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ON SOME IDENTITIES AND SYMMETRIC FUNCTIONS FOR LUCAS AND PELL NUMBERS

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Abstract. In this paper, we show how the action of the symmetrizing endomorphism operator $\delta^k_{e_1e_2}$ to the series $\sum_{n=0}^{\infty} a_n e_1^n z^n$ allows us to obtain an alternative approach for the determination of Fibonacci and Lucas-Pell numbers.

1. Introduction and Notations

The second-order linear recurrence sequence $(U_n(a,b;p,q))_{n\geq 0}$, or briefly $(U_n)_{n\geq 0}$, is defined by

$$U_{n+2} = pU_{n+1} + qU_n,\ U_0 = a,\ U_1 = b.$$

Where $a,\ b$ and $p,\ q$ are arbitrary real numbers for $n > 0$. The Binet formula for the sequence $(U_n)_{n\geq 0}$ is

$$U_n = \frac{c_1 x_1^n - c_2 x_2^n}{x_1 - x_2},$$

where $c_1 = b - ax_2$ and $c_2 = b - ax_1$ [5]. Certain sequence of numbers that appeared here are Fibonacci number $(F_n)_{n\geq 0}$, if we take $p = q = b = 1, a = 2$, Lucas number $(L_n)_{n\geq 0}$ for $p = 2, q = b = 1, a = 0$, Pell number $(P_n)_{n\geq 0}$ and Pell-Lucas number $(Q_n)_{n\geq 0}$, when one has $p = b = a = 2, q = 1$. In this paper, we show that the use of the action of the symmetric endomorphism operator $\delta^k_{e_1e_2}$ [4] to the series $\sum_{n=0}^{\infty} a_n (e_1 z)^n$, gives an alternative approach for determining the generating functions of some sequences of numbers cited above.

Let k and n be two positive integer and $\{x_1, x_2, ..., x_n\}$ are set of given variables, recall [8] that the k-th elementary symmetric function $e_k(x_1, x_2, ..., x_n)$ and the k-th complete homogeneous symmetric function $h_k(x_1, x_2, ..., x_n)$ are defined respectively by

$$e_k(x_1, x_2, ..., x_n) = \sum_{i_1+i_2+...+i_n=k} x_1^{i_1} x_2^{i_2} ... x_n^{i_n}\ ,\ 0 \leq k \leq n,$$

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with $i_1, i_2, ..., i_n = 0$ or 1.

$$h_k(x_1, x_2, ..., x_n) = \sum_{i_1+i_2+...+i_n=k} x_1^{i_1} x_2^{i_2} ... x_n^{i_n} \ , \ 0 \leq k \leq n,$$

with $i_1, i_2, ..., i_n \geq 0$.

First, we set $e_0(x_1, x_2, ..., x_n) = 1$ and $h_0(x_1, x_2, ..., x_n) = 1$ (by convention). For $k > n$ or $k < 0$, we set $e_k(x_1, x_2, ..., x_n) = 0$ and $h_k(x_1, x_2, ..., x_n) = 0$.

Lemma 1 [10] The relations

$$\begin{aligned} 1)\ F_{-n} &= (-1)^{n+1} F_n, \\ 2)\ L_{-n} &= (-1)^n L_n, \\ 3)\ P_{-n} &= (-1)^{n+1} P_n, \\ 4)\ Q_{-n} &= (-1)^n Q_n \end{aligned}$$

hold for all $n \geq 0$.

Definition 1 Let A and E be any two alphabets, then we give $S_n(A - E)$ by the following form:

$$\frac{\Pi_{e \epsilon E}(1 - ez)}{\Pi_{a \epsilon A}(1 - az)} = \sum_{n=0}^{\infty} S_n(A - E) z^n, \tag{1}$$

with the condition $S_n(A - E) = 0$ for $n < 0$ (see [1]).

Corollary 1 Taking $A = 0$ in (1.1), that gives

$$\Pi_{e \epsilon E}(1 - ez) = \sum_{n=0}^{\infty} S_n(-E) z^n. \tag{2}$$

Definition 2 [7] Given a function g on $\mathbb{R}^n$, the divided difference operator is defined as follows:

$$\partial_{x_i x_{i+1}}(g) = \frac{g(x_1, \cdots, x_i, x_{i+1}, \cdots x_n) - g(x_1, \cdots x_{i-1}, x_{i+1}, x_i, x_{i+2} \cdots x_n)}{x_i - x_{i+1}}.$$

It should be noted that the divided difference operator $\partial_{x_i x_{i+1}}$ commutes with symmetric functions at x_i, x_{i+1} and is compatible with the function S_n [6].

Definition 3 [2] The symmetrizing operator $\delta^k_{e_1 e_2}$ is defined by

$$\delta^k_{e_1 e_2}(f) = \frac{e_1^k f(e_1) - e_2^k f(e_2)}{e_1 - e_2} \text{ for all } k \in \mathbb{N}.$$

Proposition 1 [3] Let $E = \{e_1, e_2\}$ an alphabet, we define the operator $\delta^k_{e_1 e_2}$ as follows:

$$\delta^k_{e_1 e_2} f\ (e_1) = S_{k-1}(e_1 + e_2) f\ (e_1) + e_2^k \partial_{e_1 e_2} f\ (e_1),\ \textit{for all}\ k \in \mathbb{N}.$$

2. The Main Result

In our main result, we will combine all these results in a unified way such that all these obtained results can be treated as special case of the following Theorem.

Theorem 1 Given an alphabet $E = \{e_1, e_2\}$ and two sequences $\sum_{n=0}^{+\infty} a_n z^n$, $\sum_{n=0}^{+\infty} b_n z^n$ such that $\left(\sum_{n=0}^{+\infty} a_n z^n\right)\left(\sum_{n=0}^{+\infty} b_n z^n\right) = 1$, then

$$\sum_{n=0}^{\infty} a_n\, \delta_{e_1e_2}^{k+n-1}(e_1) z^n = \frac{\sum\limits_{n=0}^{k-1} b_n (e_1e_2)^n \delta_{e_1e_2}^{k-n}(e_1^{-1}) z^n - (e_1e_2z)^k \sum\limits_{n=0}^{\infty} b_{n+k+1} \delta_{e_1e_2}(e_1^n) z^{n+1}}{\left(\sum\limits_{n=0}^{\infty} b_n e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} b_n e_2^n z^n\right)}. \tag{3}$$

Proof. Let $\sum\limits_{n=0}^{\infty} a_n z^n$ and $\sum\limits_{n=0}^{\infty} b_n z^n$ be two sequences as $\sum\limits_{n=0}^{\infty} a_n z^n \times \sum\limits_{n=0}^{\infty} b_n z^n = 1$. On one hand, since $f(e_1) = \sum\limits_{n=0}^{\infty} a_n e_1^n z^n$, we have

$$\begin{aligned} \delta_{e_1e_2}^{k} f\ (e_1) &= \delta_{e_1e_2}^{k} \left(\sum_{n=0}^{\infty} a_n e_1^n z^n\right) \\ &= \sum_{n=0}^{\infty} a_n\ \delta_{e_1e_2}^{k+n-1}(e_1) z^n, \end{aligned}$$

which is the left hand side of (3). On the other hand, since

$$f(e_1) = \frac{1}{\sum\limits_{n=0}^{\infty} b_n e_1^n z^n},$$

we have that

$$\begin{aligned} \partial_{e_1e_2} f(e_1) &= \frac{1}{e_1 - e_2}\left(\frac{1}{\sum\limits_{n=0}^{\infty} b_n e_1^n z^n} - \frac{1}{\sum\limits_{n=0}^{\infty} b_n e_2^n z^n}\right) \\ &= \frac{1}{e_1 - e_2}\left(\frac{\sum\limits_{n=0}^{\infty} b_n e_2^n z^n - \sum\limits_{n=0}^{\infty} b_n e_1^n z^n}{\left(\sum\limits_{n=0}^{\infty} b_n e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} b_n e_2^n z^n\right)}\right) \\ &= \frac{\sum\limits_{n=0}^{\infty} b_n \frac{e_2^n - e_1^n}{e_1 - e_2} z^n}{\left(\sum\limits_{n=0}^{\infty} b_n e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} b_n e_2^n z^n\right)} \\ &= -\frac{\sum\limits_{n=0}^{\infty} b_n S_{n-1}(e_1 + e_2) z^n}{\left(\sum\limits_{n=0}^{\infty} b_n e_1^n z^n\right)\left(\sum\limits_{n=0}^{\infty} b_n e_2^n z^n\right)}. \end{aligned}$$

By Proposition 1, it follows that

$$\begin{aligned}\delta^k_{e_1e_2}f\ (e_1) &= S_{k-1}(e_1+e_2)f\ (e_1)+e_2^k\partial_{e_1e_2}f\ (e_1)\\ &= \frac{S_{k-1}(e_1+e_2)}{\sum\limits_{n=0}^{\infty}b_ne_1^nz^n}-e_2^k\frac{\sum\limits_{n=0}^{\infty}b_nS_{n-1}(e_1+e_2)z^n}{\left(\sum\limits_{n=0}^{\infty}b_ne_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}b_ne_2^nz^n\right)}\\ &= \frac{\sum\limits_{n=0}^{\infty}b_n\left[e_2^nS_{k-1}(e_1+e_2)-e_2^kS_{n-1}(e_1+e_2)\right]z^n}{\left(\sum\limits_{n=0}^{\infty}b_ne_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}b_ne_2^nz^n\right)}.\end{aligned}$$

Hence, we have that

$$\begin{aligned}\delta^k_{e_1e_2}f\ (e_1) &= \frac{\sum\limits_{n=0}^{k-1}b_n\left[e_2^nS_{k-1}(e_1+e_2)-e_2^kS_{n-1}(e_1+e_2)\right]z^n}{\left(\sum\limits_{n=0}^{\infty}b_ne_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}b_ne_2^nz^n\right)}+\frac{\sum\limits_{n=k+1}^{\infty}b_n\left[e_2^nS_{k-1}(e_1+e_2)-e_2^kS_{n-1}(e_1+e_2)\right]z^n}{\left(\sum\limits_{n=0}^{\infty}b_ne_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}b_ne_2^nz^n\right)}\\ &= \frac{\sum\limits_{n=0}^{k-1}b_n(e_1e_2)^n\delta^{k-n}_{e_1e_2}(e_1^{-1})z^n-(e_1e_2z)^k\sum\limits_{n=0}^{\infty}b_{n+k+1}\delta_{e_1e_2}(e_1^n)z^{n+1}}{\left(\sum\limits_{n=0}^{\infty}b_ne_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty}b_ne_2^nz^n\right)}.\end{aligned}$$

This completes the proof.

3. On the Symmetric Functions of Some numbers

In this part, we derive the new generating functions of some known numbers. Indeed, we consider Theorem 1 in order to get Fibonacci numbers, Lucas numbers and Pell-Lucas numbers with $k=1$ and $k=2$, for the case $\frac{1}{1+z}=\sum\limits_{n=0}^{\infty}(-1)^nz^n$.

Lemma 2 Given an aphabet $E=\{e_1,e_2\}$, we have

$$\sum_{n=0}^{\infty}(-1)^nh_n(e_1,e_2)z^n=\frac{1}{(1+e_1z)(1+e_2z)},\ \text{with } h_n(e_1,e_2)=S_n(e_1+e_2). \tag{4}$$

Lemma 3 Given an aphabet $E=\{e_1,e_2\}$, we have

$$\sum_{n=0}^{\infty}(-1)^nh_{n+1}(e_1,e_2)z^n=\frac{e_1+e_2+e_1e_2z}{(1+e_1z)(1+e_2z)},\ \text{with } h_{n+1}(e_1,e_2)=S_{n+1}(e_1+e_2). \tag{5}$$

By replacing e_2 by $(-e_2)$ in (4) and (5), we obtain

$$\sum_{n=0}^{\infty}(-1)^nS_n(e_1+[-e_2])z^n=\frac{1}{1+(e_1-e_2)z-e_1e_2z^2}, \tag{6}$$

$$\sum_{n=0}^{\infty}(-1)^nS_{n+1}(e_1+[-e_2])z^n=\frac{e_1-e_2-e_1e_2z}{1+(e_1-e_2)z-e_1e_2z^2}, \tag{7}$$

Choosing e_1 and e_2 such that $\begin{cases} e_1e_2 = 1 \\ e_1 - e_2 = 1 \end{cases}$ and substituting in (6) and (7) we get

$$\sum_{n=0}^{\infty} F_{-n}z^n = \frac{1}{z^2 - z - 1}, \tag{8}$$

which represent a generating function for Fibonacci numbers such that $F_{-n} = (-1)^{n+1}S_n(e_1 + [-e_2])$.

$$\sum_{n=0}^{\infty}(-1)^n S_{n+1}(e_1 + [-e_2])z^n = \frac{1 - z}{1 + z - z^2}, \tag{9}$$

which is given by Boussayoud et al [3].

Corollary 2 For $n \in \mathbb{N}$, we have

$$S_{n+2}(e_1 + [-e_2]) = S_{n+1}(e_1 + [-e_2]) + S_n(e_1 + [-e_2]).$$

Choosing e_1 and e_2 such that $\begin{cases} e_1e_2 = 1 \\ e_1 - e_2 = 2 \end{cases}$ and substituting in (6) and (7). where we have

$$\sum_{n=0}^{\infty}(-1)^n S_n(e_1 + [-e_2])z^n = \frac{1}{1 + 2z - z^2}, \tag{10}$$

which yields also new generating functions.

$$\sum_{n=0}^{\infty}(-1)^n S_{n+1}(e_1 + [-e_2])z^n = \frac{2 - z}{1 + 2z - z^2}, \tag{11}$$

Multiplying the equation (8) by 3 and subtract it from (9) we get

$$\sum_{n=0}^{\infty} L_{-n}z^n = \frac{2 + z}{1 + z - z^2},$$

which represents a new generating function for Lucas Numbers.

Corollary 3 For all $n \in \mathbb{N}$, we have

$$L_{-n} = (-1)^n \left[3S_n(e_1 + [-e_2]) - S_{n+1}(e_1 + [-e_2])\right].$$

Multiplying the equation (10) by (-2) and added to (11) we obtain

$$\sum_{n=0}^{\infty} P_{-n}z^n = \frac{z}{1 + 2z - z^2},$$

which represents a new generating function for Pell Numbers.

Corollary 4 For all $n \in \mathbb{N}$, we have

$$P_{-n} = (-1)^{n+1} \left[S_{n+1}(e_1 + [-e_2]) - 2S_n(e_1 + [-e_2])\right].$$

Multiplying the equation (10) by 6 and added to (11) by (-2), we have

$$\sum_{n=0}^{\infty} Q_{-n}z^n = \frac{2 + 2z}{1 + 2z - z^2},$$

which represents a new generating function for Pell-Lucas Numbers.

Corollary 5 For all $n \in \mathbb{N}$, we have

$$P_{-n} = (-1)^n \left[6S_n(e_1 + [-e_2]) - 2S_{n+1}(e_1 + [-e_2])\right].$$

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