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ON THE GROWTH PROPERTIES OF COMPOSITE FUNCTIONS ANALYTIC IN THE UNIT DISC FROM THE VIEW POINT OF THEIR RELATIVE L^*- TYPES AND RELATIVE L^*-WEAK TYPES

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Abstract. In this paper we introduce the idea of relative Nevanlinna L^* -type and relative Nevanlinna L^* -weak type in the Unit disc $U = \{z : |z| < 1\}$. Hence we study some comparative growth properties of composition of two analytic function in the unit disc U on the basis of relative Nevanlinna L^* -type and relative Nevanlinna L^*-weak type.

1. Introduction

A function f, analytic in the unit disc $U = \{z : |z| < 1\}$, is said to be of finite Nevanlinna order [10] if there exist a number μ such that Nevanlinna characteristic function

$$T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta$$

satisfies $T(r, f) < (1-r)^{-\mu}$ for all r in $0 < r_0(\mu) < r < 1$. The greatest lower bound of all such numbers μ is called the Nevanlinna order of f. Thus the Nevanlinna order ρ_f of f is given by

$$\rho_f = \limsup_{r\to 1} \frac{\log T(r, f)}{-\log(1-r)} .$$

Similarly, the Nevanlinna lower order λ_f of f is given by

$$\lambda_f = \liminf_{r\to 1} \frac{\log T(r, f)}{-\log(1-r)} .$$

Datta et. al. [4] introduced the notion of Nevanlinna L-order for an analytic function f in the unit disc $U = \{z : |z| < 1\}$ where $L = L\left(\frac{1}{1-r}\right)$ is a positive continuous function in the unit disc U increasing slowly i.e., $L\left(\frac{a}{1-r}\right) \sim L\left(\frac{1}{1-r}\right)$ as $r \to 1$, for every positive constant 'a', in the following manner:

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Definition 1. *If f be analytic in U, then the Nevanlinna L-order ρ_f^L and the Nevanlinna L-lower order λ_f^L of f are defined as*

$$\rho_f^L = \frac{\log T(r,f)}{\log\left(\frac{L\left(\frac{1}{1-r}\right)}{(1-r)}\right)} \quad \textit{and} \quad \lambda_f^L = \liminf_{r\to 1} \frac{\log T(r,f)}{\log\left(\frac{L\left(\frac{1}{1-r}\right)}{(1-r)}\right)} .$$

Now we introduce the concepts of relative Nevanlinna L^*-order and relative Nevanlinna L^*-lower order of an analytic function f with respect to another analytic function g in the unit disc U which are as follows:

Definition 2. *If f be analytic in U and g be entire, then the relative Nevanlinna L^*-order of f with respect to g, denoted by $\rho_g^{L^*}(f)$ is defined by*

$$\rho_g^{L^*}(f) = \inf\left\{\mu > 0 : T_f(r) < T_g\left[\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right]^{\mu} \textit{ for all } 0 < r_0(\mu) < r < 1\right\} .$$

Similarly, the relative Nevanlinna L^*-order of f with respect to g, denoted by $\lambda_g^{L^*}(f)$ is given by

$$\lambda_g^{L^*}(f) = \liminf_{r\to 1} \frac{\log T_g^{-1} T_f(r)}{\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)} .$$

When $g(z) = \exp z$, the definition coincides with the definition of the Nevanlinna L^*-order and the Nevanlinna L^*-lower order.

To compare the relative growth of two analytic functions having same non zero finite relative Nevanlinna L^*-order with respect to another entire function, one may introduce the definitions of relative Nevanlinna L^*-type and relative Nevanlinna L^*-lower type of analytic functions with respect to an entire function in the following manner:

Definition 3. *The relative Nevanlinna L^*-type and relative Nevanlinna L^*-lower type denoted respectively by $\sigma_g^{L^*}(f)$ and $\overline{\sigma}_g^{L^*}(f)$ of an analytic function f with respect to an entire function g are respectively defined as follows:*

$$\begin{aligned}
\sigma_g^{L^*}(f) &= \limsup_{r\to 1} \frac{T_g^{-1} T_f(r)}{\left[\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right]^{\rho_g^{L^*}(f)}} \textit{ and} \\
\overline{\sigma}_g^{L^*}(f) &= \liminf_{r\to 1} \frac{T_g^{-1} T_f(r)}{\left[\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right]^{\rho_g^{L^*}(f)}},\ 0 < \rho_g^{L^*}(f) < \infty .
\end{aligned}$$

Analogusly to determine the relative growth of two analytic functions having same non zero finite relative Nevanlinna L^*-lower order with respect to another entire function one may introduce the definition of relative Nevanlinna L^*-weak type of an analytic functions having finite positive relative Nevanlinna L^*-lower order with respect to an entire function in the following way:

Definition 4. *The relative Nevanlinna L^*-weak type denoted by $\tau_g^{L^*}(f)$ of an analytic function f with respect to an entire function g is defined as follows:*

$$\tau_g^{L^*}(f) = \liminf_{r\to 1} \frac{T_g^{-1}T_f(r)}{\left[\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right]^{\lambda_g^{L^*}(f)}}, \ 0 < \ \lambda_g^{L^*}(f) \ < \infty .$$

Also one may define the growth indicator $\overline{\tau}_g^{L^*}(f)$ of an analytic function f in the following manner :

$$\overline{\tau}_g^{L^*}(f) = \limsup_{r\to 1} \frac{T_g^{-1}T_f(r)}{\left[\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right]^{\lambda_g^{L^*}(f)}}, \ 0 < \ \lambda_g^{L^*}(f) \ < \infty .$$

For analytic functions, the notions of the growth indicators such as *Nevanlinna order* and *Nevanlinna type* are classical in complex analysis and during the past decades, several researchers have already been exploring their studies in the area of comparative growth properties of analytic functions in different directions using the classical growth indicators. But at that time, the concepts of *relative Nevanlinna order, relative Nevanlinna type* and *relative Nevanlinna weak type* of analytic functions are not at all known to the researchers of this area. Therefore the studies of the growths of analytic functions in the light of their *relative Nevanlinna order (relative Nevanlinna L^*-order) relative Nevanlinna type (relative Nevanlinna L^*-type) and relative weak type (relative Nevanlinna L^*-weak)* are the prime concern of this paper. In fact some light has already been thrown on such type of works in [1], [2], [5], [6], [7] , [8] and [9]. Actually in this paper we establish some new results depending on the comparative growth properties of composite analytic function in the unit disc $U = \{z : |z| < 1\}$ using *relative Nevanlinna L^*-order, relative Nevanlinna L^*- type* and *relative Nevanlinna L^*-weak type* as compared to the corresponding left and right factors. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [11].

2. Theorems.

In this section we present the main results of the paper.

Theorem 1. *If f, g be any two analytic functions and h, k be any two entire functions such that $0 < \overline{\sigma}_h^{L^*}(f\circ g) \le \sigma_h^{L^*}(f\circ g) < \infty$, $0 < \overline{\sigma}_k^{L^*}(f) \le \sigma_k^{L^*}(f) < \infty$ and $\rho_h^{L^*}(f\circ g) = \rho_k^{L^*}(f)$, then*

$$\frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(f)} \le \liminf_{r\to 1} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \le \frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\overline{\sigma}_k^{L^*}(f)}$$

$$\le \limsup_{r\to 1} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \le \frac{\sigma_h^{L^*}(f\circ g)}{\overline{\sigma}_k^{L^*}(f)} .$$

Proof. From the definition of $\sigma_k^{L^*}(f)$ and $\overline{\sigma}_h^{L^*}(f\circ g)$, we have for arbitrary positive ε and for all sufficiently large values of $\left(\frac{1}{1-r}\right)$ that

$$T_h^{-1}T_{f\circ g}(r) \ge \left(\overline{\sigma}_h^{L^*}(f\circ g) - \varepsilon\right) \left[\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right]^{\rho_h^{L^*}(f\circ g)}, \tag{1}$$

and

$$T_k^{-1}T_f(r) \leq \left(\sigma_k^{L^*}(f)+\varepsilon\right)\left[\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right]^{\rho_k^{L^*}(f)} \tag{2}$$

fNow from (1), (2) and the condition $\rho_h^{L^*}(f\circ g)=\rho_k^{L^*}(f)$, it follows for all sufficiently large values of $\left(\frac{1}{1-r}\right)$ that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \geq \frac{\overline{\sigma}_h^{L^*}(f\circ g)-\varepsilon}{\sigma_k^{L^*}(f)+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary , we obtain from above that

$$\liminf_{r\to 1}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \geq \frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(f)}\ . \tag{3}$$

Again for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity,

$$T_h^{-1}T_{f\circ g}(r) \leq \left(\overline{\sigma}_h^{L^*}(f\circ g)+\varepsilon\right)\left[\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right]^{\rho_h^{L^*}(f\circ g)} \tag{4}$$

and for all sufficiently large values of $\left(\frac{1}{1-r}\right)$,

$$T_k^{-1}T_f(r) \geq \left(\overline{\sigma}_k^{L^*}(f)-\varepsilon\right)\left[\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right]^{\rho_k^{L^*}(f)}\ . \tag{5}$$

Combining (4) and (5) and the condition $\rho_h^{L^*}(f\circ g)=\rho_k^{L^*}(f)$, we get for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \leq \frac{\overline{\sigma}_h^{L^*}(f\circ g)+\varepsilon}{\overline{\sigma}_k^{L^*}(f)-\varepsilon}\ .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to 1}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \leq \frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\overline{\sigma}_k^{L^*}(f)}\ . \tag{6}$$

Also for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity, it follows that

$$T_k^{-1}T_f(r) \leq \left(\overline{\sigma}_k^{L^*}(f)+\varepsilon\right)\left[\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right]^{\rho_k^{L^*}(f)}\ . \tag{7}$$

Now from (1), (7) and the condition $\rho_h^{L^*}(f\circ g)=\rho_k^{L^*}(f)$, we obtain for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \geq \frac{\overline{\sigma}_h^{L^*}(f\circ g)-\varepsilon}{\overline{\sigma}_k^{L^*}(f)+\varepsilon}\ .$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to 1}\frac{T_h^{-1}T_{f\circ g}\left(r\right)}{T_k^{-1}T_f\left(r\right)}\geq\frac{\overline{\sigma}_h^{L^*}\left(f\circ g\right)}{\overline{\sigma}_k^{L^*}\left(f\right)}\ . \tag{8}$$

Also for all sufficiently large values of $\left(\frac{1}{1-r}\right)$,

$$T_h^{-1}T_{f\circ g}\left(r\right)\leq\left(\sigma_h^{L^*}\left(f\circ g\right)+\varepsilon\right)\left[\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right]^{\rho_h^{L^*}(f\circ g)}\ . \tag{9}$$

In view of the condition $\rho_h^{L^*}\left(f\circ g\right)=\rho_k^{L^*}\left(f\right)$, it follows from (5) and (9) for all sufficiently large values of $\left(\frac{1}{1-r}\right)$ that

$$\frac{T_h^{-1}T_{f\circ g}\left(r\right)}{T_k^{-1}T_f\left(r\right)}\leq\frac{\sigma_h^{L^*}\left(f\circ g\right)+\varepsilon}{\overline{\sigma}_k^{L^*}\left(f\right)-\varepsilon}\ .$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to 1}\frac{T_h^{-1}T_{f\circ g}\left(r\right)}{T_k^{-1}T_f\left(r\right)}\leq\frac{\sigma_h^{L^*}\left(f\circ g\right)}{\overline{\sigma}_k^{L^*}\left(f\right)}\ . \tag{10}$$

Thus the theorem follows from (3), (6), (8) and (10). □

The following theorem can be proved in the line of Theorem 1 and so its proof is omitted.

Theorem 2. *If f, g be any two analytic functions and h,k be any two entire functions with $0<\overline{\sigma}_h^{L^*}\left(f\circ g\right)\leq\sigma_h^{L^*}\left(f\circ g\right)<\infty$, $0<\overline{\sigma}_k^{L^*}\left(g\right)\leq\sigma_k^{L^*}\left(g\right)<\infty$ and $\rho_h^{L^*}\left(f\circ g\right)=\rho_k^{L^*}\left(g\right)$, then*

$$\frac{\overline{\sigma}_h^{L^*}\left(f\circ g\right)}{\sigma_k^{L^*}\left(g\right)}\leq\liminf_{r\to 1}\frac{T_h^{-1}T_{f\circ g}\left(r\right)}{T_k^{-1}T_g\left(r\right)}\leq\frac{\overline{\sigma}_h^{L^*}\left(f\circ g\right)}{\overline{\sigma}_k^{L^*}\left(g\right)}$$
$$\leq\limsup_{r\to 1}\frac{T_h^{-1}T_{f\circ g}\left(r\right)}{T_k^{-1}T_g\left(r\right)}\leq\frac{\sigma_h^{L^*}\left(f\circ g\right)}{\overline{\sigma}_k^{L^*}\left(g\right)}\ .$$

Theorem 3. *If f, g be any two analytic functions and h,k be any two entire functions such that $0<\sigma_h^{L^*}\left(f\circ g\right)<\infty$, $0<\sigma_k^{L^*}\left(f\right)<\infty$ and $\rho_h^{L^*}\left(f\circ g\right)=\rho_k^{L^*}\left(f\right)$, then*

$$\liminf_{r\to 1}\frac{T_h^{-1}T_{f\circ g}\left(r\right)}{T_k^{-1}T_f\left(r\right)}\leq\frac{\sigma_h^{L^*}\left(f\circ g\right)}{\sigma_k^{L^*}\left(f\right)}\leq\limsup_{r\to 1}\frac{T_h^{-1}T_{f\circ g}\left(r\right)}{T_k^{-1}T_f\left(r\right)}\ .$$

Proof. From the definition of $\sigma_k^{L^*}\left(f\right)s$ we get for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$T_k^{-1}T_f\left(r\right)\geq\left(\sigma_k^{L^*}\left(f\right)-\varepsilon\right)\left[\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right]^{\rho_k^{L^*}(f)} \tag{11}$$

fNow from (9), (11) and the condition $\rho_h^{L^*}(f\circ g)=\rho_k^{L^*}(f)$, it follows for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}\leq\frac{\sigma_h^{L^*}(f\circ g)+\varepsilon}{\sigma_k^{L^*}(f)-\varepsilon}\ .$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to 1}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}\leq\frac{\sigma_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(f)}\ . \tag{12}$$

Again for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$T_h^{-1}T_{f\circ g}(r)\geq\left(\sigma_h^{L^*}(f\circ g)-\varepsilon\right)\left[\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right]^{\rho_h^{L^*}(f\circ g)}\ . \tag{13}$$

So combining (2) and (13) and in view of the condition $\rho_h^{L^*}(f\circ g)=\rho_k^{L^*}(f)$, we get for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}\geq\frac{\sigma_h^{L^*}(f\circ g)-\varepsilon}{\sigma_k^{L^*}(f)+\varepsilon}\ .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to 1}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}\geq\frac{\sigma_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(f)}\ . \tag{14}$$

Thus the theorem follows from (12) and (14). □

The following theorem can be carried out in the line of Theorem 3 and therefore we omit its proof.

Theorem 4. *If f, g be any two analytic functions and h,k be any two entire functions with $0<\sigma_h^{L^*}(f\circ g)<\infty$, $0<\sigma_k^{L^*}(g)<\infty$ and $\rho_h^{L^*}(f\circ g)=\rho_k^{L^*}(g)$, then*

$$\liminf_{r\to 1}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)}\leq\frac{\sigma_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(g)}\leq\limsup_{r\to 1}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)}\ .$$

The following theorem is a natural consequence of Theorem 1 and Theorem 3:

Theorem 5. *If f, g be any two analytic functions and h,k be any two entire functions such that $0<\overline{\sigma}_h^{L^*}(f\circ g)\leq\sigma_h^{L^*}(f\circ g)<\infty$, $0<\overline{\sigma}_k^{L^*}(f)\leq\sigma_k^{L^*}(f)<\infty$ and $\rho_h^{L^*}(f\circ g)=\rho_k^{L^*}(f)$, then*

$$\begin{aligned}\liminf_{r\to 1}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}&\leq\min\left\{\frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\overline{\sigma}_k^{L^*}(f)},\frac{\sigma_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(f)}\right\}\\&\leq\max\left\{\frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\overline{\sigma}_k^{L^*}(f)},\frac{\sigma_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(f)}\right\}\leq\limsup_{r\to 1}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}\ .\end{aligned}$$

Analogously one may state the following theorem without its proof.

Theorem 6. *If f, g be any two analytic functions and h, k be any two entire functions with $0 < \overline{\sigma}_h^{L^*}(f \circ g) \leq \sigma_h^{L^*}(f \circ g) < \infty$, $0 < \overline{\sigma}_k^{L^*}(g) \leq \sigma_k^{L^*}(g) < \infty$ and $\rho_h^{L^*}(f \circ g) = \rho_k^{L^*}(g)$, then*

$$\begin{aligned}\liminf_{r\to 1} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)} &\leq \min\left\{\frac{\overline{\sigma}_h^{L^*}(f \circ g)}{\overline{\sigma}_k^{L^*}(g)}, \frac{\sigma_h^{L^*}(f \circ g)}{\sigma_k^{L^*}(g)}\right\} \\ &\leq \max\left\{\frac{\overline{\sigma}_h^{L^*}(f \circ g)}{\overline{\sigma}_k^{L^*}(g)}, \frac{\sigma_h^{L^*}(f \circ g)}{\sigma_k^{L^*}(g)}\right\} \leq \limsup_{r\to 1} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)} .\end{aligned}$$

Now in the line of Theorem 1, Theorem 3, Theorem 5 and Theorem 2, Theorem 4, Theorem 6 respectively one can easily prove the following six theorems using the notion of relative Nevanlinna L^*-weak type of a meromorphic function with respect to an entire function and therefore their proofs are omitted.

Theorem 7. *If f, g be any two analytic functions and h, k be any two entire functions such that $0 < \tau_h^{L^*}(f \circ g) \leq \overline{\tau}_h^{L^*}(f \circ g) < \infty$, $0 < \tau_k^{L^*}(f) \leq \overline{\tau}_k^{L^*}(f) < \infty$ and $\lambda_h^{L^*}(f \circ g) = \lambda_k^{L^*}(f)$, then*

$$\begin{aligned}\frac{\tau_h^{L^*}(f \circ g)}{\overline{\tau}_k^{L^*}(f)} &\leq \liminf_{r\to 1} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \leq \frac{\tau_h^{L^*}(f \circ g)}{\tau_k^{L^*}(f)} \\ &\leq \limsup_{r\to 1} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \leq \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\tau_k^{L^*}(f)} .\end{aligned}$$

Theorem 8. *If f, g be any two analytic functions and h, k be any two entire functions with $0 < \overline{\tau}_h^{L^*}(f \circ g) < \infty$, $0 < \overline{\tau}_k^{L^*}(f) < \infty$ and $\lambda_h^{L^*}(f \circ g) = \lambda_k^{L^*}(f)$, then*

$$\liminf_{r\to 1} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} \leq \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\overline{\tau}_k^{L^*}(f)} \leq \limsup_{r\to 1} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} .$$

Theorem 9. *If f, g be any two analytic functions and h, k be any two entire functions such that $0 < \tau_h^{L^*}(f \circ g) \leq \overline{\tau}_h^{L^*}(f \circ g) < \infty$, $0 < \tau_k^{L^*}(f) \leq \overline{\tau}_k^{L^*}(f) < \infty$ and $\lambda_h^{L^*}(f \circ g) = \lambda_k^{L^*}(f)$, then*

$$\begin{aligned}\liminf_{r\to 1} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} &\leq \min\left\{\frac{\tau_h^{L^*}(f \circ g)}{\tau_k^{L^*}(f)}, \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\overline{\tau}_k^{L^*}(f)}\right\} \\ &\leq \max\left\{\frac{\tau_h^{L^*}(f \circ g)}{\tau_k^{L^*}(f)}, \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\overline{\tau}_k^{L^*}(f)}\right\} \leq \limsup_{r\to 1} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)} .\end{aligned}$$

Theorem 10. *If f, g be any two analytic functions and h, k be any two entire functions with $0 < \tau_h^{L^*}(f \circ g) \leq \overline{\tau}_h^{L^*}(f \circ g) < \infty$, $0 < \tau_k^{L^*}(g) \leq \overline{\tau}_k^{L^*}(g) < \infty$ and $\lambda_h^{L^*}(f \circ g) = \lambda_k^{L^*}(g)$, then*

$$\begin{aligned}\frac{\tau_h^{L^*}(f \circ g)}{\overline{\tau}_k^{L^*}(g)} &\leq \liminf_{r\to 1} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)} \leq \frac{\tau_h^{L^*}(f \circ g)}{\tau_k^{L^*}(g)} \\ &\leq \limsup_{r\to 1} \frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)} \leq \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\tau_k^{L^*}(g)} .\end{aligned}$$

Theorem 11. *If f, g be any two analytic functions and h, k be any two entire functions such that $0 < \overline{\tau}_h^{L^*}(f \circ g) < \infty$, $0 < \overline{\tau}_k^{L^*}(g) < \infty$ and $\lambda_h^{L^*}(f \circ g) = \lambda_k^{L^*}(g)$, then*

$$\liminf_{r \to 1} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_g(r)} \leq \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\overline{\tau}_k^{L^*}(g)} \leq \limsup_{r \to 1} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_g(r)} \ .$$

Theorem 12. *If f, g be any two analytic functions and h, k be any two entire functions with $0 < \tau_h^{L^*}(f \circ g) \leq \overline{\tau}_h^{L^*}(f \circ g) < \infty$, $0 < \tau_k^{L^*}(g) \leq \overline{\tau}_k^{L^*}(g) < \infty$ and $\lambda_h^{L^*}(f \circ g) = \lambda_k^{L^*}(g)$, then*

$$\begin{aligned} \liminf_{r \to 1} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_g(r)} &\leq \min\left\{ \frac{\tau_h^{L^*}(f \circ g)}{\tau_k^{L^*}(g)}, \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\overline{\tau}_k^{L^*}(g)} \right\} \\ &\leq \max\left\{ \frac{\tau_h^{L^*}(f \circ g)}{\tau_k^{L^*}(g)}, \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\overline{\tau}_k^{L^*}(g)} \right\} \leq \limsup_{r \to 1} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_g(r)} \ . \end{aligned}$$

We may now state the following theorems without their proofs based on relative Nevanlinna L^*- type and relative Nevanlinna L^*-weak type of a meromorphic fucntion with respect to an entire function:

Theorem 13. *If f, g be any two analytic functions and h, k be any two entire functions such that $0 < \overline{\sigma}_h^{L^*}(f \circ g) \leq \sigma_h^{L^*}(f \circ g) < \infty$, $0 < \tau_k^{L^*}(f) \leq \overline{\tau}_k^{L^*}(f) < \infty$ and $\rho_h^{L^*}(f \circ g) = \lambda_k^{L^*}(f)$, then*

$$\begin{aligned} \frac{\overline{\sigma}_h^{L^*}(f \circ g)}{\overline{\tau}_k^{L^*}(f)} \leq \liminf_{r \to 1} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_f(r)} &\leq \frac{\overline{\sigma}_h^{L^*}(f \circ g)}{\tau_k^{L^*}(f)} \\ &\leq \limsup_{r \to 1} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_f(r)} \leq \frac{\sigma_h^{L^*}(f \circ g)}{\tau_k^{L^*}(f)} \ . \end{aligned}$$

Theorem 14. *If f, g be any two analytic functions and h, k be any two entire functions with $0 < \sigma_h^{L^*}(f \circ g) < \infty$, $0 < \overline{\tau}_k^{L^*}(f) < \infty$ and $\rho_h^{L^*}(f \circ g) = \lambda_k^{L^*}(f)$, then*

$$\liminf_{r \to 1} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_f(r)} \leq \frac{\sigma_h^{L^*}(f \circ g)}{\overline{\tau}_k^{L^*}(f)} \leq \limsup_{r \to 1} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_f(r)} \ .$$

Theorem 15. *If f, g be any two analytic functions and h, k be any two entire functions such that $0 < \overline{\sigma}_h^{L^*}(f \circ g) \leq \sigma_h^{L^*}(f \circ g) < \infty$, $0 < \tau_k^{L^*}(f) \leq \overline{\tau}_k^{L^*}(f) < \infty$ and $\rho_h^{L^*}(f \circ g) = \lambda_k^{L^*}(f)$, then*

$$\begin{aligned} \liminf_{r \to 1} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_f(r)} &\leq \min\left\{ \frac{\overline{\sigma}_h^{L^*}(f \circ g)}{\tau_k^{L^*}(f)}, \frac{\sigma_h^{L^*}(f \circ g)}{\overline{\tau}_k^{L^*}(f)} \right\} \\ &\leq \max\left\{ \frac{\overline{\sigma}_h^{L^*}(f \circ g)}{\tau_k^{L^*}(f)}, \frac{\sigma_h^{L^*}(f \circ g)}{\overline{\tau}_k^{L^*}(f)} \right\} \leq \limsup_{r \to 1} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_f(r)} \ . \end{aligned}$$

Theorem 16. *If f, g be any two analytic functions and h, k be any two entire functions with $0 < \tau_h^{L^*}(f \circ g) \leq \overline{\tau}_h^{L^*}(f \circ g) < \infty$, $0 < \overline{\sigma}_k^{L^*}(f) \leq \sigma_k^{L^*}(f) < \infty$ and*

$\lambda_h^{L^*}(f\circ g)=\rho_k^{L^*}(f)$, *then*

$$\frac{\tau_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(f)}\le\liminf_{r\to1}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}\le\frac{\tau_h^{L^*}(f\circ g)}{\overline{\sigma}_k^{L^*}(f)}\le\limsup_{r\to1}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}\le\frac{\overline{\tau}_h^{L^*}(f\circ g)}{\overline{\sigma}_k^{L^*}(f)}\ .$$

Theorem 17. *If f, g be any two analytic functions and h,k be any two entire functions such that $0<\overline{\tau}_h^{L^*}(f\circ g)<\infty$, $0<\sigma_k^{L^*}(f)<\infty$ and $\lambda_h^{L^*}(f\circ g)=\rho_k^{L^*}(f)$, then*

$$\liminf_{r\to1}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}\le\frac{\overline{\tau}_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(f)}\le\limsup_{r\to1}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}\ .$$

Theorem 18. *If f, g be any two analytic functions and h,k be any two entire functions with $0<\tau_h^{L^*}(f\circ g)\le\overline{\tau}_h^{L^*}(f\circ g)<\infty$, $0<\overline{\sigma}_k^{L^*}(f)\le\sigma_k^{L^*}(f)<\infty$ and $\lambda_h^{L^*}(f\circ g)=\rho_k^{L^*}(f)$, then*

$$\liminf_{r\to1}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}\le\min\left\{\frac{\tau_h^{L^*}(f\circ g)}{\overline{\sigma}_k^{L^*}(f)},\frac{\overline{\tau}_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(f)}\right\}\le\max\left\{\frac{\tau_h^{L^*}(f\circ g)}{\overline{\sigma}_k^{L^*}(f)},\frac{\overline{\tau}_h^{L^*}(f\circ g)}{\sigma_k^{L^*}(f)}\right\}\le\limsup_{r\to1}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_f(r)}\ .$$

Theorem 19. *If f, g be any two analytic functions and h,k be any two entire functions such that $0<\overline{\sigma}_h^{L^*}(f\circ g)\le\sigma_h^{L^*}(f\circ g)<\infty$, $0<\tau_k^{L^*}(g)\le\overline{\tau}_k^{L^*}(g)<\infty$ and $\rho_h^{L^*}(f\circ g)=\lambda(g)$, then*

$$\frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\overline{\tau}_k^{L^*}(g)}\le\liminf_{r\to1}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)}\le\frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\tau_k^{L^*}(g)}\le\limsup_{r\to1}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)}\le\frac{\sigma_h^{L^*}(f\circ g)}{\tau_k^{L^*}(g)}\ .$$

Theorem 20. *If f, g be any two analytic functions and h,k be any two entire functions with $0<\sigma_h^{L^*}(f\circ g)<\infty$, $0<\overline{\tau}_k^{L^*}(g)<\infty$ and $\rho_h^{L^*}(f\circ g)=\lambda(g)$, then*

$$\liminf_{r\to1}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)}\le\frac{\sigma_h^{L^*}(f\circ g)}{\overline{\tau}_k^{L^*}(g)}\le\limsup_{r\to1}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)}\ .$$

Theorem 21. *If f, g be any two analytic functions and h,k be any two entire functions such that $0<\overline{\sigma}_h^{L^*}(f\circ g)\le\sigma_h^{L^*}(f\circ g)<\infty$, $0<\tau_k^{L^*}(g)\le\overline{\tau}_k^{L^*}(g)<\infty$ and $\rho_h^{L^*}(f\circ g)=\lambda(g)$, then*

$$\liminf_{r\to1}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)}\le\min\left\{\frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\tau_k^{L^*}(g)},\frac{\sigma_h^{L^*}(f\circ g)}{\overline{\tau}_k^{L^*}(g)}\right\}\le\max\left\{\frac{\overline{\sigma}_h^{L^*}(f\circ g)}{\tau_k^{L^*}(g)},\frac{\sigma_h^{L^*}(f\circ g)}{\overline{\tau}_k^{L^*}(g)}\right\}\le\limsup_{r\to1}\frac{T_h^{-1}T_{f\circ g}(r)}{T_k^{-1}T_g(r)}\ .$$

Theorem 22. *If f, g be any two analytic functions and h, k be any two entire functions with $0 < \tau_h^{L^*}(f \circ g) \leq \overline{\tau}_h^{L^*}(f \circ g) < \infty$, $0 < \overline{\sigma}_k^{L^*}(g) \leq \sigma_k^{L^*}(g) < \infty$ and $\lambda_h^{L^*}(f \circ g) = \rho_k^{L^*}(g)$, then*

$$\frac{\tau_h^{L^*}(f \circ g)}{\sigma_k^{L^*}(g)} \leq \liminf_{r \to 1} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_g(r)} \leq \frac{\tau_h^{L^*}(f \circ g)}{\overline{\sigma}_k^{L^*}(g)}$$
$$\leq \limsup_{r \to 1} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_g(r)} \leq \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\overline{\sigma}_k^{L^*}(g)} .$$

Theorem 23. *If f, g be any two analytic functions and h, k be any two entire functions such that $0 < \overline{\tau}_h^{L^*}(f \circ g) < \infty$, $0 < \sigma_k^{L^*}(g) < \infty$ and $\lambda_h^{L^*}(f \circ g) = \rho_k^{L^*}(g)$, then*

$$\liminf_{r \to 1} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_g(r)} \leq \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\sigma_k^{L^*}(g)} \leq \limsup_{r \to 1} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_g(r)} .$$

Theorem 24. *If f, g be any two analytic functions and h, k be any two entire functions with $0 < \tau_h^{L^*}(f \circ g) \leq \overline{\tau}_h^{L^*}(f \circ g) < \infty$, $0 < \overline{\sigma}_k^{L^*}(g) \leq \sigma_k^{L^*}(g) < \infty$ and $\lambda_h^{L^*}(f \circ g) = \rho_k^{L^*}(g)$, then*

$$\liminf_{r \to 1} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_g(r)} \leq \min \left\{ \frac{\tau_h^{L^*}(f \circ g)}{\overline{\sigma}_k^{L^*}(g)}, \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\sigma_k^{L^*}(g)} \right\}$$
$$\leq \max \left\{ \frac{\tau_h^{L^*}(f \circ g)}{\overline{\sigma}_k^{L^*}(g)}, \frac{\overline{\tau}_h^{L^*}(f \circ g)}{\sigma_k^{L^*}(g)} \right\} \leq \limsup_{r \to 1} \frac{T_h^{-1} T_{f \circ g}(r)}{T_k^{-1} T_g(r)} .$$

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