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PERIODIC AND SUBHARMONIC SOLUTIONS FOR SECOND-ORDER $p-$LAPLACIAN DIFFERENCE EQUATIONS CONTAINING BOTH ADVANCE AND RETARDATION

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Abstract. We obtain some new existence results on nontrivial periodic and subharmonic solutions for second-order p-Laplacian difference equations containing both advance and retardation. Without the nonnegativity restriction on vector field F, we greatly improve the existing results (e.g. see [4-6]).

1. Introduction

Let $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, stand the set of all real numbers, natural numbers, and integers, respectively. For $a \leq b \in \mathbb{Z}$, we define $\mathbb{Z}[a] = \{a, a+1, \ldots\}$, $\mathbb{Z}[a, b] = \{a, a+1, \ldots, b\}$.

Consider the following nonlinear discrete system

$$\Delta\big(\varphi_p(\Delta x_{n-1})\big) + f(n, x_{n+1}, x_n, x_{n-1}) = 0, \qquad \forall\, n \in \mathbb{Z}, \tag{1.1}$$

where $\Delta x_n = x_{n+1} - x_n$, $\Delta^2 x_n = \Delta(\Delta x_n)$, $\varphi_p(s) = |s|^{p-2}s\,(2 \leq p < \infty)$, $f : \mathbb{Z} \times \mathbb{R}^3 \to \mathbb{R}$, $f(t, \cdot, \cdot, \cdot) \in C(\mathbb{R}^3)$. More precisely, f is given by

$$\begin{cases} f(t,\, u,\, v,\, w) = F_2'(t-1,\, v,\, w) + F_3'(t,\, u,\, v) \\ F_2'(t-1,\, v,\, w) = \frac{\partial F(t-1,\, v,\, w)}{\partial v}, \quad F_3'(t,\, u,\, v) = \frac{\partial F(t,\, u,\, v)}{\partial v}, \end{cases}$$

for some function $F \in C^1(\mathbb{R}^3)$.

Clearly, system (1.1) may be thinking as a discrete analogue of the functional differential equation below

$$[\varphi_p(x')]' + f\big(t, x(t+1), x(t), x(t-1)\big) = 0, \quad t \in \mathbb{R},$$

which is arising in the study of the existence of solitary waves of lattice differential equations, see [1-3].

When dealing with (1.1), there are many excellent works (e.g. see [4-6]). [4] is the first paper to deal with the existence of periodic and subharmonic solutions of discrete system (1.1), and the results of [5-6] are based on [4].

In [4], by using the critical point theory, Chen and Fang consider the existence of periodic and subharmonic solutions of discrete system (1.1) for the case of $F \geq 0$.

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Employing the critical point theory, in [5], Yu, Shi, and Guo study a special case of (1.1). Without any periodic assumptions, they gain some different existence results on nontrivial homoclinic orbit for nonlinear advance and retardation difference equations.

In [6], Lin and Zhou argue some extensions of system (1.1). Then the critical theory uses, they derive some sufficient conditions on the existence and multiplicity of periodic and subharmonic solutions to a 2nth-order nonlinear advance and retardation ϕ-Laplacian difference equation. Besides, if one fetches $\phi(x) = |x|^{p-2}(1 < p < \infty)$, $r_k = 1$ and $n = 1$, Theorem 3.1 in [6] reduces to Theorem 3.1 in [4]. So the existence results in [6] are extensions in [4].

For other related works on the existence of periodic solutions, for nonlinear difference equations, we refer the authors to see [7-8], for second-order (or higher-order) discrete Hamiltonian system, one can consult to [9-11], and for the boundary value problems, to read [12] and the references cited up there.

It is remarked that, in (1.1), f depends on x_{n+1} and x_{n-1}, the traditional ways of establishing the functional in [3-5, 7-8, 10-12] are inapplicable. Besides, one discovers that $F(t, u, v) \in \mathbb{R}$. However, as far as we know, there are very few research works on (1.1) without the nonnegativity restriction on vector field F. For example, in [4-6], all the authors make the same assumption on F, i.e., $F \geq 0$.

The main purpose of this paper is to consider the existence of nontrivial periodic and subharmonic solutions of discrete system (1.1) for the case of $F \leq 0$. In (1.1), since $F \leq 0$, the methods used in [4-6] are not available. More precisely, let $p \geq 2$ and $qm \geq 5$, with the help of the constructing triple $(J(x), Y \oplus Z, \text{P})$ (given in Section 2), we obtain the existence of nontrivial periodic and subharmonic solutions of (1.1) for nonpositivity vector field F.

We assume that

(A_1) $0 < q, m \in \mathbb{N}$, $qm \geq 5$ and for every $(t, x, y, z) \in \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $f(t + m, x, y, z) = f(t, x, y, z)$;

(A_2) $F(t,\, u,\, v) \leq 0$ and satisfies

$$\lim_{\rho \to 0} \frac{F(t,\, u,\, v)}{\rho^p} = 0 \quad where \quad \rho = \sqrt{u^2 + v^2};$$

(A_3) there exist constants $a_1 > 0$, $a_2 > 0$ and $\beta \geq p + 1$, such that

$$F(t, u, v) \leq -a_1 \left(\sqrt{u^2 + v^2} \right)^{\beta} + a_2 \quad for \quad \forall\ (t,\, u,\, v) \in \mathbb{R}^3.$$

Our main result is given by

Theorem 1.1. For any given positive integers q and m, let $qm \geq 5$ and $p \in [2, \infty)$, suppose conditions $(A_1) - (A_3)$ are satisfied. Then, system (1.1) has at least two nontrivial qm-periodic solutions, here a solution $\{x_n\}$ of (1.1) is said to be periodic with period qm if $\{x_n\} \in E_{qm}$. Moreover, if $\{x_n\} \in E_{qm}$ and $x_1 = ... = x_{qm} = 0$, $\{x_n\}$ is called to be trivial.

Corollary 1.1. For $p \in [2, \infty)$, assume $F(t, u, v)$ satisfies conditions (A_1), (A_2) and

(A_4) there exist constants $\beta \geq p+1$ *and* $a_1 > 0$, such that

$$F(t,u,v) \leq -a_1\left(\sqrt{u^2+v^2}\right)^{\beta}.$$

Then, system (1.1) has at least two nontrivial qm-periodic solutions.

We give an representative example to make our results on the system (1.1) more clear.

Example 1.1. Suppose that

$$\begin{aligned} f(t,\, u,\, v,\, w) &= -2(p+1)v\Bigg[\left(1+\cos^2\frac{\pi t}{m}\right)(u^2+v^2)^p \\ &\quad +\left(1+\cos^2\frac{\pi(t-1)}{m}\right)(v^2+w^2)^p\Bigg]. \end{aligned}$$

Take

$$F(t,\, u,\, v) = -\left(1+\cos^2\frac{\pi t}{m}\right)(u^2+v^2)^{p+1}.$$

we have

$$\begin{aligned} & F_2'(t-1,\, v,\, w)+F_3'(t,\, u,\, v) \\ =\; & -2(p+1)v\Bigg[\left(1+\cos^2\frac{\pi t}{m}\right)(u^2+v^2)^p+(1+\cos^2\frac{\pi(t-1)}{m}\Bigg)(v^2+w^2)^p\Bigg]. \end{aligned}$$

Proof. It is clear that $F \in C^1(\mathbb{R}^3, \mathbb{R})$, and (A_1), (A_2) and (A_3) are satisfied, so system (1.1) has at least two nontrivial qm-periodic solutions.

2. Notations and useful lemmas

Before proving Theorem 1.1, we introduce some notions and notations initially.

- Let W be the set of sequences, i.e. $W = \big\{w = \{w_k\}_k,\ w_k \in \mathbb{R},\ k \in \mathbb{N}\big\}$ and when $x, y \in W$, $a, b \in \mathbb{R}$, $ax+by$ is given by $ax+by = \{ax_k+by_k\}_{k=-\infty}^{+\infty}$. For any given positive integers q and m, $E_{qm} \subset W$, meeting that

$$E_{qm} = \big\{w = \{w_k\} \in W \mid w_{k+qm} = w_k,\ k \in \mathbb{Z}\big\}.$$

  Then with the common Euclid inner product ($\|x\| = (\sum_{k=1}^{qm} x_k^2)^{\frac{1}{2}}$), E_{qm} is a qm-dimensional Hilbert space. Let $\|\cdot\|_\alpha$ denote by

$$\|x\|_\alpha = (\sum_{k=1}^{qm}|x_k|^{\alpha})^{\frac{1}{\alpha}},\ \alpha \in (1,\infty),$$

  then there exist positive constants $C_1 \leq C_2$, $C_3 \leq C_4$ and $C_5 \leq C_6$ such that

$$\begin{aligned} &C_1\|x\|_p \leq \|x\| \leq C_2\|x\|_p\,, \quad \forall\ p \in [2,\infty), \quad \forall\ x \in E_{qm}, \\ &C_3\|x\|_\beta \leq \|x\| \leq C_4\|x\|_\beta, \quad \forall\ \beta \in [3,\infty), \quad \forall\ x \in E_{qm}. \end{aligned}$$

- Let H be a real Hilbert space, $J \in C^1(H)$ is said to satisfy PS condition if any sequence $\{x_k\} \subset H$ for which $\{J(x_k)\}$ is bounded and $J'(x_k) \to 0$, as $k \to \infty$, possesses a convergent subsequence in H.

- Set $Z=\{(x_1,x_2,\ldots,x_{n-1},x_n,x_{n+1},\ldots,x_{qm})\in E_{qm}\ |\Delta x_{n-1}=-\Delta x_n=c,\ \Delta x_1=\ldots=\Delta x_{n-2}=\Delta x_{n+1}=\ldots=\Delta x_{qm}=0,\ c\in\mathbb{R}\}$ and $Y=Z^{\perp}$, then $E_{qm}=Y\oplus Z$.

Remark 2.1. The orthogonal direct sum decomposition related to Δx_j $(j=1,2,\ldots,qm)$, and $\{(u,u,\ldots,u)^{\top}\in E_{qm}\mid u\in\mathbb{R}\}\subseteq Y$ is very different from the known research works, and when $x\in Y$, we can conclude that $x\neq(u,u,\ldots,u)^{\top}$.

We now give some useful lemmas, which will serve us well later.

Lemma 2.1.([3]) Let $qm\geq 3$, S be a matrix

$$S=\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1\\ -1 & 2 & -1 & \cdots & 0 & 0\\ 0 & -1 & 2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{qm\times qm}.$$

If $0\neq c\in\mathbb{R}$, then the eigenvector of S associated with the eigenvalue 0, is $\xi=(c,c,\ldots,c)^{\top}$. Let $\lambda_1,\ \lambda_2,\ \ldots,\ \lambda_{qm-1}$ be the other eigenvalues of S, then $\lambda_j>0$. Moreover, for all $j\in\mathbb{Z}[1,qm-1]$

$$\lambda_{\min}=\min_j\lambda_j=2(1-\cos\frac{2\pi}{qm}),\ \lambda_{\max}=\max_j\lambda_j=\begin{cases}4, & \text{if } qm \text{ is even},\\ 2(1+\cos\frac{\pi}{qm}), & \text{if } qm \text{ is odd}.\end{cases}$$

Lemma 2.2. Let P be a matrix and $qm\geq 5$, such that for any $(\Delta x_1,\ldots,\Delta x_{qm})^{\top}\in\mathbb{R}^{qm}$,

$$(\Delta x_1,\cdots,\Delta x_{qm})^{\top}P(\Delta x_1,\cdots,\Delta x_{qm})=\sum_{k=1}^{qm}(\Delta x_k)^2+2\Delta x_{n-1}\Delta x_n,$$

where

$$P=\begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\ 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots\\ \vdots & \cdots & \ddots & \ddots & 0 & 0 & \cdots & \cdots & \cdots & \vdots\\ \vdots & \cdots & \cdots & 0 & 1 & 1 & 0 & \cdots & \cdots & \vdots\\ \vdots & \cdots & \cdots & 0 & 1 & 1 & 0 & \cdots & \cdots & \vdots\\ \vdots & \cdots & \cdots & \vdots & 0 & 0 & \ddots & 0 & \cdots & \vdots\\ \vdots & \cdots & \cdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & 0\\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}_{qm\times qm}.$$

Then the eigenvalues of P are $\underbrace{1,1,\ldots,1}_{qm-2}$, 0, 2. Moreover, matrix P has qm linearly independent eigenvectors, and when $x\in Y$, the eigenvalues of P are positive.

Proof. It is easy to compute that the eigenvalues of P are $\underbrace{1, 1, \ldots, 1}_{qm-2}$, 0, 2, and matrix P has qm linearly independent eigenvectors. Since $Z = \{(x_1, x_2, \ldots, x_{qm}) \in E_{qm} \,|\Delta x_{n-1} = -\Delta x_n = c,\ \Delta x_1 = \ldots = \Delta x_{n-2} = \Delta x_{n+1} = \ldots = \Delta x_{qm} = 0,\ c \in \mathbb{R}\}$ and $Y = Z^{\perp}$, then when $x \in Y$, the eigenvalues of P are positive.

Let $k \in \mathbb{Z}[1,\, qm]$, and $qm \geq 5$, we define our new functional $J(x)$ on E_{qm} as follows:

$$J(x) = \sum_{k=1}^{qm} \frac{2\lambda_{\max}^{\frac{p}{2}}(\frac{1}{C_1})^p}{p\lambda_{\min}\gamma_{\min}}[(\Delta x_k)^2 + 2\Delta x_{n-1}\Delta x_n] + \sum_{k=1}^{qm}[-\frac{1}{p}|\triangle x_k|^p + F(k, x_{k+1}, x_k)], \tag{2.1}$$

where $\gamma_{\min}$ is the smallest positive eigenvalue of P.

Clearly, for any $x = \{x_k\}_{k\in\mathbb{Z}} \in E_{qm}$, according to $x_0 = x_{qm}$, $x_1 = x_{qm+1}$, one computes that

$$\frac{\partial J}{\partial x_n} = \Delta\big(\varphi_p(\Delta x_{n-1})\big) + f(n,\, x_{n+1},\, x_n,\, x_{n-1}), \quad \forall\ n \in \mathbb{Z}[1, qm], \quad i = 1,\ 2.$$

Thus, the existence of critical points of J_i on E_{qm} may implies the existence of periodic solutions of system (1.1).

Remark 2.2. We have the following identity:

$$\frac{\partial\Big[(\Delta x_1, \cdots, \Delta x_{qm})^{\top} P(\Delta x_1, \cdots, \Delta x_{qm})\Big]}{\partial x_n} = \frac{\partial\Big[\sum_{k=1}^{qm}[(\Delta x_k)^2 + 2\Delta x_{n-1}\Delta x_n]\Big]}{\partial x_n} = 0.$$

Lemma 2.3. Let (A_3) be valid, then the functional $J(x)$ is bounded from above on E_{qm}.

Proof. It is clear that,

$$\sum_{k=1}^{qm}(\Delta x_k)^2 = \sum_{k=1}^{qm}(x_{k+1} - x_k)^2 = \sum_{k=1}^{qm}(2x_k^2 - 2x_k x_{k+1}). \tag{2.2}$$

Then, for all $x \in E_{qm}$, by (A_3) and Lemma 2.1, we have

$$\begin{aligned}
J(x) \quad &\leq \quad \sum_{k=1}^{qm}\Big[\frac{4\lambda_{\max}^{\frac{p}{2}}(\frac{1}{C_1})^p}{p\lambda_{\min}\gamma_{\min}}(\Delta x_k)^2\Big] + \sum_{k=1}^{qm}\Big[-\frac{1}{p}|\triangle x_k|^p + F(k, x_{k+1}, x_k)\Big] \\
&\leq \quad \frac{4\lambda_{\max}^{\frac{p}{2}}(\frac{1}{C_1})^p}{p\lambda_{\min}\gamma_{\min}}\sum_{k=1}^{qm}(\Delta x_k)^2 + \sum_{k=1}^{qm}F(k, x_{k+1}, x_k) \\
&\leq \quad \frac{4\lambda_{\max}^{\frac{p}{2}}(\frac{1}{C_1})^p}{p\lambda_{\min}\gamma_{\min}}\sum_{k=1}^{qm}2(x_k^2 - x_k x_{k+1}) - a_1\sum_{k=1}^{qm}\Big(\sqrt{x_{k+1}^2 + x_k^2}\Big)^{\beta} + a_2 qm \\
&\leq \quad \frac{4\lambda_{\max}^{\frac{p}{2}}(\frac{1}{C_1})^p}{p\lambda_{\min}\gamma_{\min}}x^{\top}Sx - a_1\sum_{k=1}^{qm}|x_k|^{\beta} + a_2 qm \\
&\leq \quad \frac{4\lambda_{\max}^{\frac{p}{2}+1}(\frac{1}{C_1})^p}{p\lambda_{\min}\gamma_{\min}}\|x\|^2 - a_1\big(\frac{1}{C_4}\big)^{\beta}\|x\|^{\beta} + a_2 qm.
\end{aligned} \tag{2.3}$$

Since $\beta \geq p+1 > 2$, from (2.3), there exists a constant $M_1 > 0$, such that $J(x) \leq M_1$ for every $x \in E_{qm}$.

Lemma 2.4. Let (A_3) hold, then $J(x)$ satisfies PS condition.

Proof. Let $x^{(j)} \in E_{qm}$, for all $j \in \mathbb{N}$, be such that $\{J(x^{(j)})\}$ is bounded. By Lemma 2.3, there exists $M_2 > 0$, such that

$$-M_2 \leq J(x^{(j)}) \leq \frac{4\lambda_{\max}^{\frac{p}{2}+1}(\frac{1}{C_1})^p}{p\lambda_{\min}\gamma_{\min}}\|x\|^2 - a_1\big(\frac{1}{C_4}\big)^{\beta}\|x\|^{\beta} + a_2 qm,$$

which implies

$$a_1\big(\frac{1}{C_4}\big)^{\beta}\|x\|^{\beta} - \frac{4\lambda_{\max}^{\frac{p}{2}+1}(\frac{1}{C_1})^p}{p\lambda_{\min}\gamma_{\min}}\|x\|^2 \leq M_2 + a_2 qm.$$

For $\beta > 2$, there exsit a constant $M_3 > 0$ such that for every $k \in \mathbb{N}$, $\|x^{(k)}\| \leq M_3$.

Therefore, $\{x^{(k)}\}$ is bounded in E_{qm}. Since E_{qm} is finite dimensional, then the P-S condition is satisfied.

Lemma 2.5. (Linking Theorem) [13, Theorem 5.3]. Let H be a real Hilbert space, $H = H_1 \oplus H_2$, where H_1 is a finite-dimensional subspace of H. Assume that $J \in C^1(H)$ satisfies the PS condition and

(D_1) there exist constants $\sigma > 0$ and $\rho > 0$, such that $J|_{\partial B_\rho \cap H_2} \geq \sigma$;

(D_2) there is an $e \in \partial B_1 \cap H_2$ and a constant $R_1 > \rho$, such that $J|_{\partial Q} \leq 0$,

where $Q = (\overline{B}_{R_1} \cap H_1) \oplus \{re| \, 0 < r < R_1\}$, B_ρ denotes the open ball in X with radius ρ and centered at 0 and ∂B_ρ represents its boundary. Then, J possesses a critical value $c \geq \sigma$, here

$$c = \inf_{h\in\Gamma}\max_{u\in Q} J(h(u)), \Gamma = \{h \in C(\overline{Q}, H) \mid h|_{\partial Q} = id\},$$

and id denotes the identity operator.

3. Proof of main result

It is time for us to give details for proving the Theorem 1.1.

Proof. For any $x \in Y$, let $\Delta x = (\Delta x_1, \Delta x_2, \ldots, \Delta x_{qm})^\top$, from (2.1)-(2.2), we have

$$\begin{aligned} J(x) \quad \geq \quad & \frac{2\lambda_{\max}^{\frac{p}{2}}(\frac{1}{C_1})^p}{p\lambda_{\min}\gamma_{\min}}\sum_{k=1}^{qm}\Big[(\Delta x_k)^2 + 2\Delta x_{n-1}\Delta x_n\Big] \\ & -\frac{1}{p}\big(\frac{1}{C_1}\big)^p\bigg([\sum_{k=1}^{qm}|\triangle x_k|^2]^{\frac{1}{2}}\bigg)^p + \sum_{k=1}^{qm}F(k, x_{k+1}, x_k) \\ = \quad & \frac{2\lambda_{\max}^{\frac{p}{2}}(\frac{1}{C_1})^p}{p\lambda_{\min}\gamma_{\min}}(\Delta x)^\top P(\Delta x) - \frac{1}{p}\big(\frac{1}{C_1}\big)^p[\sum_{k=1}^{qm}(2x_s^2 - 2x_k x_{k+1})]^{\frac{p}{2}} \\ & + \sum_{k=1}^{qm}F(k, x_{k+1}, x_k) \\ = \quad & \frac{2\lambda_{\max}^{\frac{p}{2}}(\frac{1}{C_1})^p}{p\lambda_{\min}\gamma_{\min}}(\Delta x)^\top P(\Delta x) - \frac{1}{p}\big(\frac{1}{C_1}\big)^p\bigg(x^\top Sx\bigg)^{\frac{p}{2}} + \sum_{k=1}^{qm}F(k, x_{k+1}, x_k). \end{aligned} \tag{3.1}$$

In view of condition (A_2), we have

$$\lim_{\rho\to 0}\frac{F(t,\, u,\, v)}{\rho^p} = 0, \quad \rho = \sqrt{u^2 + v^2}.$$

Now, if one chooses $\varepsilon = 2^{-\frac{p}{2}-2}(\frac{1}{p})\lambda_{\max}^{\frac{p}{2}}(\frac{1}{C_1})^p$, there exists a sufficiently small positive number δ, such that

$$|F(t,\, u,\, v)| \le 2^{-\frac{p}{2}-2}(\frac{1}{p})\lambda_{\max}^{\frac{p}{2}}\left(\sqrt{u^2+v^2}\right)^p,\ \forall\ \rho < \delta.$$

For $\|x\| \le \delta$, with the help of Lemma 2.2, from (3.1), we have

$$\begin{aligned} J(x) &\ge \frac{2\lambda_{\max}^{\frac{p}{2}}(\frac{1}{C_1})^p}{p\lambda_{\min}}\|\Delta x\|^2 - \frac{1}{p}(C_1)^p\left(x^\top Sx\right)^{\frac{p}{2}} \\ &\quad -2^{-\frac{p}{2}-2}(\frac{1}{p})\lambda_{\max}^{\frac{p}{2}}\sum_{k=1}^{qm}[2^{\frac{p}{2}}\max\{|x_{k+1}|^p,\, |x_k|^p\}] \\ &\ge \frac{2\lambda_{\max}^{\frac{p}{2}}(\frac{1}{C_1})^p}{p\lambda_{\min}}\|\Delta x\|^2 - \frac{1}{p}(\frac{1}{C_1})^p\left(x^\top Sx\right)^{\frac{p}{2}} - 2^{-\frac{p}{2}-2}(\frac{1}{p})\lambda_{\max}^{\frac{p}{2}}2^{\frac{p}{2}+1}\|x\|_p^p \\ &\ge \frac{2\lambda_{\max}^{\frac{p}{2}}(\frac{1}{C_1})^p}{p\lambda_{\min}}\|\Delta x\|^2 - \frac{1}{p}(\frac{1}{C_1})^p\left(x^\top Sx\right)^{\frac{p}{2}} \\ &\quad -2^{-\frac{p}{2}-2}(\frac{1}{p})\lambda_{\max}^{\frac{p}{2}}2^{\frac{p}{2}+1}(\frac{1}{C_1})^p\|x\|^p. \end{aligned} \tag{3.2}$$

By Lemma 2.1 and Remark 2.2, from (3.2), then

$$\begin{aligned} J(x) &\ge \frac{2\lambda_{\max}^{\frac{p}{2}}(\frac{1}{C_1})^p}{p\lambda_{\min}}\|\Delta x\|^2 - \frac{1}{p}(\frac{1}{C_1})^p\lambda_{\max}^{\frac{p}{2}}\|x\|^p - \frac{1}{2p}\lambda_{\max}^{\frac{p}{2}}(\frac{1}{C_1})^p\|x\|^p \\ &= \frac{2\lambda_{\max}^{\frac{p}{2}}(\frac{1}{C_1})^p}{p\lambda_{\min}}x^\top Sx - \frac{1}{p}(\frac{1}{C_1})^p\lambda_{\max}^{\frac{p}{2}}\|x\|^p - \frac{1}{2p}\lambda_{\max}^{\frac{p}{2}}(\frac{1}{C_1})^p\|x\|^p \\ &\ge \frac{2\lambda_{\max}^{\frac{p}{2}}(\frac{1}{C_1})^p}{p}\|x\|^2 - \frac{3}{2p}\lambda_{\max}^{\frac{p}{2}}(\frac{1}{C_1})^p\|x\|^p, \end{aligned} \tag{3.3}$$

where $x = (x_1, x_2, \ldots, x_{qm})^\top$.

Observing that δ is sufficiently small and $p \ge 2$, thus we get from (3.3)

$$J(x) \ge \frac{2}{p}\lambda_{\max}^{\frac{p}{2}}(\frac{1}{C_1})^p\|x\|^2 - \frac{3}{2p}\lambda_{\max}^{\frac{p}{2}}(\frac{1}{C_1})^p\|x\|^p = \frac{1}{2p}\lambda_{\max}^{\frac{p}{2}}(\frac{1}{C_1})^p\|x\|^2.$$

If one takes $\sigma = \frac{1}{2p}\lambda_{\max}^{\frac{p}{2}}\big(\frac{1}{C_1}\big)^p\delta^2$, then

$$J(x) \ge \sigma > 0, \qquad \forall\, x \in Y \cap \partial B_\delta.$$

So,

$$c_1 = \sup_{x\in E_{qm}} J(x) \ge \sigma > 0,$$

which hints that J satisfies the condition (D_1) in Lemma 2.5.

Finally, we verify condition (D_2) of the linking theorem. By Lemma 2.4, $J(x)$ meets P-S condition. Taking $e \in \partial B_1 \cap Y$, for any $z \in Z$, $r \in \mathbb{R}$, let $x = re + z$, from (2.6),

$$J(x) \le \frac{4\lambda_{\max}^{\frac{p}{2}+1}(\frac{1}{C_1})^p}{p\lambda_{\min}\gamma_{\min}}\|x\|^2 - a_1\big(\frac{1}{C_4}\big)^\beta\|x\|^\beta + a_2qm.$$

It is clear that, there exists a big enough constant $R_3 > 0$, such that $J(x) \leq 0$, for all $x \in \partial Q$, where

$$Q = (\overline{B}_{R_3} \cap Z) \oplus \{re| \, 0 < r < R_3\}.$$

Employing linking theorem (Lemma 2.5), J exists a critical value $c \geq \sigma > 0$, where

$$c = \inf_{h \in \Gamma} \max_{x \in Q} J(h(x)), \qquad \Gamma = \{h \in C(\overline{Q}, \, E_{qm}) | h|_{\,\partial Q} = id\}.$$

From Lemma 2.3, we get $\lim\limits_{\|x\| \to \infty} J(x) = -\infty$, so $-J$ is coercive. Set $c_1 = \sup\limits_{x \in E_{qm}} J(x)$. By the continuity of J on E_{qm}, there exists $\bar{x} \in E_{qm}$, such that $J(\bar{x}) = c_1$, and $\bar{x}$ is a critical point of J. Obviously, when $x_1 = \ldots = x_{qm}$, we have $\Delta x_1 = \ldots = \Delta x_{qm} = 0$. Employing (2.4) and $F(t, \, u, \, v) \leq 0$, we obtain

$$J(x) = \sum_{k=1}^{qm} F(k, x_{k+1}, x_k) \leq 0,$$

Thus, $J(x)$ does not acquire its maximum c_1. Then, the critical point associated with the critical value c_1 of J is a nontrivial qm-periodic solutions of system (1.1).

By now, we obtain a nontrivial qm-periodic solution. The rest of the proof of the other nontrivial qm-periodic solution is similar to that of [7, Theorem 1.1] or [4, Theorem 3.1], we omit the details. Now, the proof of our Theorem 1.1 is now complete, that means system (1.1) has at least two nontrivial qm-periodic solutions.

Remark 3.1 In the proof of Theorem 1.1, in order to obtain that $J(x)$ satisfies condition (D_1) of the linking theorem, we let $p \in [2, \infty)$.

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