

In the present paper, we obtain coefficient condition, distortion bound, extreme points, convolution and convex combination. Finally we discuss a class preserving integral operator for this class.

2. Coefficient Bound

We begin with a sufficient coefficient condition for functions in $G_H(n,\lambda,j,\alpha,\rho,t)$.

Theorem 2.1. *Let $f = h + \overline{g}$ be given by (1.1). If*

$$\sum_{k=j+1}^{\infty} \{k(1+\rho) - t(\alpha+\rho)\} |a_k| k^n C(\lambda,k) + \sum_{k=1}^{\infty} \{k(1+\rho) + t(\alpha+\rho)\} |b_k| k^n C(\lambda,k) \le 1-\alpha, \tag{2.1}$$

where $n, \lambda \in N_0, j \in N, C(\lambda, k) = \binom{k+\lambda-1}{\lambda},\ \rho \geq 0,\ 0 \leq t \leq 1$ *and* $0 \leq \alpha < 1$, *then* f *is sense-preserving, harmonic univalent in* U *and* $f \in G_H(n, \lambda, j, \alpha, \rho, t)$.

Proof. If $z_1 \neq z_2$, then

$$\begin{aligned}
\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1 - \left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| \\
&= 1 - \left|\frac{\sum_{k=1}^{\infty} b_k(z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=j+1}^{\infty} a_k(z_1^k - z_2^k)}\right| \qquad (2.2) \\
&> 1 - \frac{\sum_{k=1}^{\infty} k|b_k|}{1 - \sum_{k=j+1}^{\infty} k|a_k|} \\
&\geq 1 - \frac{\sum_{k=1}^{\infty} \frac{[k(1+\rho)+t(\alpha+\rho)]k^n C(\lambda,k)|b_k|}{1-\alpha}}{1 - \sum_{k=j+1}^{\infty} \frac{[k(1+\rho)-t(\alpha+\rho)]k^n C(\lambda,k)|a_k|}{1-\alpha}} \\
&\geq 0
\end{aligned}$$

which proves univalence. Note that f is sense-preserving in U. This is because

$$\begin{aligned}
|h'(z)| &\geq 1 - \sum_{k=j+1}^{\infty} k|a_k||z|^{k-1} \\
&> 1 - \sum_{k=j+1}^{\infty} \frac{\{k(1+\rho)-t(\alpha+\rho)\}k^n C(\lambda,k)|a_k|}{1-\alpha} \\
&\geq \sum_{k=1}^{\infty} \frac{\{k(1+\rho)+t(\alpha+\rho)\}k^n C(\lambda,k)|b_k|}{1-\alpha} \qquad (2.3) \\
&> \sum_{k=1}^{\infty} \frac{\{k(1+\rho)+t(\alpha+\rho)\}k^n C(\lambda,k)|b_k||z|^{k-1}}{1-\alpha} \\
&\geq \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \geq |g'(z)|.
\end{aligned}$$

Using the fact that $\Re w > \alpha$, if and only if $|1-\alpha+w| \geq |1+\alpha-w|$ it suffices to show that

$$|(1-\alpha)+(1+\rho e^{i\eta})\frac{D_{\lambda}^{n+1}f(z)}{D_{\lambda}^{n}f_t(z)}-\rho e^{i\eta}| - |(1+\alpha)-(1+\rho e^{i\eta})\frac{D_{\lambda}^{n+1}f(z)}{D_{\lambda}^{n}f_t(z)}+\rho e^{i\eta}| \geq 0. \quad (2.4)$$

Substituting the values of $D_{\lambda}^{n+1}f(z)$ and $D_{\lambda}^{n}f_t(z)$ in (2.4), we obtain

$$\begin{aligned}
&|(1-\alpha-\rho e^{i\eta})D_{\lambda}^{n}f_t(z)+(1+\rho e^{i\eta})D_{\lambda}^{n+1}f(z)| - \\
&\quad |-(1+\alpha+\rho e^{i\eta})D_{\lambda}^{n}f_t(z)+(1+\rho e^{i\eta})D_{\lambda}^{n+1}f(z)| \\
&= |(2-\alpha)z+\sum_{k=j+1}^{\infty}\{k(1+\rho e^{i\eta})+t(1-\alpha-\rho e^{i\eta})\}k^nC(\lambda,k) \\
&\quad a_kz^k-(-1)^n\sum_{k=1}^{\infty}\{k(1+\rho e^{i\eta})-t(1-\alpha-\rho e^{i\eta})\}k^nC(\lambda,k)b_kz^k| \\
&\quad -|-\alpha z+\sum_{k=j+1}^{\infty}\{k(1+\rho e^{i\eta})-t(1+\alpha+\rho e^{i\eta})\}k^nC(\lambda,k)a_kz^k \\
&\quad -(-1)^n\sum_{k=1}^{\infty}\{k(1+\rho e^{i\eta})+t(1+\alpha+\rho e^{i\eta})\}k^nC(\lambda,k)\overline{b_kz^k}| \\
&\geq 2(1-\alpha)|z|\Big[1-\sum_{k=j+1}^{\infty}\frac{\{k(1+\rho)-t(\alpha+\rho)\}k^nC(\lambda,k)|a_k||z|^{k-1}}{1-\alpha} \\
&\quad -\sum_{k=1}^{\infty}\frac{\{k(1+\rho)+t(\alpha+\rho)\}k^nC(\lambda,k)|b_k||z|^{k-1}}{1-\alpha}\Big] \\
&> 2(1-\alpha)|z|\Big[1-\sum_{k=j+1}^{\infty}\frac{\{k(1+\rho)-t(\alpha+\rho)\}k^nC(\lambda,k)|a_k|}{1-\alpha} \\
&\quad -\sum_{k=1}^{\infty}\frac{\{k(1+\rho)+t(\alpha+\rho)\}k^nC(\lambda,k)|b_k|}{1-\alpha}\Big]. \quad (2.5)
\end{aligned}$$

This last expressions is non-negative by (2.1), and so the proof is completed.

The harmonic function

$$f(z) = z+\sum_{k=j+1}^{\infty}\frac{(1-\alpha)}{\{k(1+\rho)-t(\alpha+\rho)\}k^nC(\lambda,k)}x_kz^k + \sum_{k=1}^{\infty}\frac{(1-\alpha)}{\{k(1+\rho)+t(\alpha+\rho)\}k^nC(\lambda,k)}\overline{y_kz^k} \quad (2.6)$$

where $n, \lambda \in N_0$, $j \in N$, $0 \leq t \leq 1$, $\rho \geq 0, 0 \leq \alpha < 1$ and $\sum_{k=j+1}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$ shows that the coefficient bound given by (2.1) is sharp. The functions of the form (2.6) are in $G_H(n, \lambda, j, \alpha, \rho, t)$ because

$$\sum_{k=j+1}^{\infty} \frac{k(1+\rho) - t(\alpha+\rho)}{1-\alpha} k^n C(\lambda, k)|a_k| + \sum_{k=1}^{\infty} \frac{k(1+\rho) + t(\alpha+\rho)}{1-\alpha} k^n C(\lambda, k)|b_k|$$
$$= \sum_{k=j+1}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1. \quad (2.7)$$

In the following theorem, it is shown that the condition (2.1) is also necessary for functions $f_n = h + \overline{g}_n$, where h and g_n are of the form (1.4). □

Theorem 2.2. *Let $f_n = h + \overline{g}_n$ be given by (1.4). Then $f_n \in \overline{G}_H(n, \lambda, j, \alpha, \rho, t)$, if and only if*

$$\sum_{k=j+1}^{\infty} \frac{k(1+\rho) - t(\alpha+\rho)}{1-\alpha} k^n C(\lambda, k)|a_k| + \sum_{k=1}^{\infty} \frac{k(1+\rho) + t(\alpha+\rho)}{1-\alpha} k^n C(\lambda, k)|b_k| \leq 1, \quad (2.8)$$

where $n, \lambda \in N_0, j \in N$, $C(\lambda, k) = \binom{k+\lambda-1}{\lambda}, \rho \geq 0, 0 \leq \alpha < 1, 0 \leq t \leq 1$.

Proof. Since $\overline{G}_H(n, \lambda, j, \alpha, \rho, t) \subset G_H(n, \lambda, j, \alpha, \rho, t)$, we only need to prove the "only if" part of the theorem. To this end, for functions f_n of the form (1.4), we notice that the condition (1.3) is equivalent to

$$
\begin{aligned}
&\Re\left\{\left(1+\rho e^{i\eta}\right)\frac{D_{\lambda}^{n+1}f(z)}{D_{\lambda}^{n}f_t(z)}-\left(\rho e^{i\eta}+\alpha\right)\right\}\geq 0\\
&\Rightarrow \Re\frac{\left\{\left(1+\rho e^{i\eta}\right)D_{\lambda}^{n+1}f(z)-\left(\rho e^{i\eta}+\alpha\right)D_{\lambda}^{n}f_t(z)\right\}}{D_{\lambda}^{n}f_t(z)}\geq 0\\
&\Rightarrow \Re\left\{\frac{\begin{array}{c}\left(1+\rho e^{i\eta}\right)\left(z-\displaystyle\sum_{k=j+1}^{\infty}k^{n+1}C(\lambda,k)|a_k|z^k+(-1)^{2n+1}\sum_{k=1}^{\infty}k^{n+1}|b_k|C(\lambda,k)\overline{z}^k\right)\\
-\left(\rho e^{i\eta}+\alpha\right)\left(z-\displaystyle\sum_{k=j+1}^{\infty}k^{n}C(\lambda,k)t|a_k|z^k+(-1)^{2n}\sum_{k=1}^{\infty}k^{n}t|b_k|C(\lambda,k)\overline{z}^k\right)\end{array}}{z-\displaystyle\sum_{k=j+1}^{\infty}k^{n}C(\lambda,k)t|a_k|z^k+(-1)^{2n}\sum_{k=1}^{\infty}k^{n}C(\lambda,k)t|b_k|\overline{z}^k}\right\}\geq 0\\
&\Rightarrow \Re\left\{\frac{\begin{array}{c}(1-\alpha)z-\displaystyle\sum_{k=j+1}^{\infty}k^{n}[k\left(1+\rho e^{i\eta}\right)-t\left(\rho e^{i\eta}+\alpha\right)]C(\lambda,k)|a_k|z^k\\
+(-1)^{2n+1}\displaystyle\sum_{k=1}^{\infty}k^{n}[k\left(1+\rho e^{i\eta}\right)+t\left(\rho e^{i\eta}+\alpha\right)]C(\lambda,k)|b_k|\overline{z}^k\end{array}}{z-\displaystyle\sum_{k=j+1}^{\infty}k^{n}C(\lambda,k)t|a_k|z^k+(-1)^{2n}\sum_{k=1}^{\infty}k^{n}C(\lambda,k)t|b_k|\overline{z}^k}\right\}\geq 0\\
&\Rightarrow \Re\left\{\begin{array}{c}\dfrac{(1-\alpha)-\displaystyle\sum_{k=j+1}^{\infty}k^{n}[k\left(1+\rho e^{i\eta}\right)-t\left(\rho e^{i\eta}+\alpha\right)]C(\lambda,k)|a_k|z^{k-1}}{1-\displaystyle\sum_{k=j+1}^{\infty}k^{n}C(\lambda,k)t|a_k|z^{k-1}+\frac{\overline{z}}{z}(-1)^{2n}\sum_{k=1}^{\infty}k^{n}C(\lambda,k)t|b_k|\overline{z}^{k-1}}\\
-\dfrac{\frac{\overline{z}}{z}(-1)^{2n}\displaystyle\sum_{k=1}^{\infty}k^{n}[k\left(1+\rho e^{i\eta}\right)+t\left(\rho e^{i\eta}+\alpha\right)]C(\lambda,k)|b_k|\overline{z}^{k-1}}{1-\displaystyle\sum_{k=j+1}^{\infty}k^{n}C(\lambda,k)t|a_k|z^{k-1}+\frac{\overline{z}}{z}(-1)^{2n}\sum_{k=1}^{\infty}k^{n}C(\lambda,k)t|b_k|\overline{z}^{k-1}}\end{array}\right\}\geq 0
\end{aligned}
\tag{2.9}
$$

The above condition (2.9) must hold for all values of z on the positive real axes, where, $0 \leq |z| = r < 1$, we must have

$$\Re\left\{\begin{array}{c}\dfrac{(1-\alpha)-\sum\limits_{k=j+1}^{\infty}k^n(k-t\alpha)C(\lambda,k)|a_k|r^{k-1}-(-1)^{2n}\sum\limits_{k=1}^{\infty}k^n(k+t\alpha)C(\lambda,k)|b_k|r^{k-1}}{1-\sum\limits_{k=j+1}^{\infty}k^nC(\lambda,k)t|a_k|r^{k-1}+(-1)^{2n}\sum\limits_{k=1}^{\infty}k^nC(\lambda,k)t|b_k|r^{k-1}}\\ -\rho e^{i\eta}\dfrac{\sum\limits_{k=j+1}^{\infty}k^n(k-t)C(\lambda,k)t|a_k|r^{k-1}+(-1)^{2n}\sum\limits_{k=1}^{\infty}k^n(k+t)C(\lambda,k)t|b_k|r^{k-1}}{1-\sum\limits_{k=j+1}^{\infty}k^nC(\lambda,k)t|a_k|r^{k-1}+(-1)^{2n}\sum\limits_{k=1}^{\infty}k^nC(\lambda,k)t|b_k|r^{k-1}}\end{array}\right\}\geq 0$$

Since $\Re(-e^{i\eta}) \geq -|e^{i\eta}| = -1$, the above inequality reduce to

$$\left\{(1-\alpha)-\sum_{k=j+1}^{\infty}k^n\{(k(1+\rho)-t(\rho+\alpha))\}C(\lambda,k)|a_k|r^{k-1}-\sum_{k=1}^{\infty}k^n\{(k(1+\rho)+t(\rho+\alpha)\}C(\lambda,k)|b_k|r^{k-1}\right\}.$$

$$\left\{1-\sum_{k=j+1}^{\infty}k^nC(\lambda,k)t|a_k|r^{k-1}+\sum_{k=1}^{\infty}k^nC(\lambda,k)t|b_k|r^{k-1}\right\}^{-1}\geq 0. \tag{2.10}$$

If the condition (2.8) does not hold, then the numerator in (2.10) is negative for r sufficiently close to 1. Hence there exist a, $z_0 = r_0$ in (0,1) for which the quotient in (2.10) is negative. This contradicts the condition for $f_n \in \overline{G}_H(n,\lambda,j,\alpha,\rho,t)$ and so the proof is complete. □

3. Distortion Bounds

In this section, we will obtain distortion bounds for functions in $\overline{G}_H(n,\lambda,j,\alpha,\rho,t)$.

Theorem 3.1. *Let $f_n \in \overline{G}_H(n,\lambda,j,\alpha,\rho,t)$. Then for $|z| = r < 1$, we have*

$$|f_n(z)| \leq (1+|b_1|+|b_2|r+\ldots+|b_j|r^{j-1})r+\frac{1-\alpha}{((j+1)(1+\rho)-t(\rho+\alpha))(j+1)^nC(\lambda,j+1)}$$
$$\left[1-\frac{(1+(t+1)\rho+\alpha)}{1-\alpha}|b_1|-\frac{(2(1+\rho)+t(\rho+\alpha))}{1-\alpha}|b_2|2^nC(\lambda,2)-\ldots-\frac{(j(1+\rho)+t(\rho+\alpha))}{1-\alpha}|b_j|j^nC(\lambda,j)\right]r^{(j+1)}$$
$$|f_n(z)| \geq (1-|b_1|-|b_2|r-\ldots+|b_j|r^{j-1})r-\frac{1-\alpha}{((j+1)(1+\rho)-t(\rho+\alpha))(j+1)^nC(\lambda,j+1)}$$
$$\left[1-\frac{(1+(t+1)\rho+\alpha)}{1-\alpha}|b_1|-\frac{(2(1+\rho)+t(\rho+\alpha))}{1-\alpha}|b_2|2^nC(\lambda,2)-\ldots-\frac{(j(1+\rho)+t(\rho+\alpha))}{1-\alpha}|b_j|j^nC(\lambda,j)\right]r^{(j+1)}$$

Proof. We only prove the left-hand inequality. The proof for the right-hand inequality is similar and is thus omitted. Let $f_n \in \overline{G}_H(n,\lambda,j,\alpha,\rho,t)$. Taking the absolute value of f_n, we obtain

$$|f_n(z)| = \left| z - \sum_{k=j+1}^{\infty} a_k z^k + (-1)^n \sum_{k=1}^{\infty} b_k \overline{z}^k \right|$$

$$\leq (1 + |b_1| + |b_2|r + \ldots + |b_j|r^{j-1})r + \sum_{k=j+1}^{\infty} (|a_k| + |b_k|)r^k$$

$$\leq (1 + |b_1| + |b_2|r + \ldots + |b_j|r^{j-1})r + \sum_{k=j+1}^{\infty} (|a_k| + |b_k|)r^{j+1}$$

$$\leq (1 + |b_1| + |b_2|r + \ldots + |b_j|r^{j-1})r + \frac{1-\alpha}{\left((j+1)(1+\rho) - t(\rho+\alpha)\right)(j+1)^n C(\lambda, j+1)}$$
$$\sum_{k=j+1}^{\infty} \left(\frac{\left((j+1)(1+\rho) - t(\rho+\alpha)\right)(j+1)^n C(\lambda, j+1)}{1-\alpha} |a_k| + \frac{\left((j+1)(1+\rho) - t(\rho+\alpha)\right)(j+1)^n C(\lambda, j+1)}{1-\alpha} |b_k| \right) r^{(j+1)}$$

$$\leq (1 + |b_1| + |b_2|r + \ldots + |b_j|r^{j-1})r + \frac{1-\alpha}{\left((j+1)(1+\rho) - t(\rho+\alpha)\right)(j+1)^n C(\lambda, j+1)}$$
$$\sum_{k=j+1}^{\infty} \left[\frac{(k(1+\rho) - t(\rho+\alpha))k^n C(\lambda,k)}{1-\alpha} |a_k| + \frac{(k(1+\rho) + t(\rho+\alpha))k^n C(\lambda,k)}{1-\alpha} |b_k| \right] r^{(j+1)}$$

$$\leq (1 + |b_1| + |b_2|r + \ldots + |b_j|r^{j-1})r + \frac{1-\alpha}{\left((j+1)(1+\rho) - t(\rho+\alpha)\right)(j+1)^n C(\lambda, j+1)}$$

$$\left[1 - \frac{(1+(t+1)\rho+\alpha)}{1-\alpha}|b_1| - \frac{(2(1+\rho)+t(\rho+\alpha))}{1-\alpha}|b_2|2^n C(\lambda,2) - \ldots - \frac{(j(1+\rho)+t(\rho+\alpha))}{1-\alpha}|b_j|j^n C(\lambda,j) \right] r^{(j+1)}$$

□

4. Convolution, Convex Combination and Extreme Points

In this section, we show that the class $\overline{G}_H(n,\lambda,j,\alpha,\rho,t)$ is invariant under convolution and convex combination.

For harmonic functions

$$f_n(z) = z - \sum_{k=j+1}^{\infty} |a_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k| \overline{z}^k$$

and

$$F_n(z) = z - \sum_{k=j+1}^{\infty} |A_k| z^k + (-1)^n \sum_{k=1}^{\infty} |B_k| \overline{z}^k,$$

the convolution of f_n and F_n is given by

$$(f_n * F_n)(z) = f_n(z) * F_n(z) = z - \sum_{k=j+1}^{\infty} |a_k A_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k B_k| \overline{z}^k. \tag{4.1}$$

Theorem 4.1. *For* $0 \leq \beta \leq \alpha < 1$, $n, \lambda \in N_0, j \in N$, $\rho \geq 0$, $0 \leq t \leq 1$ *let* $f_n \in \overline{G}_H(n, \lambda, j, \alpha, \rho, t)$ *and* $F_n \in \overline{G}_H(n, \lambda, j, \beta, \rho, t)$. *Then* $f_n * F_n \in \overline{G}_H(n, \lambda, j, \alpha, \rho, t) \subset \overline{G}_H(n, \lambda, j, \beta, \rho, t)$.

Proof. We wish to show that the coefficient of $f_n * F_n$ satisfy the required condition given in Theorem 2.2. For $F_n \in \overline{G}_H(n, \lambda, j, \beta, \rho, t)$, we note that $|A_k| \leq 1$ and $|B_k| \leq 1$. Now, for the convolution function $f_n * F_n$, we obtain

$$\begin{aligned}
&\sum_{k=j+1}^{\infty} \frac{\{k(1+\rho) - t(\alpha+\rho)\} k^n C(\lambda, k)}{1-\alpha} |a_k||A_k| \\
&\quad + \sum_{k=1}^{\infty} \frac{\{k(1+\rho) + t(\alpha+\rho)\} k^n C(\lambda, k)}{1-\alpha} |b_k||B_k| \\
&\leq \sum_{k=j+1}^{\infty} \frac{\{k(1+\rho) - t(\alpha+\rho)\} k^n C(\lambda, k)}{1-\alpha} |a_k| \\
&\quad + \sum_{k=1}^{\infty} \frac{\{k(1+\rho) + t(\alpha+\rho)\} k^n C(\lambda, k)}{1-\alpha} |b_k| \\
&\leq \sum_{k=j+1}^{\infty} \frac{\{k(1+\rho) - t(\alpha+\rho)\} k^n C(\lambda, k)}{1-\alpha} |a_k| \\
&\quad + \sum_{k=1}^{\infty} \frac{\{k(1+\rho) + t(\alpha+\rho)\} k^n C(\lambda, k)}{1-\alpha} |b_k| \\
&\leq 1.
\end{aligned}$$

Since $0 \leq \beta \leq \alpha < 1$ and $f_n \in \overline{G}_H(n, \lambda, j, \alpha, \rho, t)$. Therefore $f_n * F_n \in \overline{G}_H(n, \lambda, j, \alpha, \rho, t) \subset \overline{G}_H(n, \lambda, j, \beta, \rho, t)$.

We now examine the convex combination of $\overline{G}_H(n, \lambda, j, \alpha, \rho, t)$.

Let the functions $f_{n_i}(z)$ be defined, for $i = 1, 2, \ldots, m$, by

$$f_{n_i}(z) = z - \sum_{k=j+1}^{\infty} |a_{k,i}| z^k + (-1)^n \sum_{k=1}^{\infty} |b_{k,i}| \overline{z}^k. \tag{4.2}$$

□

Theorem 4.2. *Let the functions $f_{n_i}(z)$ defined by (4.2) be in the class $\overline{G}_H(n,\lambda,j,\alpha,\rho,t)$ for every $i=1,2,\dots,m$. Then the functions $t_i(z)$ defined by*

$$t_i(z)=\sum_{i=1}^{m} c_i f_{n_i}(z),\ 0\le c_i\le 1 \tag{4.3}$$

are also in the class $\overline{G}_H(n,\lambda,j,\alpha,\rho,t)$, where $\sum_{i=1}^{m} c_i=1$.

Proof. According to the definition of t_i, we can write

$$t_i(z)=z-\sum_{k=j+1}^{\infty}\left(\sum_{i=1}^{m} c_i|a_{k,i}|\right)z^k+(-1)^n\sum_{k=1}^{\infty}\left(\sum_{i=1}^{m} c_i|b_{k,i}|\right)\overline{z}^k.$$

Further, since $f_{n_i}(z)$ are in $\overline{G}_H(n,\lambda,j,\alpha,\rho,t)$ for every $i=1,2,\dots,m$, then

$$\sum_{k=j+1}^{\infty}\Bigg[(k(1+\rho)-t(\alpha+\rho))\left(\sum_{i=1}^{m} c_i|a_{k,i}|\right)$$
$$+\sum_{k=1}^{\infty}(k(1+\rho)+t(\alpha+\rho))\left(\sum_{i=1}^{m} c_i|b_{k,i}|\right)\Bigg]k^nC(\lambda,k)$$
$$=\sum_{i=1}^{m} c_i\left(\sum_{k=j+1}^{\infty}(k(1+\rho)-t(\alpha+\rho))k^nC(\lambda,k)|a_{k,i}|+\sum_{k=1}^{\infty}(k(1+\rho)+t(\alpha+\rho))|b_{k,i}|k^nC(\lambda,k)\right)$$
$$\le\sum_{i=1}^{m} c_i(1-\alpha)\le(1-\alpha).$$

Hence the Theorem 4.2 follows. □

Next we determine the extreme points of closed convex hulls of $\overline{G}_H(n,\lambda,j,\alpha,\rho,t)$ denoted by clco $\overline{G}_H(n,\lambda,j,\alpha,\rho,t)$.

Theorem 4.3. *Let f_n be given by (1.4). Then $f_n\in\overline{G}_H(n,\lambda,j,\alpha,\rho,t)$, if and only if*

$$f_n(z)=\sum_{k=j}^{\infty}X_kh_k(z)+\sum_{k=1}^{\infty}Y_kg_{n_k}(z), \tag{4.4}$$

where

$$h_j(z)=z,\ h_k(z)=z-\left(\frac{1-\alpha}{(k(1+\rho)-t(\alpha+\rho))k^nC(\lambda,k)}\right)z^k,\ k=j+1,j+2,\dots.$$

$$g_{n_k}(z)=z+(-1)^n\left(\frac{1-\alpha}{(k(1+\rho)+t(\alpha+\rho))k^nC(\lambda,k)}\right)\overline{z}^k,\ k=1,2,3,\dots.$$

and $\sum_{k=j}^{\infty}X_k+\sum_{k=1}^{\infty}Y_k=1,\ X_k\ge 0,\ Y_k\ge 0$. In particular, the extreme points of $\overline{G}_H(n,\lambda,j,\alpha,\rho,t)$ are $\{h_k\}$ and $\{g_{n_k}\}$.

Proof. For the function f_n of the form (4.4) we have

$$f_n(z) = \sum_{k=j}^{\infty} X_k h_k(z) + \sum_{k=1}^{\infty} Y_k g_{n_k}(z)$$

$$= z - \sum_{k=j+1}^{\infty} \frac{1-\alpha}{(k(1+\rho)-t(\alpha+\rho))k^n C(\lambda,k)} X_k z^k + (-1)^n \sum_{k=1}^{\infty} \frac{1-\alpha}{(k(1+\rho)+t(\alpha+\rho))k^n C(\lambda,k)} Y_k \overline{z}^k$$

Then

$$\sum_{k=j+1}^{\infty} \frac{(k(1+\rho)-t(\alpha+\rho))k^n C(\lambda,k)}{1-\alpha}|a_k| + \sum_{k=1}^{\infty} \frac{(k(1+\rho)+t(\alpha+\rho))k^n C(\lambda,k)}{1-\alpha}|b_k|$$

$$= \sum_{k=j+1}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k$$

$$= 1 - X_j \leq 1$$

and so $f_n \in \text{clco}\, \overline{G}_H(n,\lambda,j,\alpha,\rho,t)$.

Conversely, suppose that $f_n \in \text{clco}\, \overline{G}_H(n,\lambda,j,\alpha,\rho,t)$. Setting

$$X_k = \frac{(k(1+\rho)-t(\alpha+\rho))k^n C(\lambda,k)}{1-\alpha}|a_k|,\ 0 \leq X_k \leq 1\ k = j+1, j+2\ldots$$
$$Y_k = \frac{(k(1+\rho)+t(\alpha+\rho))k^n C(\lambda,k)}{1-\alpha}|b_k|,\ 0 \leq Y_k \leq 1\ k = 1,2,3\ldots \tag{4.5}$$

and $X_j = 1 - \sum_{k=j+1}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k$. Therefore, f_n can be written as

$$f_n(z) = z - \sum_{k=j+1}^{\infty} |a_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k| \overline{z}^k$$

$$= z - \sum_{k=j+1}^{\infty} \frac{(1-\alpha)X_k}{(k(1+\rho)-t(\alpha+\rho))k^n C(\lambda,k)} z^k$$

$$+ (-1)^n \sum_{k=1}^{\infty} \frac{(1-\alpha)Y_k}{(k(1+\rho)+t(\alpha+\rho))k^n C(\lambda,k)} \overline{z}^k$$

$$= z + \sum_{k=j+1}^{\infty} (h_k(z)-z)X_k + \sum_{k=1}^{\infty} (g_{n_k}(z)-z)Y_k$$

$$= \sum_{k=j+1}^{\infty} h_k(z)X_k + \sum_{k=1}^{\infty} g_{n_k}(z)Y_k + z\left(1 - \sum_{k=j+1}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k\right)$$

$$= \sum_{k=j}^{\infty} X_k h_k(z) + \sum_{k=1}^{\infty} Y_k g_{n_k}(z),\ \text{as required.} \tag{4.6}$$

This completes the proof of Theorem 4.3. □

5. A Family of Class Preserving Integral Operator

Let $f(z) = h(z) + \overline{g(z)}$ be defined by (1.1) then $F(z)$ defined by the relation

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}h(t)dt + \overline{\frac{c+1}{z^c}\int_o^z t^{c-1}g(t)dt}, \ (c > -1). \tag{5.1}$$

Theorem 5.1. *Let $f_n(z) = h(z) + \overline{g_n(z)} \in S_H$ be given by (1.4) and $f_n(z) \in \overline{G}_H(n, \lambda, j, \alpha, \rho, t)$ then $F(z)$ be defined by (5.1) also belong to $\overline{G}_H(n, \lambda, j, \alpha, \rho, t)$.*

Proof. From the representation of (5.1) of $F(z)$, it follows that

$$F(z) = z - \sum_{k=j+1}^{\infty} \frac{c+1}{c+k}|a_k|z^k + (-1)^n \sum_{k=1}^{\infty} \frac{c+1}{c+k}|b_k|\overline{z}^k. \tag{5.2}$$

Since $f_n(z) \in \overline{G}_H(n, \lambda, j, \alpha, \rho, t)$, then by Theorem 2.2 we have

$$\sum_{k=j+1}^{\infty} \frac{(k(1+\rho) - t(\alpha+\rho))k^n C(\lambda, k)}{1-\alpha}|a_k| + \sum_{k=1}^{\infty} \frac{(k(1+\rho) + t(\alpha+\rho))k^n C(\lambda, k)}{1-\alpha}|b_k| \leq 1.$$

Now

$$\sum_{k=2}^{\infty} \frac{(k(1+\rho) - t(\alpha+\rho))k^n C(\lambda, k)}{1-\alpha}\frac{c+1}{c+k}|a_k| + \sum_{k=1}^{\infty} \frac{(k(1+\rho) + t(\alpha+\rho))k^n C(\lambda, k)}{1-\alpha}\frac{c+1}{c+k}|b_k|$$

$$\leq \sum_{k=2}^{\infty} \frac{(k(1+\rho) - t(\alpha+\rho))k^n C(\lambda, k)}{1-\alpha}|a_k| + \sum_{k=1}^{\infty} \frac{(k(1+\rho) + t(\alpha+\rho))k^n C(\lambda, k)}{1-\alpha}|b_k|$$

$$\leq 1.$$

Thus $F(z) \in \overline{G}_H(n, \lambda, j, \alpha, \rho, t)$. □

Acknowledgement

The authors are thankful to the referee for his/her valuable comments and observations which helped in improving the paper.

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