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A NEW SEQUENCE REALIZING LUCAS NUMBERS, AND THE LUCAS BOUND

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Abstract. We define a set $L(n)$ of vectors with positive integral entries. We show that the cardinality of $L(n)$ is the *nth* Lucas number L_n, for $n \geq 1$. We then show that the number $l(n)$ of M-sequences of length n is bounded by the Lucas number L_n, for $n \geq 1$. This is an analogue of similar statements with Fibonacci numbers F_n instead of Lucas numbers.

1. Introduction

First, we recall the well-known definitions of the Fibonacci and Lucas numbers.

The Fibonacci number F_n is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, n \geq 2,$$

with seed values $F_0 = 0$ and $F_1 = 1$.

The Lucas number L_n is defined by the recurrence relation

$$L_n = L_{n-1} + L_{n-2}, n \geq 2,$$

with seed values $L_0 = 2$ and $L_1 = 1$.

In recent years, there has been much interest in applications of Fibonacci and Lucas numbers. They appear in many branches of mathematics. These include group theory, calculus, applied mathematics, linear algebra, etc. Also, these numbers have many important applications to diverse fields such as computer science, physics, biology, and statistics. We can see applications of the Fibonacci sequence in group theory in [4,6] and also see some generalized Fibonacci and Lucas sequences in [3,5].

In this work, we give an application of Lucas numbers and M-sequences. We recall some definitions.

A *multiset* is a finite sequence of elements that may contain repeated elements. The order of the elements is irrelevant, but their multiplicities are part of the structure. So, a multiset S with elements in $\mathbb{N}$ can be uniquely written in the form $S = \{b_1, b_2, \ldots, b_{k-1}, b_k\}_{\leq}$, where $b_i \in \mathbb{N}$, $k \geq 0$, and the subscript "$\leq$" indicates that we have arranged the elements in weakly increasing order: $b_1 \leq b_2 \leq \ldots \leq b_k$. The *dimension* of a multiset S is the number of elements in S, counting

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multiplicities. So the multiset S above has dimension $|S| = k$, and we call it a k-*multiset*. If $k = 0$ then S is empty and $|S| = 0$.

A *multicomplex* is a finite collection of multisets that is closed under taking submultisets. The *dimension* of a multicomplex is the greatest dimension of a multiset it contains. The *m-vector* of a multicomplex is $(m_0, m_1, m_2, \ldots, m_d)$, where m_{i} is the number of multisets of dimension i in the multicomplex. We have $m_0 = 1$.

Let M be a collection of monomials $x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}, i_j \geq 0$, such that if a is a monomial in M, and b is a monomial which divides a, then b is in M. Then M is a multicomplex where the mutisets are encoded as monomials. The dimension of the multiset is the degree of the monomial it defines.

A sequence is called an M-*sequence* if it is the *m-vector* of the multicomplex M, where m_i is the number of monomials of degree i in M. Rather than bounding the degree and number of variables, we fix the *length* $\lambda(m) := \sum_i m_i$. This is the number of the elements in the multicomplex M.

Following [2], for $n \geq 1$, let $l(n)$ denote the number of M-sequences of length n. In fact, the authors of [2] define the set

$$M(n) = \{m = (m_0, m_1, \ldots) | \, m \textit{ is an } M\textit{-sequence and } \lambda(m) = n\},$$

and set $l(n) = |M(n)|$ for its cardinality. The sets $M(1)$, $M(2)$, $\ldots$, are mutually disjoint.

Using Macaulay's Theorem, the authors of [1, Theorem 4.2.10] constructed all possible M-sequences of a given length. In Table 1, we list the M-sequences of length at most 7, from which we deduce that the first few terms of the sequence $\{l(n)\}_{n \geq 1}$ of [2] are the numbers 1, 1, 2, 3, 5, 8, 12. They are bounded by, but not equal to, the Fibonacci numbers.

λ	1	2	3	4	5	6	7
	1	11	111	1111	11111	111111	1111111
			12	121	1211	12111	121111
				13	122	1221	12211
					131	123	1231
					14	1311	1222
						132	13111
						141	1321
						15	133
							1411
							142
							151
							16
Total	1	1	2	3	5	8	12

Table 1. The M-sequences of length at most 7 obtained by using Definition 2.1. We write $t_0 t_1 t_2 \ldots t_s$ for the M-sequence $(t_0, t_1, \ldots, t_s)$. E.g., 1221 is the M-sequence (1, 2, 2, 1). For example, when $\lambda = 4$, the m-vector of $M = \{x_1^3, x_1^2, x_1, 1\}$ is $(1, 1, 1, 1)$, of $M = \{x_1^2, x_1, x_2, 1\}$ is $(1, 2, 1)$, of $M = \{x_1, x_2, x_3, 1\}$ is $(1, 3)$.

Let us recall the Fibonacci bound of [2] before we define the Lucas bound.

2. Fibonacci bound

Definition 2.1. For $n \geq 1$, define recursively the Fibonacci set $F(n)$ of vectors with entries in $\mathbb{N}$ as follows: F(1) ={(1)}; F(2) ={(1,1)};

For $n \geq 3$ define $F(n) := C(n) \cup D(n)$ where (see [2])

$$C(n) := \{(1, t_1, ..., t_s, 1) \mid (1, t_1, ..., t_s) \in F(n-1)\},$$

$$D(n) := \left\{\left(1, t_1, ..., t_{s-1}, t_s + 1\right) \middle| (1, t_1, ..., t_s) \in F(n-1), \text{with } t_{s-1} > 1 \text{ or } s = 1\right\}.$$

The sets $C(n)$, $D(n)$ are disjoint for each n. The sets $F(1)$, $F(2)$, ..., are mutually disjoint.

For example,

$$\begin{aligned}
F(3) &= \{(1,1,1),(1,2)\};\\
F(4) &= \{(1,1,1,1),(1,2,1),(1,3)\};\\
F(5) &= \{(1,1,1,1,1),(1,2,1,1),(1,3,1),(1,2,2),(1,4)\};\\
F(6) &= \{(1,1,1,1,1,1),(1,2,1,1,1),(1,3,1,1),(1,2,2,1)(1,4,1),(1,3,2),(1,2,3),\\
&\quad (1,5)\};\\
F(7) &= \{(1,1,1,1,1,1,1),(1,2,1,1,1,1),(1,3,1,1,1),(1,2,2,1,1),(1,4,1,1),\\
&\quad (1,3,2,1),(1,2,3,1),(1,5,1),(1,2,2,2),(1,4,2),(1,3,3),(1,2,4),(1,6)\};\\
F(8) &= \{(1,1,1,1,1,1,1,1),(1,2,1,1,1,1,1),(1,3,1,1,1,1),(1,2,2,1,1,1),(1,4,1,1,1),\\
&\quad (1,3,2,1,1),(1,2,3,1,1),\ (1,5,1,1),(1,2,2,2,1),(1,4,2,1),(1,3,3,1),(1,2,4,1),\\
&\quad (1,6,1),(1,3,2,2),(1,2,3,2),(1,5,2),(1,2,2,3),(1,4,3),(1,3,4),(1,2,5),(1,7)\}.
\end{aligned}$$

Theorem 2.2. For $n \geq 1$, the cardinality of $F(n)$ is equal to the nth Fibonacci number F_n (see [2], Lemma 2.3).

Theorem 2.3. For all $n \geq 1$, $M(n) \subseteq F(n)$. In particular, the sequence $l(n)$ is bounded above by the Fibonacci sequence $l(n) \leq F_n$ (see [2], Theorem 2.4).

3. Lucas bound

Definition 3.1. For $n \geq 1$, define recursively the Lucas set $L(n)$ of vectors with entries in $\mathbb{N}$ as follows:

(1) $L(1) = \{(1)\};; L(2) = \{(1,1,1), (1), (1,2)\}$.
(2) For $n \geq 3$ define $L(n) := C(n) \cup D(n)$ where

$$C(n) := \{(1, t_1, ..., t_s, 1) \mid (1, t_1, ..., t_s) \in L(n-1)\}$$

$$D(n) := \left\{\left(1, t_1, ..., t_{s-1}, t_s + 1\right) \middle| (1, t_1, ..., t_s) \in L(n-1), \text{ with } t_{s-1} > 1 \text{ or } s = 1\right\}.$$

Remark 3.2. The sets $C(n)$ and $D(n)$ of Definition 3.1 form a disjoint set partition of $L(n)$. The sets $L(1)$, $L(2)$, ..., are mutually disjoint.

The first few sets $L(n)$ are

$L(1) = \{(1)\};$

$L(2) = \{(1,1,1),(1),(1,2)\};$

$L(3) = \{(1,1,1,1),(1,1),(1,2,1),(1,3)\};$

$L(4) = \{(1,1,1,1,1),(1,1,1),(1,2,1,1),(1,2,2),(1,3,1),(1,2),(1,4)\};$

$L(5) = \{(1,1,1,1,1,1),(1,1,1,1),(1,2,1,1,1),(1,2,2,1),(1,3,1,1),(1,2,1),(1,4,1),$
$(1,2,3),(1,3,2),(1,3),(1,5)\};$

$L(6) = \{(1,1,1,1,1,1,1),(1,1,1,1,1),(1,2,1,1,1,1),(1,2,2,1,1),(1,3,1,1,1),(1,2,1,1),$
$(1,4,1,1),(1,2,3,1),(1,3,2,1),(1,3,1),(1,5,1),(1,2,2,2),(1,2,2),(1,4,2),(1,2,4),$
$(1,3,3),(1,4),(1,6)\};$

$L(7) = \{(1,1,1,1,1,1,1,1),(1,1,1,1,1,1),(1,2,1,1,1,1,1),(1,2,2,1,1,1),(1,3,1,1,1,1),$
$(1,2,1,1,1),(1,4,1,1,1),(1,2,3,1,1),(1,3,2,1,1),(1,3,1,1),(1,5,1,1),(1,2,2,2,1),$
$(1,2,2,1),(1,4,2,1),(1,2,4,1),(1,3,3,1),(1,4,1),(1,6,1),(1,2,3,2),(1,3,2,2),$
$(1,3,2),(1,5,2),(1,2,2,3),(1,2,3),(1,4,3),(1,2,5),(1,3,4),(1,5),(1,7)\}.$

In Table 2, we find the M-sequences of length at most 5 and the first few terms of the sequence.

λ	1	2	3	4	5
	1	111	1111	11111	111111
		1	11	111	1111
		12	121	1211	12111
			13	131	1221
				122	1311
				14	123
				12	132
					141
					121
					13
					15
Total	1	3	4	7	11

Table 2. The M-sequences of length at most 5 obtained by using Definition 3.1. We write $t_0t_1t_2\dots\ t_s$ for the M-sequence $(t_0,\ t_1,\dots,\ t_s)$.

Theorem 3.3. The cardinality of $L(n)$ is the $n\ th$ Lucas number L_n, $n \geq 1$.
Proof. We prove this by induction on n. We have $L_n = L_{n-1} + L_{n-2}$, $n \geq 2$, $L_0 = 2$ and $L_1 = 1$. Hence

$$L_0 = 2\ ,\quad L_1 = 1\ , L_2 = 3\ , L_3 = 4\ , L_4 = 7, \cdots .$$

For $n = 1$, the claim is true, since $|L(1)| = L_1 = 1$.

For $n = 2$, the claim is true, since $|L(2)| = L_2 = 3$.

Suppose the claim is true for all $r < k$, that is $|L(r)| = L_r$ for all $r < k$.

We have to prove that the claim is true for $n = k$, that is, $|L(k)| = L_k$.

We have $|L(k-2)| = L_{k-2}$ and $|L(k-1)| = L_{k-1}$.

From these equalities we have

$$|L(k-2)| + |L(k-1)| = L_{k-2}\ + L_{k-1} =\ L_k.$$

Hence

$$|L(k)| = L_k.$$

□

Theorem 3.4. For all $n \geq 2$, $M(n+1) \subseteq L(n)$. In particular, the sequence $l(n+1)$ is bounded from above by the Lucas sequence.

Proof. By the construction one can check that $L(n)$ is the set of all integer vectors $(1, t_1, ..., t_s)$ with $1 + t_1 + t_2 + \cdots + t_s = n+1$ and the property that if $t_i = 1$ then $t_j = 1$ for all $j \geq i$.

Let $m = (1, m_1, ..., m_s) \in M(n+1)$ be an M- sequence of length n .

Using Macaulay's Theorem it is not hard to see that if $m_i = 1$ for some $i \geq 1$, then $m_j = 1$ for all $j \geq i$. We prove this by induction on n. We have

$$M(n+1) \subseteq L(n), n \geq 2,$$

For $n = 2$, the claim is true, since $M(3) \subseteq L(2)$.

$M(3) = \{(1,1,1), (1,2)\}$ and $L(2) = \{(1,1,1), (1), (1,2)\}$.

For $n = 3$, the claim is true, since $M(4) \subseteq L(3)$.

$M(4) = \{(1,1,1,1), (1,2,1), (1,3)\}$ and $L(3) = \{(1,1,1,1), (1,1), (1,2,1), (1,3)\}$.

By induction we assume that $M(k) \subseteq L(k-1)$ and $M(k+1) \subseteq L(k)$.

We have to prove that the claim is true for $n = k+1$, that is, $M(k+2) \subseteq L(k+1)$.

Let $s(S)$ denote the number of elements of a set S.

We have $M(k) \subseteq L(k-1) \Rightarrow s(M(k)) \leq s(L(k-1))$,

$$M(k+1) \subseteq L(k) \Rightarrow s(M(k+1)) \leq s(L(k)),$$

$$s(M(k) \cup M(k+1)) = s(M(k)) + s(M(k+1)) - s(M(k) \cap M(k+1)),$$

$$s(M(k) \cap M(k+1)) = 0,$$

hence

$$s(M(k) \cup M(k+1)) = s(M(k)) + s(M(k+1)).$$

From these equalities we have

$$M(k) \cup M(k+1) \subseteq L(k-1) \cup L(k).$$

In the same way, we have

$$s(L(k-1) \cup L(k)) = s(L(k-1)) + s(L(k)) - s(L(k-1) \cap L(k)),$$

$$s(L(k-1) \cap L(k)) = 0,$$

$$s(L(k-1) \cup L(k)) = s(L(k-1)) + s(L(k))$$

and

$$s(M(k)) + s(M(k+1)) \leq s(L(k-1)) + s(L(k)).$$

Hence

$$s(M(k)) + s(M(k+1)) = s(M(k+2)), s(L(k-1)) + s(L(k)) = s(L(k+1)).$$

We know

$$s(M(k+2)) \leq s(L(k+1)).$$

Hence

$$M(k+2) \subseteq L(k+1).$$

Thus $M(n+1) \subseteq L(n)$. □

Theorem 3.5. For all $n \geq 2$, we have the relation

$$L(n) \setminus M(n+1) = M(n-1).$$

Proof. We prove this by induction on n.

For $n = 2$, the claim is true, since $L(2) \backslash M(3) = M(1)$.

For $n = 3$, the claim is true, since $L(3) \backslash M(4) = M(2)$.

Suppose the claim is true for some $n = k \geq 3$, that is $L(k) \backslash M(k+1) = M(k-1)$.

We have to prove that the claim is true for $n = k+1$, that is,

$$L(k+1) \backslash M(k+2) = M(k).$$

The identity $L(k) \backslash M(k+1) = M(k-1)$ implies $L(k) = M(k-1) \cup M(k+1)$.

From these equalities, we can easily see that

$$L(k+1) \backslash M(k+2) = M(k).$$

□

Corollary 3.6. For all $n \geq 2$, we have $|L(n)| - |F(n+1)| = |F(n-1)|$.

Proof. We prove that by induction on n.

For $n = 2$, the claim $|L(2)| - |F(3)| = |F(1)|$ is true, since $|L(2)| = 3$, $|F(3)| = 2$ and $|F(1)| = 1$.

Suppose the claim is true for all $n \leq k$ for some $k \geq 2$, that is

$$|L(n)| - |F(n+1)| = |F(n-1)|.$$

We have to prove that the claim is true for $n = k+1$, that is

$$|L(k+1)| - |F(k+2)| = |F(k)|.$$

Thus by induction we have

$$|L(k-1)| = |F(k-2)| + |F(k)|$$

$$|L(k)| = |F(k-1)| + |F(k+1)|$$

From these equalities, we have

$$|L(k-1)| + |L(k)| = |F(k-2)| + |F(k-1)| + |F(k)| + |F(k+1)|.$$

From the last equality,

$$|L(k+1)| - |F(k+2)| = |F(k)|$$

since $|L(k-1)| + |L(k)| = |L(k+1)|$, $|F(k-2)| + |F(k-1)| = |F(k)|$ and

$$|F(k)| + |F(k+1)| = |F(k+2)|.$$

□

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