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EIGENVALUES AND EIGENFUNCTIONS OF NON-LOCAL BOUNDARY VALUE PROBLEMS OF THE STURM-LIOUVILLE EQUATION

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ABSTRACT. The initial and boundary value problems of differential equations with nonlocal initial and nonlocal boundary conditions are of interests in applications. In this paper we study the existence and some general properties of eigenvalues and eigenfunctions of the boundary value problem of the Sturm-Liouville differential equation non-local boundary conditions.

1. INTRODUCTION

The problems of differential equations with nonlocal conditions have been studied by some authors (see for example [1] and [3]-[5]). Also theoretical investigation of Sturm-Liouville boundary value problems with various types of nonlocal boundary conditions is a topical problem and recently has been paid much attention to them in the scientific literature (see for exampl [6] and [7] -[9]).
Consider the two non-local boundary value problems of the Sturm-Liouville equation

$$-y'' + q(x)y = \lambda^2 y,\ 0 \le x \le \pi \tag{1}$$

with each one of the two non-local conditions

$$y(0) = 0,\ y(\xi) = 0,\ \xi \in (0, \pi], \tag{2}$$

and

$$y(\eta) = 0,\ y(\pi) = 0,\ \eta \in [0, \pi) \tag{3}$$

where the non-negative real function $q(x)$ has a second piecewise integrable derivatives on $(0, \pi)$ and λ is spectral parameter.
Here we study the existence and some general properties of the eigenvalues and eigenfunctions of the two non-local boundary value problems (1) and (2) and (1)

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and (3). Comparison with the local boundary value problem problem of equation (1) with the local boundary value problem

$$y(0) = 0,\ y(\pi) = 0$$

will be given.

2. General properties

Here we prove some results concerning the eigenvalues and eigenfunctions of the two non-local problems (1)-(2) and (1)-(3).

lemma 2.1 The eigenvalues of the non-local boundary value problem (1) and (2) are real.

Proof. Let $y_0(x)$ be the eigenfunction that corresponds to the eigenvalue λ_0 of the problem (1) and (2), then

$$-y_0'' + q(x)\ y_0 = \lambda_0^2 y_0\ (0 \le x \le \pi), \tag{4}$$

and

$$y_0(0) = y_0(\xi) = 0 \tag{5}$$

Multiplying both sides of (4) by $\bar{y_0}$ and then integrating form 0 to ξ with respect to x, we have

$$-\bar{y_0}\ \acute{y_0}\ |_0^{\xi} + \int_0^{\xi} |\acute{y_0}|^2\ dx + \int_0^{\xi} q(x)\ |y_0|^2 dx = \lambda_0^2 \int_0^{\xi} |y_0|^2\ dx.$$

using the boundary condition (5), we have

$$\lambda_0^2\ = \frac{\int_0^{\xi}\ [\ q(x)\ |y_0|^2\ +\ |\acute{y_0}|^2\ ]\ dx}{\int_0^{\xi}\ |y_0|^2\ dx}.$$

From which it follows the reality of λ_0^2 .□

By the same way we can prove the following lemma.

lemma 2.2 The eigenvalues of the non-local boundary value problem (1) and (3) are real.

lemma 2.3 The eigenfunctions that corresponds to two different eigenvalues of the non-local boundary value problem (1) and (2) are orthogonal.

Proof. Let $\lambda_1 \neq \lambda_2$ be two different eigenvalues of the non-local boundary value problem (1) and (2). Let $y_1(x), y_2(x)$ be the corresponding eigenfunctions, then

$$-y_1'' + q(x)\ y_1 =\ \lambda_1^2\ y_1\ (0 \le x \le \pi), \tag{6}$$

$$y_1(0) = y_1(\xi) = 0 \tag{7}$$

and

$$-y_2'' + q(x)\ y_2 =\ \lambda_2^2\ y_2\ (0 \le x \le \pi), \tag{8}$$

$$y_2(0) = y_2(\xi) = 0 \tag{9}$$

Multiplying both sides of (6) by $\bar{y}_2$ and integrating with respect to x, we obtain

$$-\int_0^{\xi} y''_1\ \bar{y}_2\ dx + \int_0^{\xi} q(x)\ y_1\ \bar{y}_2\ dx\ =\ \lambda_1^2 \int_0^{\xi} y_1\ \bar{y}_2\ dx. \tag{10}$$

By taking the complex conjugate of (8) and multiply it by y_1 and integrate the resulting expression with respect to x, we have

$$-\int_0^{\xi} y_1\ \bar{y}''_2\ dx + \int_0^{\xi} q(x)\ y_1\ \bar{y}_2\ dx\ =\ \lambda_1^2 \int_0^{\xi} y_1\ \bar{y}_2\ dx. \tag{11}$$

Subtracting (10) from (11) and using the boundary conditions of (7) and (9) we obtain

$$(\lambda_1^2 - \lambda_2^2) \int_0^{\xi} y_1 \ \bar{y}_2 \ dx \ = \ 0, \ \lambda_1^2 \neq \lambda_2^2.$$

which completes the proof.□
By the same way we can prove the following lemma
lemma 2.4 The eigenfunctions that corresponds to two different eigenvalues of the non-local boundary value problem (1) and (3) are orthogonal.

3. The asymptotic formulas for the solution

Here we study the asymptotic formulas for the solutions of problems (1) and (2) and (1) and (3).
Lemma 1.1 deals with the nature the eigenvalues. Let be $\varphi(x,\lambda)$ the solution of equation (1) and (2) satisfying the initial conditions

$$\varphi(0,\lambda) = 0, \ \varphi'(0,\lambda) = 1 \tag{12}$$

and by $\psi(x,\lambda)$ the solution of the same equation, satisfying the initial conditions

$$\psi(0,\lambda) = 1, \ \psi'(0,\lambda) = 0 \tag{13}$$

We notes that $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are linearly independent if and only if $W(\lambda) \neq 0$.

$$W(\lambda) = \varphi(x,\lambda)\psi'(x,\lambda) - \varphi'(x,\lambda)\psi(x,\lambda).$$

The characteristic equation will be

$$W(\lambda) = \varphi(\xi,\lambda) \tag{14}$$

lemma 3.5 The solution $\varphi(x,\lambda)$ of problem (1) and (2) satisfy the integral equations

$$\varphi(x,\lambda) = \frac{\sin\lambda x}{\lambda} + \int_0^x \frac{\sin\lambda(x-\tau)}{\lambda} q(\tau)\varphi(\tau,\lambda)d\tau, \tag{15}$$

Proof. First we obtain formula (15) Indeed,with solution of the form $q(x) = 0$. (1) becomes becomes $-y'' = \lambda^2 y$ by means of variation of parameter method, we have

$$\varphi(x,\lambda) = C_1(x,\lambda)\cos\lambda x + C_2(x,\lambda)\sin\lambda x \tag{16}$$

and the direct calculation of $C_1(x,s)$ and $C_2(x,s)$, we have

$$\begin{aligned} C_1(x,\lambda) &= -\int_0^x \frac{\sin\lambda\tau}{\lambda} q(\tau)\varphi(\tau,\lambda)d\tau, \\ C_2(x,\lambda) &= \frac{1}{\lambda} + \int_0^x \frac{\cos\lambda\tau}{\lambda} q(\tau)\psi(\tau,\lambda)d\tau. \end{aligned} \tag{17}$$

substituting from (17) into (16) equation (15) follows. Second we show that the integral representation (15) satisfies the problem (1) and (12). Let $\varphi(x,\lambda)$ be the solution of (1), so that

$$q(x) \ \varphi(x,\lambda) = \varphi''(x,\lambda) + \lambda^2\varphi(x,\lambda).$$

We multiply both sides by

$$\frac{\sin\lambda(x-\tau)}{\lambda}$$

and integrating with respect to τ from 0 to x we obtain

$$\int_0^x \frac{\sin\lambda(x-\tau)}{\lambda}\ q(x)\ \varphi(\tau,\lambda)\ d\tau = \int_0^x \frac{\sin\lambda(x-\tau)}{\lambda}\ \varphi''(\tau,\lambda)\ d\tau \tag{18}$$

$$+\lambda^2 \int_0^x \frac{\sin\lambda(x-\tau)}{\lambda}\ \varphi(\tau,\lambda)\ d\tau.$$

Integrating by parts twice and using the condition (12), we have

$$\int_0^x \frac{\sin\lambda(x-\tau)}{\lambda}\ \varphi''(\tau,\lambda)\ d\tau \tag{19}$$

$$= \varphi(x,\lambda) - \frac{\sin\lambda x}{\lambda} - \lambda \int_0^x \sin\lambda(x-\tau)\ \varphi(\tau,\lambda)\ d\tau.$$

By substituting from (19) into (18) we get the required formula (15).□
By the same way we can prove the following lemma
lemma 3.6 The solutions $\psi(x,\lambda)$ of problem (1)and (3) satisfy the integral equations

$$\psi(x,\lambda) = \frac{\sin\lambda(\eta-x)}{\lambda} + \int_\eta^x \frac{\sin\lambda(x-\tau)}{\lambda}\ q(\tau)\ \psi(\tau,\lambda)\ d\tau. \tag{20}$$

lemma 3.7 Let $\lambda = \sigma + it$. Then there exists $\lambda_0 > 0$, such that $|\lambda| > \lambda_0$ the following inequalities for the solutions $\varphi(x,\lambda)$ of boundary value problem (1) and (2) hold true

$$\varphi(x,\lambda) = \frac{\sin\lambda x}{\lambda} + O\left(\frac{e^{|Im\lambda|x}}{|\lambda|^2}\right), \tag{21}$$

Proof. We show first that

$$\varphi(x,\lambda) = O\left(\frac{e^{|t|x}}{|\lambda|}\right),$$

where the inequality is uniformly with respect to x. Form the integral equation (15) we have

$$|\varphi(x,\lambda)| \le \frac{e^{|t|x}}{\lambda} + \frac{e^{|t|x}}{\lambda} \int_0^x e^{|t|x} |q(\tau)||\varphi(\tau,\lambda)|\ d\tau. \tag{22}$$

By using the notation $\varphi(x,\lambda)e^{-|t|x} = F(x,\lambda)$, equation (22) takes the form

$$|F(x,\lambda)| \le \frac{1}{\lambda} + \frac{1}{\lambda}\int_0^\pi |q(\tau)||F(\tau,\lambda)|\ d\tau. \tag{23}$$

Let $\mu = \max_{0\le x\le\pi} F(x,\lambda)$, so that from (23) it follows that

$$\mu \ \le\ \frac{\frac{1}{\lambda}}{1-\frac{1}{\lambda}\int_0^\pi\ |q(\tau)|\ d\tau}.$$

For $|\lambda| > \lambda_0 = \int_0^\pi |q(\tau)| d\tau$ it follows from the last inequality that $F(x,\lambda) \le$ constant $/|\lambda|$ and this implies that

$$\varphi(x,\lambda) = O\left(\frac{e^{|t|x}}{|\lambda|}\right), \tag{24}$$

By the aid of (23) we find that

$$\int_0^x \frac{\sin\lambda(x-\tau)}{\lambda}\ q(\tau)\ \varphi(\tau,\lambda)\ d\tau =\ O\left(\frac{e^{|t|x}}{|\lambda|^2}\right). \tag{25}$$

From (15) and (23) it follows that, $\varphi(x,\lambda)$ has the asymptotic formula (21), as in [2] .□

By the same way we can prove the following lemma

lemma 3.8 Let $\lambda=\sigma+it$. Then there exists $\lambda_0>0$, such that $|\lambda|>\lambda_0$ the following inequalities for the solutions $\psi(x,\lambda)$ of boundary value problem (1) and (3) hold true

$$\psi(x,\lambda)=\frac{\sin(\eta-x)}{\lambda}\ +\ O\left(\frac{e^{|Im\lambda|(\eta-x)}}{|\lambda|^2}\right). \tag{26}$$

Theorem 3.1 Let $\lambda=\sigma+it$ and suppose that $q(x)$ has a second order piecewise differentiable derivatives on $[0,\pi]$. Then the solution $\varphi(x,\lambda)$ of non-local boundary value (1) and (2) have the following asymptotic formula

$$\varphi(x,\lambda)=\frac{\sin\lambda x}{\lambda}-\frac{\alpha_1(x)}{\lambda^2}\cos\lambda x-\frac{\alpha_2(x)}{\lambda^3}\sin\lambda x\ +O\left(\frac{e^{|Im\lambda|x}}{|\lambda^4|}\right) \tag{27}$$

where

$$\begin{aligned}\alpha_1(x)\ &=\ \frac{1}{2}\int_0^x q(t)\ dt,\\ \alpha_2(x)\ &=\ \frac{1}{4}\left(\int_0^x q(t)\ dt\right)^2-\frac{[q(x)+q(0)]}{4}.\end{aligned} \tag{28}$$

Proof. By substituting from (21) into the integral equation (15), we have

$$\begin{aligned}\varphi(x,\lambda)=\frac{\sin\lambda x}{\lambda}-\frac{\cos\lambda x}{2\lambda^2}\int_0^x\ q(t)\ dt+\frac{1}{2\lambda^2}\int_0^x\cos\lambda(x-2t)\ q(t)\ dt\\ +\ O\left(\frac{e^{|Im\lambda|x}}{|\lambda|^3}\right).\end{aligned} \tag{29}$$

Integrating the last integration of (29) by parts and noticing that there exists $q'(x)$ such that $q'\in L_1[0,\pi]$

$$\begin{aligned}&\frac{1}{2\lambda^2}\int_0^x\cos\lambda(x-2t)\ q(t)\ dt\\ &=\frac{1}{2}[q(x)-q(0)]\frac{\cos\lambda x}{\lambda}-\frac{1}{2\lambda}\int_0^x\sin\lambda(x-2t)\ dt\\ &=\ O\left(\frac{e^{|Im\lambda|x}}{|\lambda|^3}\right).\end{aligned} \tag{30}$$

substituting from (30) into (29) , we get

$$\varphi(x,\lambda)=\frac{\sin\lambda x}{\lambda}-\frac{\alpha_1(x)}{\lambda^2}\cos\lambda x\ +\ O\left(\frac{e^{|Im\lambda|x}}{|\lambda|^3}\right). \tag{31}$$

where $\alpha_1(x)$ is defined by (28) . In order to make $\varphi(x,\lambda)$ more precise we repeat this procedure again by substituting from the last result (31) into the same integral equation (15), we have

$$\begin{aligned}\varphi(x,\lambda)=\frac{\sin\lambda x}{\lambda}+\int_0^x\frac{\sin\lambda(x-t)\sin\lambda t}{\lambda^2}\ q(t)\ dt\\ -\int_0^x\frac{\sin\lambda(x-t)\cos\lambda t}{\lambda^3}\ q(t)\alpha_1(t)\ dt\end{aligned}$$

$$+\int_0^x \frac{\sin\lambda(x-t)}{\lambda}\ q(t)O\left(\frac{e^{|Im\lambda|x}}{|\lambda|^3}\right)\ dt. \tag{32}$$

Now we estimate each term in (32). Integrating by parts twice the first term of (32), and noticing that $q'' \in L_1[0,\pi]$, we have

$$\int_0^x \frac{\sin\lambda(x-t)\sin\lambda t}{\lambda^2}\ q(x)\ dt = -\frac{\alpha_1(x)}{\lambda^2}\cos\lambda x + \frac{q(x)-q(0)}{4\lambda^3}\sin\lambda x + O\left(\frac{e^{|Im\lambda|x}}{|\lambda|^4}\right) \tag{33}$$

Similarly, we have

$$-\int_0^x \frac{\sin\lambda(x-t)\cos\lambda t}{\lambda^3}\ q(x)\ \alpha_1(x)\ dt = -\frac{1}{2}\int_0^x \alpha_1(x)\ q(x)\ dt\frac{\sin\lambda x}{\lambda^3} + O\left(\frac{e^{|Im\lambda|x}}{|\lambda|^4}\right) \tag{34}$$

Substituting from (33) and (34) into (32) we get the required formula (27).□
By the same way we can prove the following theorem
Theorem 3.2 Let $\lambda = \sigma + it$ and suppose that $q(x)$ has a second order piecewise differentiable derivatives on $[0,\pi]$. Then the solution $\psi(x,\lambda)$ of non-local boundary value (1) and (3) have the following asymptotic formula

$$\psi(x,\lambda) = \frac{\sin\lambda(x-\eta)}{\lambda} - \frac{\gamma_1(x)}{\lambda^2}\cos\lambda(x-\eta) - \frac{\gamma_2(x)}{\lambda^3}\sin\lambda(x-\eta) + O\left(\frac{e^{|Im\lambda|(x-\eta)}}{|\lambda^4|}\right) \tag{35}$$

where

$$\begin{aligned}\gamma_1(x) &= \frac{1}{2}\int_0^x q(t)\ dt,\\ \gamma_2(x) &= \frac{1}{4}\left(\int_0^x q(t)\ dt\right)^2 - \frac{[q(x)+q(0)]}{4}.\end{aligned} \tag{36}$$

Now inserting the values of the functions $\varphi(x,\lambda)$ from the estimate (27) into the second of the boundary conditions in (2), we obtain the following equation for the determination of the eigenvalues: Equation (21) is the characteristic equation which gives roots of λ

$$\lambda_n^0 = \frac{n\pi}{\xi},\ n = 0, \pm1, \pm2, ...$$

Then the $W(\lambda)$ has the same root of the function $sin\lambda\xi$ (By Rouche's theorem)

$$\lambda_n = \lambda_n^0 + \varepsilon_n,\ n = 0, 1, 2, ... \tag{37}$$

Theorem 3.3 Let $q \in L_1(0,\pi)$, then we have the following asymptotic formulas for λ_n of non-local boundary value (1) and (2)

$$\lambda_n = \frac{n\pi}{\xi} + \frac{\alpha_1}{n\pi} + O\left(\frac{1}{n^2}\right). \tag{38}$$

where $\alpha_1(x)$ defined in (28).
Proof.

$$W(x,\lambda) = \frac{\sin\lambda\xi}{\lambda} - \frac{\alpha_1(x)}{\lambda^2}\cos\lambda\xi - \frac{\alpha_2(x)}{\lambda^3}\sin\lambda\xi + O\left(\frac{e^{|Im\lambda|\xi}}{|\lambda^4|}\right) \tag{39}$$

It follows from (39) that

$$\sin \lambda\xi - \frac{\alpha_1}{\lambda} \cos \lambda\xi - \frac{\alpha_2}{\lambda^2} \sin \lambda\xi + O\left(\frac{e^{|Im\lambda|\xi}}{|\lambda^3|}\right) = 0 \tag{40}$$

From equation (40), we have

$$[1 - \frac{\alpha_2}{\lambda^2}] \sin \lambda\xi - \frac{\alpha_1}{\lambda} \cos \lambda\xi = 0 \tag{41}$$

Dividing (41) by $\cos \lambda\xi$ we obtain

$$[1 - \frac{\alpha_2}{\lambda^2}] \tan \lambda\xi = \frac{\alpha_1}{\lambda}$$

since imaginary $\lambda = O\left(\frac{1}{n}\right)$, then

$$\tan \lambda_n \xi = \frac{\alpha_1}{\lambda_n} + O\left(\frac{1}{n^2}\right) \tag{42}$$

From (37), (42) after elementary calculation, we obtain

$$\varepsilon_n = \frac{\alpha_1}{n\pi} + O\left(\frac{1}{n^2}\right) \tag{43}$$

From (37) and (43) , we have

$$\lambda_n = \frac{n\pi}{\xi} + \frac{\alpha_1}{n\pi} + O\left(\frac{1}{n^2}\right).$$

By the same way we can prove the following theorem
Theorem 3.4 Let $q \in L_1(0, \pi)$, then we have the following asymptotic formulas for λ_n of non-local boundary value (1) and (3)

$$\lambda_n = \frac{n\pi}{(\pi - \eta)} + \frac{\gamma_1}{n\pi} + O\left(\frac{1}{n^2}\right). \tag{44}$$

where γ_1 defined in (36).
Corollary 3.1 If $\xi = \pi$, then the eigenvalues of (38), we obtain

$$\lambda_n = n + \frac{\alpha_1}{n\pi} + O\left(\frac{1}{n^2}\right).$$

and, if $\eta = 0$, then the eigenvalues of (44), we obtain

$$\lambda_n = n + \frac{\gamma_1}{n\pi} + O\left(\frac{1}{n^2}\right).$$

Which meets with the result obtained in [11].

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