CHAOTIC WEIGHTED COMPOSITION OPERATOR

S.PANAYAPPAN, K.KAVITHA

ABSTRACT. Hypercyclic weighted composition operator on $H(\Omega)$ of all holomorphic functions on an arbitrary domain Ω in C is studied. Examples are given. Further, equivalent conditions for a weighted composition operator to be chaotic on $H(\Omega)$ of all holomorphic functions on U, the open disc in C are established.

1. INTRODUCTION

Dynamical systems is a branch of mathematics concerned with time evolutions of natural and iterative processes. Some dynamical systems are predictable whereas others are not. One of the remarkable discoveries of the 20th century mathematicians is that simple systems, even systems depending on only one variable may behave unpredictably - the reason for unpredictable behaviour has been called 'Chaos'. Chaos can be an obstacle in some cases (like limiting the predictability of weather) and an asset in others (such as industrial mixing applications)

The study of operators that generate chaotic semigroups (dynamical systems) has attracted many researchers right from 1929. Birkoff [2] studied the hypercyclicity of translation operators on the Frechet space of entire functions. This result was generalised by Godefroy and Shapiro [4] to examine the chaotic translation operators.

Translation operators are a special case of composition operators. So, a study of chaotic composition operators not only enriches the theory of chaos but also has applications in chaotic dynamical systems.

Let Ω be an arbitrary domain in C, that is, a non-empty connected open set. Consider the space $H(\Omega)$ of all holomorphic functions on Ω, which is a separable Frechet space. An automorphism of Ω is a bijective holomorphic function $\Psi : \Omega \to \Omega$. Its inverse is then also holomorphic.

A weighted composition operator $C_{\Phi,\Psi}$ defined on $H(\Omega)$ maps $f \in H(\Omega)$ into $C_{\Phi,\Psi}(f(z)) = \Phi(z)f(\Psi(z))$ where Ψ is an automorphism of Ω, $\Phi \in H(\Omega)$ and $z \in \Omega$. When $\Phi \equiv 1$, $C_{\Phi,\Psi} = M_\Phi C_\Psi$ becomes the composition operator $C_\Psi f = f \circ \Psi$ for

every $f \in H(\Omega)$. The class of weighted composition operators includes two important classes : composition operators and multiplication operators. For more details on weighted composition operators refer to [3, 7].

There has been substantial interest from differential geometry and dynamical system in chaotic operators whose iterations generate dense subsets and periodic points form dense subsets. In particular, a study of chaotic nature of weighted composition operators entails a study of the iterate behaviour of holomorphic self maps.

Rezaei [6] studied chaotic weighted composition operators, Yousefi and Rezaei [9] studied hypercyclic weighted composition operators on the space $H(U)$ of holomorphic functions on U, the open disc in C. In this article we study weighted composition operators on the space $H(\Omega)$ of holomorphic functions on Ω, an arbitrary domain in C and on the space $H(U)$.

2. **Hypercyclic weighted composition operators on** $H(\Omega)$

We begin with basic definitions An operator T on a Frechet space X is called hypercyclic if there is some $x \in X$, whose orbit under T is dense in X. In such a case x is called a hypercyclic vector for T.

We now consider the problem of determining which weighted composition operators are hypercyclic.

Bernel and Montes [1] coined the term 'runaway sequence' and proved that runaway property is a necessary condition for the hypercyclicity of composition operators. Let Ω be a domain in C and $\Phi_n : \Omega \to \Omega$, $n \geq 1$, holomorphic maps. Then the sequence $(\Phi_n)_n$ is called a run-away sequence if, for any compact subset $K \subset \Omega$, there is some $n \in N$ such that $\Phi_n(K) \cap K = \phi$. The following examples are well-known. Let $\Omega = C$. Then the automorphic function of C are the functions $\Phi(z) = az + b$, $a \neq 0$, $b \in C$ and $(\Phi^n)_n$ is runaway if and only if $a = 1$, $b \neq 0$. Let $\Omega = C^* = C/\{0\}$ be the punctured plane. The automorphism of C^* are the functions $\Phi(z) = az$ or $\Phi(z) = \frac{a}{z}$, $a \neq 0$ and $(\Phi^n)_n$ is runaway if and only if $\Phi(z) = az$ with $|a| \neq 1$.

In [1], it is shown that the run-away property is a necessary condition for the hypercyclicity of the composition operator. Now we extend this result to weighted composition operators. Let Ω be a domain in C and Ψ be an automorphism of Ω. Further let Φ be a bounded holomorphic map on Ω. If $C_{\Phi,\Psi}$ is hypercyclic then $(\Psi^n)_n$ is a run-away sequence. In particular, if λC_Ψ is hypercyclic then $(\Psi^n)_n$ is a run-away sequence.

Proof. Let $(\Psi^n)_n$ be not run-away sequence. Then there exists a compact set $K \subset \Omega$ and elements $z_n \in K$ such that $\Psi^n(z_n) \in K$, $n \in N$. Assume $f \in H(\Omega)$ be hypercyclic vector for C_Ψ and let $M_1 = Sup_{z \in K}|f(z)|$ and $|\Phi(z)| < M_2$ for $z \in K$. Then

$inf_{z \in K}|((C_{\Phi,\Psi})^n f)(z)| \leq |((C_{\Phi,\Psi})^n f)(z_n)|$

$= |(\Pi_{i=0}^{n-1}(\Phi \circ \psi^i)(f \circ \Psi^n)(z_n)|$

$= |(\Pi_{i=0}^{n-1}(\Phi(\Psi^i(z_n))(f(\Psi^n(z_n))|$

$< M_1 M_2^n = M$, say, so that the function $(C_{\Phi,\Psi})^n f$ cannot approximate uniformly on K, a contradiction. $\square$

Using example 2 and theorem 2 we have
$C_{\Phi,\Psi}$ can be hypercyclic on $H(C^*)$ if $\Phi(z) = az$ with $|a| \neq 1$.

3. Chaotic weighted composition operators on $H(U)$

In this section, we relate hypercyclic $C_{\Phi,\Psi}$ with chaotic $C_{\Phi,\Psi}$. An operator T is said to be chaotic if it satisfies the following conditions
(i) T is hypercyclic.
(ii) T has dense set of periodic points. Consider two continuous linear operators T and S on the seperable Frechet spaces X and Y respectively. S is called a quasifactor of T if there exists a continuous linear operator $V : X \to Y$ which has dense range and satisfies $VT = SV$.
Clearly if T is chaotic on X, then S is chaotic on Y. Let U be the open disc in C, Ψ be a holomorphic selfmap on U and $\Phi \in H(U)$. Then $C_{\Phi,\Psi}$ is hypercyclic if and only if $C_{\Phi,\Psi}$ is chaotic, assuming the existence of non zero eigen values for $C_{\Phi,\Psi}$.

Proof. By proposition 2.1 of [9], if $C_{\Phi,\Psi}$ is hypercyclic then Ψ has no fixed point in U. This, then shows that λC_Ψ is chaotic if, $\lambda \neq 0$ by Theorem 2.2 of [5].
Let λ be a non zero eigen value of $C_{\Phi,\Psi}$. Then there is a non zero holomorphic function g with $\Phi.g \circ \Psi = \lambda g$ and so
$C_{\Phi,\Psi} M_g = M_\Phi C_\Psi M_g$
$= M_\Phi M_{g \circ \Psi} C_\Psi = \lambda M_g C_\Psi = M_g(\lambda C_\Psi)$
If λ is unimodular and g has no zero on U then M_g is one-to-one and has dense range. Then $M_g(\lambda C_\Psi) = C_{\Phi,\Psi} M_g$ shows that $C_{\Phi,\Psi}$ is a quasi factor of λC_Ψ. Since λC_Ψ is chaotic, $C_{\Phi,\Psi}$ is also chaotic. $\square$
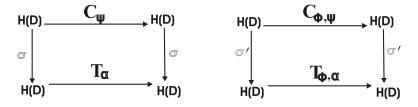
Let T and S be two continuous linear operators on the separable Frechet spaces X and Y respectively. T is called quasi conjugate to S if there exists a continuous map $\sigma : Y \to X$ with dense range such that $T \circ \sigma = \sigma \circ S$, that is, the diagram

commutes. On the space $H(D)$, the translation operator T_a given by
$T_a f(z) = f(z + a)$, $0 \neq a \in C$ is called Birkoff operator. For $\Phi \in H(D)$, weighted Birkoff operator is given by
$T_{\Phi,a} f(z) = \Phi(z) f(z+a)$ C_Ψ is quasi conjugate to a Birkoff operator T_a if and only if $C_{\Phi,\Psi}$ is quasi conjugate to the weighted Birkoff operator $T_{\Phi,a}$ where $\Phi \in H(D)$ has dense range.

Proof. Let C_Ψ be quasi conjugate to T_a. Then there exists $\sigma : H(D) \to H(D)$, a continuous map with dense range such that $T_a \circ \sigma = \sigma \circ C_\Psi$.

Define $\acute{\sigma} : H(D) \to H(D)$ by $\acute{\sigma}(\Phi(z)f(\Psi(z))) = \Phi(z)\sigma(f(\Psi(z)))$ for every $\Phi \in H(D)$ with dense range. Then $\acute{\sigma}$ is continuous with dense range. It is easy to verify that $T_a \circ \sigma = \sigma \circ C_\Psi$ if and only if $T_{\Phi,a} \circ \acute{\sigma} = \acute{\sigma} \circ C_{\Phi,\Psi}$ as desired.
Using the results from [1] and the results obtained in this article, we state.   □

Let U be the open disc in C, Ψ be a holomorphic self-map on U and $\Phi \in H(U)$ be bounded with dense range. Assume $C_{\Phi,\Psi}$ has non zero eigen value. Then the following are equivalent.
(i) $C_{\Phi,\Psi}$ is hypercyclic.
(ii) $C_{\Phi,\Psi}$ is chaotic.
(iii) $(\Psi^n)_n$ is a run-away sequence.
(iv) Ψ has no fixed point in U.
(v) $C_{\Phi,\Psi}$ is quasiconjugate to a weighted Birkoff operator $T_{\Phi,a}$. Remark : The last condition of the theorem shows that any property of a weighted Birkoff operator that is preserved under quasi conjugancies will transmit to all weighted composition operators.
Rezaei [5] has shown that if $C^*_{\Phi,\Psi}$ is hypercyclic on X^*, then M^*_Φ is also hypercyclic on X^*. We now extend this to chaotic case.

Let Φ be a non constant bounded holomorphic function on U and let M^*_Φ be the corresponding adjoint multiplier on H^2. If $C^*_{\Phi,\Psi}$ is chaotic, then M^*_Φ is also chaotic.

Proof. If $C^*_{\Phi,\Psi}$ is chaotic, then it is hypercyclic. Then by [5], M^*_Φ is also hypercyclic. By Godefroy-Shapiro criterion[4], M^*_Φ is chaotic.   □

REFERENCES

1. L.Bernal-Gonzalez and A.Montes-Rodriguez, Universal function for composition operators, Complex variables theory Appl, 27, 47-56, 1995.
2. G.D.Birkhoff, Demonstration d'un theoreme elementaire surles fonctions entieres, C.R.Acad.Sci. Paris, 189, 473-475, 1929 .
3. C.C.Cowen and B.D.Maccluer, Composition Operators On spaces Of Analytic Functions, CRC Press, Boca Raton, 1995.
4. G.Godefroy and J.H.Shapiro, Operators with dense invariant, cyclic vector manifolds, J.Func Anal.98, 229-269, 1991
5. H.Rezaei, Notes on dynamics of the adjoint of weighted composition operators, Taiwanese Journal of mathematics vol 14, No 4, 1377-1384, 2010
6. H.Rezaei, Chaotic property of weighted composition operators, Bull. Korean.Math Soc 48, 1119-1124, 2011.
7. J.M.Shapiro, Composition Operators And Classical Function Theory, Springer Verlag, Newyork, 1983.
8. J.M.Shapiro, Notes On Dynamics Of Linear Operators, http://www.mth.msu.edu/shapiro, 2001.

9. B.Yousefi and H.Rezaei, Hypercyclic property of weighted composition operators, Proc.Amer.Math Soc 135, 10, 3263-3271, 2007.

S.Panayappan

Post Graduate and Research Department of Mathematics, Government Arts College (Autonomous), Coimbatore-641 018, Tamil Nadu, India

E-mail address: panayappan@gmail.com