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GLOBAL EXISTENCE AND ENERGY DECAY OF SOLUTIONS TO A VISCOELASTIC TIMOSHENKO BEAM SYSTEM WITH A NONLINEAR TIME VARYING DELAY TERM IN THE WEAKLY NONLINEAR INTERNAL FEEDBACKS

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Abstract. In this paper, we consider the viscoelastic Timoshenko system with a delay term in the weakly nonlinear internal feedback in a bounded domain:

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) + \mu_1(t)g_1(\psi_t(x,t)) \\ \qquad + \int_0^t h(t-s)\psi_{xx}(x,s)\,ds + \mu_2(t)g_2(\psi_t(x,t-\tau(t))) = 0, \end{cases}$$

We prove a global existence result using the energy method combined with the Faedo- Galerkin procedure under a condition between the weight of the delay term in the feedback and the weight of the term without delay. Furthermore, we establish a decay rate estimate for the energy by introducing suitable Lyapunov functionals.

1. Introduction

We investigate the existence and decay properties of solutions for the initial boundary value problem of the nonlinear Timoshenko system. The system is given by the two coupled hyperbolic equations
(P)

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0 & \text{in }]0,1[\times]0,+\infty[, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) + \mu_1(t)g_1(\psi_t(x,t)) & \\ \qquad + \int_0^t h(t-s)\psi_{xx}(x,s)\,ds + \mu_2(t)g_2(\psi_t(x,t-\tau(t))) = 0 & \text{in }]0,1[\times]0,+\infty[, \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0 & t \geq 0, \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x) & x \in]0,1[, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x) & x \in]0,1[, \\ \psi_t(x,t-\tau(0)) = f_0(x,t-\tau(0)) & \text{in }]0,1[\times]0,\tau(0)[, \end{cases}$$

where t denotes the time variable and x is the space variable along the beam of length $L = 1$ in its equilibrium configuration.The unknowns $\varphi = \varphi(x,t)$ and

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$\psi = \psi(x,t)$ represent, respectively, the transverse displacement of the beam and the rotation angle of the filament of the beam. In (P), $\rho_1 = \rho$, $\rho_2 = I_\rho$, $b = EI$, where ρ, I_ρ, E, I and K are, respectively, the density (the mass per unit length), the polar moment of inertia of a cross-section, Young's modulus of elasticity, the moment of inertia of a cross-section and the shear modulus.

where h is positive non-increasing function defined on R^+, g_1, g_2 and μ_1, μ_2 are four functions, $\tau(t) > 0$ is a time varying delay, and the initial data $(\varphi_0, \varphi_1, \psi_0, \psi_1, f_0)$ belong to a suitable function space.

In absence of the viscoelastic term (that is, if $h = 0$), problem (P) has been studied by many mathematicians. It is well known that in the further absence of a damping mechanism, the delay term causes instability of system (see, for instance [12]). In contrast, in absence of the delay term, the damping term assures global existence for arbitrary initial data and energy decay estimates depending on the rate of growth of g_1 (see [5, 8, 9, 10, 17, 18, 23, 25, 26, 28]). In addition, we would like to mention the most recent work in this direction due to Alabau-Boussouira [3] which is the pioneer in establishing very general explicit decay rate estimates for solutions.

In recent years, PDEs with time delay effects have become an active area of research and arise in many practical problems (see, e.g. [1, 29, 31]). To stabilize a hyperbolic system involving delay terms, additional control terms are necessary (see [7, 20, 27]).

For instance, in [27] the authors studied a wave equation with a linear internal damping term with constant delay and determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially decaying if $\mu_2 < \mu_1$ and they also found a sequence of delays for which the corresponding solution of (P) will be instable if $\mu_2 \geq \mu_1$. The main approach used in [27], is an observability inequality obtained with a Carleman estimate. Laskri and Said-Houari [20] examined problem (P) in a linear situation with $h \equiv 0$, $g_1(s) = g_2(s) = s$ for all $s \in R$. Under the assumption $\mu_2 \leq \mu_1$ on the weights of the two feedbacks, they proved the well-posedness of the system. They also established for $\mu_2 < \mu_1$ an exponential decay result for the case of equal wave propagation speeds. Very recently, Benaissa and Bahlil [7] extended the result of [20] to the nonlinear case.

In the presence of the viscoelastic term ($h \neq 0$), Benaissa et al. [6] studied a viscoelastic wave equation in the presence of nonlinear delay term. They obtained an explicit decay rate. In Guesmia et al. [14] considered (P) for $g_2 \equiv 0$ and studied the influence of these dissipations on the decay rate of solutions. Precisely, they obtained an explicit and general decay rate, depending on g_1 and h, for the energy of solutions without any growth assumption on g_1 at the origin and under weaker conditions on the relaxation function h.

Our goal in this paper is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to the problem (P) for a nonlinear damping and a delay term. We should mention here that, to the best of our knowledge, there is no result concerning Timochenko beam system with the presence of nonlinear degenerate delay term.

To obtain global solutions to the problem (P), we use the argument combining the Galerkin approximation scheme (see [21]) with the energy estimate method. The technic based on the theory of nonlinear semigroups used in [27] and the variable norm of Kato in Refs. [15] and [16] do not seem to be applicable in the nonlinear case.

To prove decay estimates, we use a perturbed energy method and some properties of convex functions. These arguments of convexity were introduced and developed by [11] and [19] and used by Liu and Zuazua [22] , Eller et al [13] and Alabau-Boussouira [3].

2. Preliminaries and main results

For the relaxation function, the damping and the delay functions, we make the following hypotheses:

(H1) $(*)$ $h : R_+ \to R_+$ is a C^2 function satisfying

$$h(0) = h_0 > 0, \qquad l = \int_0^{+\infty} h(s)ds < 1.$$

$(**)$ There exists a non-increasing differentiable function $\zeta : R_+ \to R_+$ such that

$$h'(s) \le -\zeta(s)h(s), \qquad \forall s \ge 0.$$

and

$$\int_0^{+\infty} \zeta(s)\, ds = +\infty.$$

(H2) $\mu_1 : R_+ \to]0, +\infty[$ is a non-increasing function of class $C^1(R_+)$ satisfying

$$\int_0^{+\infty} \mu_1(\tau)\, d\tau = +\infty, \tag{1}$$

$$|\mu_1'(t)| \le c\mu_1(t). \tag{2}$$

(H3) $\mu_2 : R_+ \to R$ is a function of class $C^1(R_+)$, which is not necessarily positive or monotone, such that

$$\big|\mu_2(t)\big| \le \beta\mu_1(t), \tag{3}$$

$$\big|\mu_2'(t)\big| \le \tilde{c}\mu_1(t), \tag{4}$$

for some $0 < \beta < 1$ and $\tilde{c} > 0$.

(H4) $g_1 : R \to R$ is a non-decreasing function of the class $C(R)$ such that there exist ϵ_1, c_1, $c_2 > 0$ and a convex and increasing function $H : R_+ \to R_+$ of the class $C^1(R_+) \cap C^2(]0, \infty[)$ satisfying $H(0) = 0$, and H linear on $[0, \epsilon_1]$ or ($H'(0) = 0$ and $H'' > 0$ on $]0, \epsilon_1]$), such that

$$c_1|s| \le |g_1(s)| \le c_2|s| \quad \text{if } |s| \ge \epsilon_1, \tag{5}$$

$$g_1^2(s) \le H^{-1}(sg_1(s)) \quad \text{if } |s| \le \epsilon_1. \tag{6}$$

$g_2 : R \to R$ is an odd non-decreasing function of the class $C^1(R)$ such that there exist $c_3, \alpha_1, \alpha_2 > 0$

$$|g_2'(s)| \le c_3 \tag{7}$$

$$\alpha_1\ sg_2(s) \le G_2(s) \le \alpha_2\ sg_1(s), \tag{8}$$

where

$$G_2(s) = \int_0^s g_2(r)\, dr$$

(H5) τ is a function such that

$$\tau \in W^{2,\infty}([0,T]), \forall T > 0 \tag{9}$$

$$0 < \tau_0 \le \tau(t) \le \tau_1, \quad \forall t > 0 \tag{10}$$

where τ_0 and τ_1 are two positive constants.

$$\tau'(t) \le d < 1, \quad \forall t > 0, \tag{11}$$

where d is a positive constant.

(H6) The weight of dissipation and the delay satisfy:

$$\beta < \frac{\alpha_1(1-d)}{\alpha_2(1-\alpha_1 d)}. \tag{12}$$

We introduce as in [27] the new variable

$$z(x,\rho,t) = \psi_t(x, t-\tau(t)\rho), x \in (0,1),\ \rho \in (0,1), \quad t > 0. \tag{13}$$

Then, we have

$$\tau(t)z_t(x,\rho,t) + (1-\tau'(t)\rho)z_\rho(x,\rho,t) = 0, \text{ in } (0,1)\times(0,1)\times(0,+\infty). \tag{14}$$

Therefore, problem (P) is equivalent to:

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0 & in\ ]0,1[\times]0,+\infty[, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) + \mu_1(t)g_1(\psi_t(x,t)) & \\ \qquad + \int_0^t h(t-s)\psi_{xx}(x,s)ds + \mu_2(t)g_2(z(x,1,t)) = 0 & in\ ]0,1[\times]0,+\infty[, \\ \tau(t)z_t(x,\rho,t) + (1-\tau'(t)\rho)z_\rho(x,\rho,t) = 0, & in\ ]0,1[\times]0,1[\times]0,+\infty[, \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0 & t \ge 0, \\ z(x,0,t) = \psi_t(x,t) & on\ ]0,1[\times[0,+\infty[, \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x) & x \in]0,1[, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x) & x \in]0,1[, \\ z(x,\rho,0) = f_0(x,-\rho\tau(0)) & in\ ]0,1[\times]0,1[, \end{cases} \tag{15}$$

Let $\bar{\xi}$ be a positive constant such that

$$\frac{\beta(1-\alpha_1)}{\alpha_1(1-d)} < \bar{\xi} < \frac{1-\alpha_2\beta}{\alpha_2}. \tag{16}$$

Now, inspired by [17], we define the energy functional as

$$\begin{aligned} E(t) = E(t,z,\varphi,\psi) &= \tfrac{1}{2}\int_0^1 \left\{\rho_1\varphi_t^2 + \rho_2\psi_t^2 + K|\varphi_x+\psi|^2\right\}dx \\ &+ \tfrac{1}{2}\left(b - \int_0^t h(s)\,ds\right)\int_0^1 \psi_x^2(x)\,dx + \tfrac{1}{2}(h\circ\psi_x)(t) \\ &+ \xi(t)\tau(t)\int_0^1\int_0^1 G_2(z(x,\rho,t))\,d\rho\,dx. \end{aligned} \tag{17}$$

where

$$\xi(t) = \bar{\xi}\mu_1(t).$$

We have the following theorem.

Theorem 1 Let $(\varphi_0,\varphi_1), (\psi_0,\psi_1) \in (H^2(0,1)\cap H_0^1(0,1)) \times H_0^1(0,1)$, $f_0 \in H_0^1((0,1);H^1(0,1))$ satisfy the compatibility condition

$$f_0(.,0) = \psi_1.$$

Assume that the hypothesis $(\mathbf{H1}) - (\mathbf{H2}) - (\mathbf{H3}) - (\mathbf{H4})$ holds. Then the problem (P) admits a unique weak solution

$$\begin{aligned} &\psi,\varphi \in L^\infty([0,\infty);H^2(0,1)\cap H_0^1(0,1)), \quad \psi_t,\varphi_t \in L^\infty([0,\infty);H_0^1(0,1)), \\ &\psi_{tt},\varphi_{tt} \in L^\infty([0,\infty);L^2(0,1)) \end{aligned}$$

and, for some constants ω_1, ω_2 and ω_3, ϵ_0 we obtain the following decay property:

$$E(t) \le H_1^{-1}\left(\omega\int_0^t \mu_1(s)\zeta(s)\,ds\right), \quad \forall t > 0, \tag{18}$$

and

$$H_1(t) = \int_t^1 \frac{1}{H_2(s)}\, ds \tag{19}$$

and

$$H_2(t) = \begin{cases} t & \text{if } H \text{ is linear on } [0, \epsilon'], \\ tH'(\epsilon_0 t) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0, \epsilon']. \end{cases}$$

Example. Let g be given by $g(s) = s^p(-\ln s)^q$, where $0 \le p \le 1$ and $q \in R$ on $(0, \epsilon_1]$. Then $g'(s) = s^{p-1}(-\ln s)^{q-1}(p(-\ln s) - q)$ which is an increasing function in a right neighborhood of 0 (if $q = 0$ we can take $\epsilon_1 = 1$). The function H is defined in the neighborhood of 0 by

$$H(s) = \begin{cases} cs^{\frac{p+1}{2p}}(-\ln s)^{-\frac{q}{p}} & \text{if } 0 < p < 1, \quad q \in R \\ cs(-\ln s)^{-q} & \text{if } p = 1, \quad q > 0 \\ c\sqrt{s}\; e^{-s^{\frac{1}{2q}}} & \text{if } p = 0, \quad q < 0 \end{cases}$$

and we have

$$H'(s) = \begin{cases} cs^{\frac{1-p}{2p}}(-\ln s)^{-\frac{p+q}{p}}\left(\frac{p+1}{2p}(-\ln s) + \frac{q}{p}\right) & \text{if } 0 < p \le 1, \quad q \in R \\ c\frac{1}{\sqrt{s}}\left(1 - \frac{1}{q}\; s^{\frac{1}{2q}}\right)e^{-s^{\frac{1}{2q}}} & \text{if } p = 0, \quad q < 0 \end{cases} \quad \text{when } s \text{ is near } 0.$$

Thus

$$\varphi(s) = \begin{cases} cs^{\frac{p+1}{2p}}(-\ln s)^{-\frac{p+q}{p}}\left(\frac{p+1}{2p}(-\ln s) + \frac{q}{p}\right) & \text{if } 0 < p \le 1, \quad q \in R \\ c\sqrt{s}\left(1 - \frac{1}{q}\; s^{\frac{1}{2q}}\right)e^{-s^{\frac{1}{2q}}} & \text{if } p = 0, \quad q < 0 \end{cases} \quad \text{when } s \text{ is near } 0.$$

and

$$\begin{aligned} \psi(t) &= c\int_t^1 \frac{1}{s^{\frac{p+1}{2p}}(-\ln s)^{-\frac{p+q}{p}}\left(\frac{p+1}{2p}(-\ln s) + \frac{q}{p}\right)}\, ds \\ &= c\int_1^{\frac{1}{t}} \frac{z^{\frac{1-3p}{2p}}}{(\ln z)^{-\frac{p+q}{p}}\left(\frac{p+1}{2p}\ln z + \frac{q}{p}\right)}\, dz \text{ when } t \text{ is near } 0. \end{aligned}$$

and

$$\begin{aligned} \psi(t) &= c\int_t^1 \frac{1}{\sqrt{s}\left(1 - \frac{1}{q}\; s^{\frac{1}{2q}}\right)e^{-s^{\frac{1}{2q}}}}\, ds \\ &= c\int_1^{\frac{1}{t}} \frac{e^{(\frac{1}{z})^{\frac{1}{2q}}}}{z^{\frac{3}{2}}(1 - \frac{1}{q}\;(\frac{1}{z})^{\frac{1}{2q}})}\, dz, \quad p = 0, q < 0, \text{ when } t \text{ is near } 0 \end{aligned}$$

We obtain in a neighborhood of 0

$$\psi(t) \equiv \begin{cases} ct^{\frac{p-1}{2p}}(-\ln t)^{\frac{q}{p}} & \text{if } 0 < p < 1, \quad q \in R \\ c(-\ln t)^{1+q} & \text{if } p = 1,\ q > 0, \\ ct^{\frac{q-2}{2q}}e^{t^{\frac{1}{2q}}} & \text{if } p = 0,\ q < 0 \end{cases}$$

and then in a neighborhood of $+\infty$ (see Appendix)

$$\psi^{-1}(t) \equiv \begin{cases} ct^{\frac{2p}{p-1}}(\ln t)^{-\frac{2q}{p-1}} & \text{if } 0 < p \le 1, \quad q \in R, \\ ce^{-t^{\frac{1}{1+q}}} & \text{if } p = 1,\ q > 0, \\ c(\ln t)^{2q} & \text{if } p = 0,\ q < 0. \end{cases}$$

Using the fact that $h(t)=t$ as t goes to infinity, then

$$E(t)\leq\begin{cases} c\sigma(t)^{-\frac{2}{p-1}}(\ln\tilde{\xi}(t))^{-\frac{2q}{p-1}} & \text{if } 0<p\leq 1, \quad q\in R,\\ ce^{-\sigma(t)^{\frac{1}{1+q}}} & \text{if } p=1,\ q<1,\\ c(\ln\sigma(t))^{2q} & \text{if } p=0,\ q<0,\\ ce^{-\sigma(t)} & \text{if } p>1 \text{ or } p=1 \text{ and } q\leq 0.\end{cases}$$

where

$$\sigma(t)=\int_0^t \mu_1(s)\zeta(s)\,ds$$

We finish this section by giving an explicit upper bound for the derivative of the energy.

Lemma 1 Let (φ,ψ,z) be a solution of the problem (15). Then, the energy functional defined by (17) satisfies

$$\begin{aligned} E'(t) \quad \leq \quad & -\mu_1(t)\left(1-\bar{\xi}\alpha_2-\beta\alpha_2\right)\int_0^1 \psi_t g_1(\psi_t)\,dx\\ & -\mu_1(t)\left(\bar{\xi}(1-\tau'(t))\alpha_1-\beta(1-\alpha_1)\right)\int_0^1 z(x,1,t)g_2(z(x,1,t))\,dx\\ & -\tfrac{1}{2}h(t)\|\psi(x,t)\|_2^2+\tfrac{1}{2}(h'\circ\psi_x)(t)\\ \leq \quad & 0 \end{aligned} \tag{20}$$

Proof. Multiplying the first equation in (15) by φ_t, the second equation by ψ_t, integrating over $(0,1)$ and using integration by parts, we get

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\left(\int_0^1\left\{\rho_1\varphi_t^2+\rho_2\psi_t^2+K|\varphi_x+\psi|^2+b\psi_x^2\right\}dx\right)+\mu_1(t)\int_0^1\psi_t g_1(\psi_t)\,dx\\ &=-\mu_2(t)\int_0^1\psi_t(x,t)g_2(z(x,1,t))\,dx+\int_0^1\int_0^t h(t-s)\psi_x(x,s)\psi_{xt}\,ds\,dx=0. \end{aligned} \tag{21}$$

The term in the right-hand side of (21) can be rewritten as follows

$$\begin{aligned} &\int_0^1\int_0^t h(t-s)\psi_x(x,s)\psi_{xt}(x,t)\,ds\,dx+\tfrac{1}{2}h(t)\|\psi_x(x,t)\|_2^2\\ &=\tfrac{1}{2}\tfrac{d}{dt}\left[\int_0^t h(s)\,ds\|\psi_x(x,t)\|_2^2-(h\circ\psi_x)(t)\right]+\tfrac{1}{2}(h'\circ\psi_x)(t). \end{aligned}$$

Consequently, equality (21) becomes

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\left(\int_0^1\left\{\rho_1\varphi_t^2+\rho_2\psi_t^2+K|\varphi_x+\psi|^2\right\}dx\right)+\tfrac{1}{2}\left(b-\int_0^t h(s)\,ds\right)\int_0^1\psi_x^2\,ds\\ &+\mu_1(t)\int_0^1\psi_t g_1(\psi_t)\,dx\\ &=-\mu_2(t)\int_0^1\psi_t(x,t)g_2(z(x,1,t))\,dx-\tfrac{1}{2}h(t)\|\psi(x,t)\|_2^2+\tfrac{1}{2}(h'\circ\psi_x)(t). \end{aligned} \tag{22}$$

We multiply the third equation in (15) by $\xi g_2(z(x,\rho,t))$ and integrate the result over $(0,1)\times(0,1)$, to obtain:

$$\xi(t)\tau(t)\int_0^1\int_0^1 z'g_2(z(x,\rho,t))\,d\rho\,dx=-\xi(t)\int_0^1\int_0^1(1-\tau'(t)\rho)\frac{\partial}{\partial\rho}G_2(z(x,\rho,t))\,d\rho\,dx. \tag{23}$$

Consequently,

$$\begin{aligned} &\tfrac{d}{dt}\left(\xi(t)\tau(t)\int_0^1\int_0^1 G_2(z(x,\rho,t))\,d\rho\,dx\right)=\\ &-\xi(t)\int_0^1\int_0^1\tfrac{\partial}{\partial\rho}((1-\tau'(t)\rho)G_2(z(x,\rho,t)))\,d\rho\,dx+\xi'(t)\tau(t)\int_0^1\int_0^1 G_2(z(x,\rho,t)))\,dxd\rho.\\ &=\xi(t)\int_0^1(G_2(z(x,0,t))-G_2(z(x,1,t)))\,dx+\xi(t)\tau'(t)\int_0^1 G_2(z(x,1,t)))\,dx\\ &+\xi'(t)\tau(t)\int_0^1\int_0^1 G_2(z(x,\rho,t)))\,dxd\rho. \end{aligned} \tag{24}$$

From (21), (24) and using Young inequality we get

$$\begin{array}{l}E'(t) = -\left(\mu_1(t) - \xi(t)\alpha_2\right)\int_0^1 \psi_t g_1(\psi_t)\,dx - \xi(t)(1-\tau'(t))\int_0^1 G_2(z(x,1,t))\,dx \\ \quad -\mu_2(t)\int_0^1 \psi_t(t) g_2(z(x,1,t))\,dx - \frac{1}{2}h(t)\|\psi(x,t)\|_2^2 + \frac{1}{2}(h'\circ\psi_x)(t).\end{array} \tag{25}$$

Let us denote G_2^* to be the conjugate function of the convex function G_2, i.e., $G_2^*(s) = \sup_{t\in R^+}(st - G_2(t))$. Then G_2^* is the Legendre transform of G_2, which is given by (see Arnold [4], p. 61-62, and Lasiecka [11])

$$G_2^*(s) = s(G_2')^{-1}(s) - G_2[(G_2')^{-1}(s)], \quad \forall s \geq 0 \tag{26}$$

and satisfies the following inequality

$$st \leq G_2^*(s) + G_2(t), \quad \forall s,t \geq 0. \tag{27}$$

Then, from the definition of G_2, we get

$$G_2^*(s) = s g_2^{-1}(s) - G_2(g_2^{-1}(s)).$$

Hence

$$\begin{array}{rcl}G_2^*(g_2(z(x,1,t))) &=& z(x,1,t)g_2(z(x,1,t)) - G_2(z(x,1,t)) \\ &\leq& (1-\alpha_1)z(x,1,t)g_2(z(x,1,t)).\end{array} \tag{28}$$

Making use of (25) and (27), we have

$$\begin{array}{l}E'(t) \leq -\left(\mu_1(t) - \xi(t)\alpha_2\right)\int_0^1 \psi_t g_1(\psi_t)\,dx - \xi(t)(1-\tau'(t))\int_0^1 G_2(z(x,1,t))\,dx \\ \quad +\mu_2(t)\int_0^1 \left(G_2(\psi_t) + G_2^*(g_2(z(x,1,t)))\right)\,dx - \frac{1}{2}h(t)\|\psi(x,t)\|_2^2 + \frac{1}{2}(h'\circ\psi_x)(t)\end{array} \tag{29}$$

From (8),(28), and (16), we obtain

$$\begin{array}{rcl}E'(t) &\leq& -\left(\mu_1(t) - \xi(t)\alpha_2 - \mu_2(t)\alpha_2\right)\int_0^1 \psi_t g_1(\psi_t)\,dx \\ && -\left(\xi(t)(1-\tau'(t))\alpha_1 - (1-\alpha_1)\mu_2(t)\right)\int_0^1 z(x,1,t)g_2(z(x,1,t))\,dx \\ && -\frac{1}{2}h(t)\|\psi(x,t)\|_2^2 + \frac{1}{2}(h'\circ\psi_x)(t).\end{array}$$

Using (11) and (16)), we obtain

$$\begin{array}{rcl}E'(t) &\leq& -\mu_1(t)\left(1 - \bar{\xi}\alpha_2 - \beta\alpha_2\right)\int_0^1 \psi_t g_1(\psi_t)\,dx \\ && -\mu_1(t)\left(\bar{\xi}(1-\tau'(t))\alpha_1 - \beta(1-\alpha_1)\right)\int_0^1 z(x,1,t)g_2(z(x,1,t))\,dx \\ && -\frac{1}{2}h(t)\|\psi(x,t)\|_2^2 + \frac{1}{2}(h'\circ\psi_x)(t) \\ &\leq& 0\end{array}$$

3. Global Existence

We are now ready to prove Theorem 1 in the next two sections.

Throughout this section we assume $\varphi_0, \psi_0 \in H^2 \cap H_0^1(0,1), \varphi_1, \psi_1 \in H_0^1(0,1)$ and $f_0 \in H_0^1((0,1); H^1(0,1))$.

We employ the Galerkin method to construct a global solution. We follow the method in [6] with the necessary changes due to the presence of a coupled system of hyperbolic equations.

Let $T > 0$ be fixed and denote by V_k the space generated by $\{w_1, w_2, \ldots, w_k\}$ where the set $\{w_k, k \in N\}$ is a basis of $H^2 \cap H_0^1$.

Now, we define for $1 \leq j \leq k$ the sequence $\phi_j(x,\rho)$ as follows:

$$\phi_j(.,0) = w_j.$$

Then, we can extend to an element of $H^2 \cap H_0^1((0,1); H^1(0,1))$ and denote Z_k the linear space generated by $\{\phi_1, \phi_2, \ldots, \phi_k\}$.

We construct approximate solutions $(\varphi_k, \psi_k, z_k), k = 1, 2, 3, \dots,$ in the form

$$\varphi_k(t) = \sum_{j=1}^{k} g_{jk} w_j, \quad \psi_k(t) = \sum_{j=1}^{k} \tilde{g}_{jk} w_j, \quad z_k(t) = \sum_{j=1}^{k} h_{jk} \phi_j,$$

where $g_{jk}, \tilde{g}_{jk}$ and $h_{jk}, j = 1, 2, \dots, k$, are determined by the following ordinary differential equations:

$$\rho_1(\varphi_k''(t), w_j) - K(\varphi_{kx}(t), w_{jx}) - k(\psi_{kx}(t), w_j) = 0, \quad 1 \le j \le k, \tag{30}$$

$$\varphi_k(0) = \varphi_{0k} = \sum_{j=1}^{k} (\varphi_0, w_j) w_j \to \varphi_0 \text{ in } H^2 \cap H_0^1 \text{ as } k \to +\infty, \tag{31}$$

$$\varphi_k'(0) = \varphi_{1k} = \sum_{j=1}^{k} (\varphi_1, w_j) w_j \to \varphi_1 \text{ in } H_0^1 \text{ as } k \to +\infty. \tag{32}$$

$$\begin{cases} \rho_2(\psi_k''(t), w_j) + b(\psi_{kx}(t), w_{jx}) + K((\varphi_{kx} + \psi)(t), w_j) + \int_0^t h(t-s)(\psi_{kx}(s), w_{jx})\, ds \\ +\mu_1(t)(g_1(\psi_k'), w_j) + \mu_2(t)(g_2(z_k(.,1)), w_j) = 0 \quad 1 \le j \le k, \\ z_k(x, 0, t) = \psi_k'(x, t) \end{cases} \tag{33}$$

$$\psi_k(0) = \psi_{0k} = \sum_{j=1}^{k} (\psi_0, w_j) w_j \to \psi_0 \text{ in } H^2 \cap H_0^1 \text{ as } k \to +\infty, \tag{34}$$

$$\psi_k'(0) = \psi_{1k} = \sum_{j=1}^{k} (\psi_1, w_j) w_j \to \psi_1 \text{ in } H_0^1 \text{ as } k \to +\infty. \tag{35}$$

and

$$\begin{cases} (\tau(t) z_{kt} + (1 - \tau'(t)\rho) z_{k\rho}, \phi_j) = 0, \\ 1 \le j \le k, \end{cases} \tag{36}$$

$$z_k(\rho, 0) = z_{0k} = \sum_{j=1}^{k} (f_0, \phi_j) \phi_j \to f_0 \text{ in } H_0^1((0,1); H^1(0,1)) \text{ as } k \to +\infty. \tag{37}$$

By virtue of the theory of ordinary differential equations, the system (30)-(37) has a unique local solution which is extended to a maximal interval $[0, T_k[$ (with $0 < T_k \le +\infty$) by Zorn lemma since the nonlinear terms in (33) are locally Lipschitz continuous. Note that $(\varphi_k(t), \psi_k(t))$ is from the class C^2.

In the next step we obtain a priori estimates for the solution, such that it can be extended outside $[0, T_k[$ to obtain one solution defined for all $t > 0$.

We can utilize a standard compactness argument for the limiting procedure and it suffices to derive some a priori estimates for (φ_k, ψ_k, z_k).

The first estimate. Since the sequences $\varphi_{0k}, \varphi_{1k}, \psi_{0k}, \psi_{1k}$ and z_{0k} converge, then standard calculations, using (30)-(37), similar to those used to derive (20), yield C independent of k such that

$$\begin{aligned} E_k(t) + \int_0^t \int_0^1 a_1(s) \psi_k' g_1(\psi_k') dx\, ds + \int_0^t \int_0^1 a_2(s) z_k(x, 1, t) g_2(z_k(x, 1, t)) dx\, ds \\ + \tfrac{1}{2} h(t) \|\psi_k(x, t)\|_2^2 - \tfrac{1}{2}(h' \circ \psi_{kx})(t) \le E_k(0) \le C, \end{aligned} \tag{38}$$

where

$$E_k(t) = \frac{1}{2}\int_0^1\{\rho_1{\varphi'_k}^2 + \rho_2{\psi'_k}^2 + K|\varphi_{kx}+\psi_k|^2 + b\psi_{kx}^2\}dx$$
$$+\frac{1}{2}\left(b-\int_0^t h(s)\,ds\right)\int_0^1 \psi_{kx}^2\,dx + \frac{1}{2}(h\circ\psi_{kx})(t) + \xi(t)\tau(t)\int_0^1\int_0^1 G_2(z_k(x,\rho,t))\,d\rho\,dx.$$

$$a_1(t) = \mu_1(t)\left(1-\bar{\xi}\alpha_2-\beta\alpha_2\right) \text{ and } a_2(t) = \mu_1(t)\left(\bar{\xi}(1-\tau'(t))\alpha_1 - \beta(1-\alpha_1)\right).$$

for some C independent of k. These estimates imply that the solution (φ_k, ψ_k, z_k) exists globally in $[0,+\infty[$.
Estimate (38) yields

$$\varphi_k, \psi_k \text{ are bounded in } L^\infty_{loc}(0,\infty; H_0^1(0,1)) \tag{39}$$

$$\varphi'_k, \psi'_k \text{ are bounded in } L^\infty_{loc}(0,\infty; L^2(0,1)) \tag{40}$$

$$\mu_1(t)\psi'_k(t)g_1(\psi'_k(t)) \text{ is bounded in } L^1((0,1)\times(0,T)) \tag{41}$$

$$G_2(z_k(x,\rho,t)) \text{ is bounded in } L^\infty_{loc}(0,\infty; L^1((0,1)\times(0,1))) \tag{42}$$

$$\mu_1(t)z_k(x,1,t)g_2(z_k(x,1,t)) \text{ is bounded in } L^1((0,1)\times(0,T)) \tag{43}$$

The second estimate. First, we estimate $\varphi''_k(0)$ and $\psi''_k(0)$. Testing (30) by $g''_{jk}(t)$, (33) by $\tilde{g}''_{jk}(t)$ and choosing $t=0$ we obtain

$$\rho_1\|\varphi''_k(0)\|_2 \le K(\|\varphi_{0kxx}\|_2 + \|\psi_{0kx}\|_2)$$

and

$$\rho_2\|\psi''_k(0)\|_2 \le b\|\psi_{0kxx}\|_2 + K(\|\varphi_{0kx}\|_2 + \|\psi_{0k}\|_2) + \mu_1(0)\|g_1(\psi_{1k})\|_2 + \mu_2(0)\|g_2(z_{0k})\|_2.$$

Hence from (31), (32) and (37):

$$\|\varphi''_k(0)\|_2 \le C.$$

Since $g_1(\psi_{1k}), g_2(z_{0k})$ are bounded in $L^2(0,1)$ by (**H4**), (31), (34), (35) and (37) yield

$$\|\psi''_k(0)\|_2 \le C.$$

Differentiating (30) and (33) with respect to t, we get

$$(\rho_1\varphi'''_k(t) - K\varphi'_{kxx}(t) - K\psi'_{kx}(t), w_j) = 0 \tag{44}$$

and

$$\begin{aligned}&(\rho_2\psi'''_k(t) - b\psi'_{kxx}(t) + K\varphi'_{kx}(t) + K\psi'_k(t) + \tfrac{d}{dt}\left(\int_0^t h(t-s)\psi_{kx}(s)\,ds\right)\\ &+\mu_1(t)\psi''_k(t)g'_1(\psi'_k(t)) + \mu'_1(t)g_1(\psi'_k(t)) + \mu_2(t)z'_k(x,1,t)g'_2(z_k(x,1,t))\\ &+\mu'_2(t)g_2(z_k(x,1,t)), w_j) = 0.\end{aligned} \tag{45}$$

Multiplying (44) by $g''_{jk}(t)$ and (45) by $\tilde{g}''_{jk}(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi''_k(t)\|_2^2\right) - K\int_0^1(\varphi'_{kx}+\psi'_k)_x\varphi''_k\,dx = 0 \tag{46}$$

$$\begin{aligned}&\tfrac{1}{2}\tfrac{d}{dt}\left(\rho_2\|\psi''_k(t)\|_2^2 + b\|\psi'_{kx}(t)\|_2^2\right) + K\int_0^1(\varphi'_{kx}+\psi'_k)\psi''_k\,dx - h(0)\tfrac{d}{dt}(\psi_{kx}(t), \psi'_{kx}(t))\\ &+h(0)\|\psi'_{kx}(t)\|_2^2 - \tfrac{d}{dt}\int_0^t h'(t-s)(\psi_{kx}(t), \psi'_{kx}(t))\,ds + h'(0)(\psi_{kx}(t), \psi'_{kx}(t))\\ &+\int_0^t h''(t-s)(\psi_{kx}(s), \psi'_{kx}(t))\,ds + \mu_1(t)\int_0^1 {\psi''}_k^2(t)g'_1(\psi'_k(t))\,dx\\ &+\mu'_1(t)\int_0^1 {\psi''}_k(t)g_1(\psi'_k(t))\,dx + \mu_2(t)\int_0^1 {\psi''}_k(t)z'_k(x,1,t)g'_2(z_k(x,1,t))\,dx\\ &\qquad\qquad+\mu'_2(t)\int_0^1 {\psi''}_k(t)g_2(z_k(x,1,t))\,dx = 0\end{aligned} \tag{47}$$

Differentiating (36) with respect to t, we get

$$\left(\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)' z_k' + \frac{\tau(t)}{1-\tau'(t)\rho} z_k''(t) + \frac{\partial}{\partial\rho} z_k', \phi_j\right) = 0.$$

Multiplying by $h'_{jk}(t)$, summing over j from 1 to k, it follows that

$$\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)' \|z_k'(t)\|_2^2 + \frac{1}{2}\frac{\tau(t)}{1-\tau'(t)\rho}\frac{d}{dt}\|z_k'(t)\|_2^2 + \frac{1}{2}\frac{d}{d\rho}\|z_k'(t)\|_2^2 = 0. \quad (48)$$

Then, we have

$$\frac{1}{2}\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)' \|z_k'(t)\|_2^2 + \frac{1}{2}\frac{d}{dt}\left(\frac{\tau(t)}{1-\tau'(t)\rho}\|z_k'(t)\|_2^2\right) + \frac{1}{2}\frac{d}{d\rho}\|z_k'(t)\|_2^2 = 0. \quad (49)$$

Taking the sum of (46), (47) and (48), we obtain

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\big(\rho_1\|\varphi_k''(t)\|_2^2 + \rho_2\|\psi_k''(t)\|_2^2 + b\|\psi'_{kx}(t)\|_2^2 + K\|\varphi'_{kx}(t) + \psi_k'\|_2^2 \\
&+ \textstyle\int_0^1 \tfrac{\tau(t)}{1-\tau'(t)\rho}\|z_k'(x,\rho,t)\|^2_{L^2(0,1)}\,d\rho\big) + h(0)\|\psi'_{kx}(t)\|_2^2 \\
&+\mu_1(t)\textstyle\int_0^1 \psi''^2_k(t) g_1'(\psi_k'(t))\,dx + \tfrac{1}{2}\int_0^1 |z_k'(x,1,t)|^2\,dx \\
&= h(0)\tfrac{d}{dt}(\psi_{kx}(t),\psi'_{kx}(t)) + \tfrac{d}{dt}\textstyle\int_0^t h'(t-s)(\psi_{kx}(t),\psi'_{kx}(t))\,ds - h'(0)(\psi_{kx}(t),\psi'_{kx}(t)) \\
&- \textstyle\int_0^t h''(t-s)(\psi_{kx}(s),\psi'_{kx}(t))\,ds - \tfrac{1}{2}\int_0^1\left(\tfrac{\tau(t)}{1-\tau'(t)\rho}\right)'\|z_k'(x,\rho,t)\|_2^2\,d\rho \\
&-\mu_2(t)\textstyle\int_0^1 \psi''_k(t) z_k'(x,1,t) g_2'(z_k(x,1,t))\,dx - \mu_1'(t)\int_0^1 \psi''_k(t) g_1(\psi_k'(t))\,dx \\
&-\mu_2'(t)\textstyle\int_0^1 \psi''_k(t) g_2(z_k(x,1,t))\,dx + \tfrac{1}{2}\|\psi_k''(t)\|_2^2.
\end{aligned} \quad (50)$$

As in [6], using again Cauchy-Schwarz and Youngs inequalities, we conclude the following estimates:

$$|h'(0)(\psi_{kx}(t),\psi'_{kx}(t))| \le \varepsilon\|\psi_{kx}(t)\|_2^2 + \frac{h'(0)^2}{4\varepsilon}\|\psi'_{kx}(t)\|_2^2, \quad (51)$$

$$\begin{aligned}
\left|\textstyle\int_0^t h''(t-s)(\psi_{kx}(t),\psi'_{kx}(t))\,ds\right| &\le \|\psi'_{kx}(t)\|_2 \textstyle\int_0^t |h''(t-s)|\|\psi_{kx}(s)\|_2\,ds \\
&\le \tfrac{1}{4\varepsilon}\|\psi'_{kx}(t)\|_2^2 + \varepsilon\|h''\|_{L^1}\textstyle\int_0^t |h''(t-s)|\|\psi_{kx}(s)\|_2^2\,ds,
\end{aligned} \quad (52)$$

$$|h(0)(\psi_{kx}(t),\psi'_{kx}(t))| \le \varepsilon\|\psi'_{kx}(t)\|_2^2 + \frac{h(0)^2}{4\varepsilon}\|\psi_{kx}(t)\|_2^2, \quad (53)$$

$$|\int_0^t h'(t-s)(\psi_{kx}(t),\psi'_{kx}(t))\,ds| \le \varepsilon\|\psi'_{kx}(t)\|_2^2 + \frac{\xi(0)\|h\|_{L^1}\|h\|_{L^\infty}}{4\varepsilon}\int_0^t \|\psi_{kx}(s)\|_2^2\,ds. \quad (54)$$

Replacing (51)-(52) in (50), we get

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\big(\rho_1\|\varphi_k''(t)\|_2^2 + \rho_2\|\psi_k''(t)\|_2^2 + b\|\psi'_{kx}(t)\|_2^2 + K\|\varphi'_{kx}(t) + \psi_k'\|_2^2 \\
&+ \textstyle\int_0^1 \tfrac{\tau(t)}{1-\tau'(t)\rho}\|z_k'(x,\rho,t)\|^2_{L^2(0,1)}\,d\rho\big) + \mu_1(t)\int_0^1 \psi''^2_k(t) g_1'(\psi_k'(t))\,dx \\
&+c\textstyle\int_0^1 |z_k'(x,1,t)|^2\,dx + h(0)\|\psi'_{kx}(t)\|_2^2 \\
&\le c'\|\psi_k''(t)\|_2^2 + c''\textstyle\int_0^1 \tfrac{\tau(t)}{1-\tau'(t)\rho}\|z_k'(x,\rho,t)\|^2_{L^2(0,1)}\,d\rho \\
&+|\mu_1'(t)|\textstyle\int_0^1 |\psi''_k(t)||g_1(\psi_k'(t))|\,dx + |\mu_2'(t)|\int_0^1 |\psi''_k(t)||g_2(z_k(x,1,t))|\,dx \\
&+\varepsilon\|\psi_{kx}(t)\|_2^2 + \tfrac{h'(0)^2}{4\varepsilon}\|\psi'_{kx}(t)\|_2^2 + \tfrac{1}{4\varepsilon}\|\psi'_{kx}(t)\|_2^2 + \varepsilon\|h''\|_{L^1}\textstyle\int_0^t |h''(t-s)|\|\psi_{kx}(s)\|_2^2\,ds \\
&+h(0)\tfrac{d}{dt}(\psi_{kx}(t),\psi'_{kx}(t)) + \tfrac{d}{dt}\textstyle\int_0^t h'(t-s)(\psi_{kx}(t),\psi'_{kx}(t))\,ds.
\end{aligned}$$

Integrating the last inequality over $(0,t)$ and using Gronwall's lemma, we get

$$\begin{aligned}
&\tfrac{1}{2}\Big(\rho_1\|\varphi''_k(t)\|_2^2+\rho_2\|\psi''_k(t)\|_2^2+b\|\psi'_{kx}(t)\|_2^2+K\|\varphi'_{kx}(t)+\psi'_k(t)\|_2^2\\
&+\int_0^1\tfrac{\tau(t)}{1-\tau'(t)\rho}\|z'_k(x,\rho,t)\|^2_{L^2(0,1)}\,d\rho\Big)+\mu_1(t)\int_0^t\int_0^1{\psi''}^2_k(t)g'_1(\psi'_k(t))\,dx\,ds\\
&+c\int_0^t\int_0^1|z'_k(x,1,t)|^2\,dx\,ds\\
&\le\tfrac{1}{2}\left(\rho_1\|\varphi''_k(0)\|_2^2+\rho_2\|\psi''_k(0)\|_2^2+b\|\psi'_{kx}(0)\|_2^2+K\|\varphi'_{kx}(0)+\psi'_k(0)\|_2^2\right)\\
&+\tfrac{1}{2}\left(\int_0^1\tfrac{\tau(0)}{1-\tau'(0)\rho}\|z'_k(x,\rho,0)\|^2_{L^2(0,1)}\,d\rho\right)+c'\int_0^t\|\psi''_k(s)\|_2^2\,ds\\
&+h(0)(\psi_{kx}(t),\psi'_{kx}(t))-h(0)(\psi_{kx}(0),\psi'_{kx}(0))+\int_0^t h'(t-s)(\psi_{kx}(t),\psi'_{kx}(t))\,ds\\
&+\left(\tfrac{1}{4\varepsilon}+\tfrac{h'(0)^2}{4\varepsilon}-h(0)\right)\int_0^t\|\psi'_{kx}(s)\|_2^2\,ds+(\varepsilon+\varepsilon\|h''\|_{L^1})\int_0^t\|\psi_{kx}(s)\|_2^2\,ds,
\end{aligned}\tag{55}$$

Then from (55) and (53)-(54) , after choosing ε small enough and using Gronwall's lemma, we obtain

$$\begin{aligned}
&\rho_1\|\varphi''_k(t)\|_2^2+\rho_2\|\psi''_k(t)\|_2^2+b\|\psi'_{kx}(t)\|_2^2+K\|\varphi'_{kx}(t)+\psi'_k(t)\|_2^2\\
&+\int_0^1\tfrac{\tau(t)}{1-\tau'(t)\rho}\|z'_k(x,\rho,t)\|^2_{L^2(0,1)}\,d\rho+\mu_1(t)\int_0^t\int_0^1{\psi''}^2_k(t)g'_1(\psi'_k(t))\,dx\,ds\\
&+c\int_0^t\int_0^1|z'_k(x,1,t)|^2\,dx\,ds\le M,
\end{aligned}$$

for all $t\in[0,T]$ and M is a positive constant independent of $k\in N$. Therefore, we conclude that

$$\varphi''_k,\psi''_k \text{ is bounded in } L^\infty_{loc}(0,\infty;L^2)\tag{56}$$

$$\varphi'_k,\psi'_k \text{ is bounded in } L^\infty_{loc}(0,\infty;H^1_0)\tag{57}$$

$$\tau(t)z'_k \text{ is bounded in } L^\infty_{loc}(0,\infty;L^2((0,1)\times(0,1)))\tag{58}$$

The third estimate. Replacing w_j by $-w_{jxx}$ in (30) and (33), multiplying the result by $g'_{jk}(t)$ and $\tilde{g}'_{jk}(t)$, summing over j from 1 to k, it follows that

$$\frac{1}{2}\frac{d}{dt}\left(\rho_1\|\varphi'_{kx}(t)\|_2^2\right)+K\int_0^1(\varphi_x(t)+\psi(t))_x\varphi'_{kxx}(t)\,dx=0.\tag{59}$$

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\left(\rho_2\|\psi'_{kx}(t)\|_2^2+b\|\psi_{kxx}(t)\|_2^2\right)-K\int_0^1(\varphi_x(t)+\psi(t))\psi'_{kxx}(t)\,dx\\
&+\mu_1(t)\int_0^1|\psi'_{kx}(t)|^2g'_1(\psi'_k(t))\,dx-\int_0^t h(t-s)(\psi_{kxx}(s),\psi'_{kxx}(t))\,ds\\
&+\mu_2(t)\int_0^1{\psi'}_{kx}(t)z_{kx}(x,1,t)g'_2(z_k(x,1,t))\,dx=0.
\end{aligned}\tag{60}$$

$$\begin{aligned}
&\int_0^t h(t-s)(\psi_{xx}(s),\psi'_{xx}(t))\,ds+\tfrac{1}{2}h(t)\|\psi_{xx}(t)\|_2^2\\
&=\tfrac{1}{2}\tfrac{d}{dt}\left[\int_0^t h(s)\,ds\|\psi_{xx}(t)\|_2^2-(h\circ\psi_{xx})(t)\right]+\tfrac{1}{2}(h'\circ\psi_{xx})(t).
\end{aligned}$$

Consequently, equality (60) becomes

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\left(\|\psi'_{kx}(t)\|_2^2+\left(b-\int_0^t h(s)\,ds\right)\|\psi_{kxx}(t)\|_2^2+(h\circ\psi_{kxx})(t)\right)\\
&-K\int_0^1(\varphi_x+\psi)\psi'_{kxx}\,dx\\
&+\mu_1(t)\int_0^1|\psi'_{kx}(t)|^2g'_1(\psi'_k(t))\,dx+h(t)\|\psi_{kxx}(t)\|_2^2\\
&-\tfrac{1}{2}(h'\circ\psi_{kxx})(t)+\mu_2(t)\int_0^1{\psi'}_{kx}(t)z_{kx}(x,1,t)g'_2(z_k(x,1,t))\,dx=0.
\end{aligned}\tag{61}$$

Replacing ϕ_j by $-\phi_{jxx}$ in (36), multiplying the resulting equation by $h_{jk}(t)$, summing over j from 1 to k, it follows that

$$\frac{\tau(t)}{2(1-\tau'(t)\rho)}\frac{d}{dt}\|z_{kx}(t)\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|z_{kx}(t)\|_2^2=0. \tag{62}$$

Then

$$\frac{1}{2}\frac{d}{dt}\left(\frac{\tau(t)}{1-\tau'(t)\rho}\|z_{kx}(t)\|_2^2\right)-\frac{1}{2}\left(\frac{\tau(t)}{1-\tau'(t)\rho}\right)'\|z_{kx}(t)\|_2^2+\frac{1}{2}\frac{d}{d\rho}\|z_{kx}(t)\|_2^2=0. \tag{63}$$

From (59), (60) and (62),(63), we have

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\Big(\rho_1\|\varphi'_{kx}(t)\|_2^2+\rho_2\|\psi'_{kx}(t)\|_2^2+K\|\varphi_{kxx}+\psi_{kx}(t)\|_2^2+\Big(b-\textstyle\int_0^t h(s)\,ds\Big)\|\psi_{kxx}(t)\|_2^2\\
&+b\|\psi_{kxx}(t)\|_2^2\Big)+\tfrac{1}{2}\tfrac{d}{dt}\Big(\textstyle\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\|z_{kx}(x,\rho,t)\|^2_{L^2(0,1)}\,d\rho\Big)\\
&+h(t)\|\psi_{kxx}(t)\|_2^2-\tfrac{1}{2}(h'\circ\psi_{kxx})(t)+\mu_1(t)\textstyle\int_0^1|\psi'_{kx}(t)|^2g'_1(\psi'_k(t))\,dx+\tfrac{1}{2}\textstyle\int_0^1|z_{kx}(x,1,t)|^2\,dx\\
&=-\mu_2(t)\textstyle\int_0^1\psi'_{kx}(t)z_{kx}(x,1,t)g'_2(z_k(x,1,t))\,dx+\tfrac{1}{2}\textstyle\int_0^1\Big(\frac{\tau(t)}{1-\tau'(t)\rho}\Big)'\|z_{kx}(x,\rho,t)\|_2^2\,d\rho\\
&\qquad+\tfrac{1}{2}\|\nabla\psi'_{kx}(t)\|_2^2.
\end{aligned}$$

Using (7), Cauchy-Schwartz and Young's inequalities, we obtain

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{d}{dt}\Big(\rho_1\|\varphi'_{kx}(t)\|_2^2+\rho_2\|\psi'_{kx}(t)\|_2^2+K\|\varphi_{kxx}+\psi_{kx}(t)\|_2^2+b\|\psi_{kx}(t)\|_2^2\\
&+\textstyle\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\|z_{kx}(x,\rho,t)\|^2_{L^2(0,1)}\,d\rho\Big)+\mu_1(t)\textstyle\int_0^1|\psi'_{kx}(t)|^2g'_1(\psi'_k(t))\,dx\\
&\qquad+c\textstyle\int_0^1|z_{kx}(x,1,t)|^2\,dx\\
&\le c'\|\psi'_{kx}(t)\|_2^2+c''\textstyle\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\|z_{kx}(x,\rho,t)\|^2_{L^2(0,1)}\,d\rho.
\end{aligned}$$

Integrating the last inequality over $(0,t)$ and using Gronwall's Lemma, we have

$$\begin{aligned}
&\rho_1\|\varphi'_{kx}(t)\|_2^2+\rho_2\|\psi'_{kx}(t)\|_2^2+K\|\varphi_{kxx}+\psi_{kx}(t)\|_2^2+b\|\psi_{kxx}(t)\|_2^2\\
&+\textstyle\int_0^1\frac{\tau(t)}{1-\tau'(t)\rho}\|z_{kx}(x,\rho,t)\|^2_{L^2(0,1)}d\rho\le\\
&e^{cT}\Big(\rho_1\|\varphi'_{kx}(0)\|_2^2+\rho_2\|\psi'_{kx}(0)\|_2^2+K\|\varphi_{kxx}(0)+\psi_{kx}(0)\|_2^2+b\|\psi_{kx}(0)\|_2^2\\
&+\textstyle\int_0^1\frac{\tau(0)}{1-\tau'(0)\rho}\|z_{kx}(x,\rho,0)\|^2_{L^2(0,1)}d\rho\Big)
\end{aligned}$$

for all $t\in R_+$, therefore, we conclude that

$$\varphi_k,\psi_k \text{ are bounded in } L^\infty_{loc}(0,\infty;H^2\cap H^1_0(0,1)), \tag{64}$$

$$z_k \text{ is bounded in } L^\infty_{loc}(0,\infty;H^1_0(0,1;L^2(0,1))). \tag{65}$$

Applying Dunford-Petti's theorem we conclude from (39), (40), (41), (42), (56), (57), (58), (64) and (65), after replacing the sequences φ_k,ψ_k and z_k with a subsequence if needed, that

$$\begin{cases}\varphi_k\to\varphi \text{ weak-star in } L^\infty_{loc}(0,\infty;H^2\cap H^1_0(0,1))\\ \psi_k\to\psi \text{ weak-star in } L^\infty_{loc}(0,\infty;H^2\cap H^1_0(0,1)),\end{cases} \tag{66}$$

$$\begin{cases}\varphi'_k\to\varphi' \text{ weak-star in } L^\infty_{loc}(0,\infty;H^1_0(0,1))\\ \psi_k\to\psi' \text{ weak-star in } L^\infty_{loc}(0,\infty;H^1_0(0,1)),\end{cases}$$

$$\begin{cases}\varphi''_k\to\varphi'' \text{ weak-star in } L^\infty_{loc}(0,\infty;L^2(0,1))\\ \psi''_k\to\psi'' \text{ weak-star in } L^\infty_{loc}(0,\infty;L^2(0,1)),\end{cases} \tag{67}$$

$$\begin{aligned}
g_1(\psi'_k)&\to\chi \text{ weak-star in } L^2((0,1)\times(0,T);\mu_1(t)),\\
z_k&\to z \text{ weak-star in } L^\infty_{loc}(0,\infty;H^1_0((0,1);L^2(0,1))),\\
z'_k&\to z' \text{ weak-star in } L^\infty_{loc}(0,\infty;L^2((0,1)\times(0,1))),\\
g_2(z_k(x,1,t))&\to\psi \text{ weak-star in } L^2((0,1)\times(0,T);\mu_1(t))
\end{aligned} \tag{68}$$

for suitable functions $\varphi, \psi \in L^\infty(0,T; H^2 \cap H_0^1(0,1)), z \in L^\infty(0,T; L^2((0,1) \times (0,1)))$,
$\chi \in L^2((0,1)\times(0,T); \mu_1(t)), \psi \in L^2((0,1)\times(0,T);\mu_1(t))$ for all $T \geq 0 \Big(L^2((0,1) \times (0,T); \mu_1)$ is the space of square-summable functions with weight $\mu_1 \Big)$. We have to show that (φ, ψ, z) is a solution of (15).

From (39) and (40) we have (ψ_k') is bounded in $L^\infty(0,T; H_0^1(0,1))$. Then (ψ_k') is bounded in $L^2(0,T; H_0^1)$. Since (ψ_k'') is bounded in $L^\infty(0,T; L^2(0,1))$, then (ψ_k'') is bounded in $L^2(0,T; L^2(0,1))$. Consequently (ψ_k') is bounded in $H^1(Q)$,where $Q = (0,1) \times (0,T)$.

Since the embedding $H^1(Q) \hookrightarrow L^2(Q)$ is compact, using Aubin-Lions theorem [21] we can extract a subsequence (ψ_ν) of (ψ_k) such that

$$\psi_\nu' \to \psi' \text{ strongly in } L^2(Q).$$

Therefore

$$\psi_\nu' \to \psi' \text{ strongly and a.e on } Q. \tag{69}$$

Similarly we obtain

$$z_\nu \to z \text{ strongly and a.e on } Q. \tag{70}$$

Lemma 2 For each $T > 0$, $g_1(\psi'), g_2(z(x,1,t)) \in L^1(Q)$ and
$\|g_1(\psi')\|_{L^1(Q)}, \|g_2(z(x,1,t))\|_{L^1(Q)} \leq K_1$, where K_1 is a constant independent of t.

Proof. By **(H3)** and (69) we have

$$g_1(\psi_k'(x,t)) \to g_1(\psi'(x,t)) \text{ a.e. in } Q,$$

$$0 \leq g_1(\psi_k'(x,t))\psi_k'(x,t) \to g_1(\psi'(x,t))\psi'(x,t) \text{ a.e. in } Q$$

Hence, by (41) and Fatou's lemma we have

$$\int_0^T \int_0^1 \mu_1(t)\psi'(x,t) g_1(\psi'(x,t))\, dx\, dt \leq K \text{ for } T > 0. \tag{71}$$

By Cauchy-Schwarz inequality and using (71), we have

$$\begin{aligned} \int_0^T \int_0^1 \mu_1(t)|g_1(\psi'(x,t))|\, dx\, dt &\leq c|Q|^{\frac{1}{2}} \left(\int_0^T \int_0^1 \mu_1(t)\psi' g_1(\psi')\, dx\, dt \right)^{\frac{1}{2}} \\ &\leq c|Q|^{\frac{1}{2}} K^{\frac{1}{2}} \equiv K_1 \end{aligned}$$

Lemma 3 $g_1(\psi_k') \to g_1(\psi')$ in $L^1((0,1)\times(0,T);\mu_1(t))$ and
$g_2(z_k) \to g_2(z)$ in $L^1((0,1)\times(0,T);\mu_1(t))$.

Proof. Let $E \subset (0,1) \times [0,T]$ and set

$$E_1 = \left\{ (x,t) \in E;\ g_1(\psi_k'(x,t)) \leq \frac{1}{\sqrt{|E|}} \right\}, \quad E_2 = E \setminus E_1,$$

where $|E|$ is the measure of E. If $M(r) := \inf\{|s|;\ s \in R \text{ and } |g_1(s)| \geq r\}$,

$$\int_E \mu_1(t)|g_1(\psi_k')|\, dxdt \leq \sqrt{|E|} + \left(M\left(\frac{1}{\sqrt{|E|}} \right) \right)^{-1} \int_{E_2} \mu_1(t)|\psi_k' g_1(\psi_k')|\, dxdt.$$

Applying (41) we deduce that $\sup_k \int_E |g_1(\psi_k')|\,dxdt \to 0$ as $|E| \to 0$.
From Vitali's convergence theorem we deduce that
$g_1(\psi_k') \to g_1(\psi')$ in $L^1((0,1)\times(0,T);\mu_1(t))$, hence

$$g_1(\psi_k') \to g_1(\psi') \text{ weak star in } L^2(Q;\mu_1(t)).$$

Similarly, we have

$$g_2(z_k') \to g_2(z') \text{ weak star in } L^2(Q;\mu_1(t)),$$

and this imply that

$$\begin{aligned}&\textstyle\int_0^T \int_0^1 \mu_1(t) g_1(\psi_k') v\,dx\,dt \\ &\to \textstyle\int_0^T \int_0^1 \mu_1(t) g_1(\psi') v\,dx\,dt \text{ for all } v \in L^2(0,T;H_0^1(0,1);\mu_1(t))\end{aligned} \tag{72}$$

$$\int_0^T \int_0^1 g_2(z_k) v\,dx\,dt \to \int_0^T \int_0^1 g_2(z) v\,dx\,dt \text{ for all } v \in L^2(0,T;H_0^1) \tag{73}$$

as $k \to +\infty$. It follows at once from (66), (67), (72), (73) and (68) that for each fixed $u, v \in L^2(0,T;H_0^1(0,1);\mu_1(t))$ and $w \in L^2(0,T;H_0^1((0,1)\times(0,1));\mu_1(t))$

$$\begin{aligned}&\textstyle\int_0^T \int_0^1 (\rho_1 \varphi_k'' - K(\varphi_{kx} + \psi_k)_x)\,u\,dx\,dt \\ &\to \textstyle\int_0^T \int_0^1 (\rho_1 \varphi'' - K(\varphi_x + \psi)_x)\,u\,dx\,dt\end{aligned}$$

$$\begin{aligned}&\textstyle\int_0^T \int_0^1 \Big(\rho_2 \psi_k'' - b\psi_{kxx} + K(\varphi_{kx} + \psi_k) + \int_0^t h(t-s)\psi_{kxx}(x,s)\,ds \\ &+\mu_1(t) g_1(\psi_k') + \mu_2(t) g_2(z_k)\big)\,v\,dx\,dt \\ &\to \textstyle\int_0^T \int_0^1 \Big(\rho_2 \psi'' - b\psi_{xx} + K(\varphi_x + \psi) + \int_0^t h(t-s)\psi_{xx}(x,s)\,ds \\ &+\mu_1(t) g_1(\psi') + \mu_2(t) g_2(z)\big)\,v\,dx\,dt\end{aligned}$$

and

$$\begin{aligned}&\textstyle\int_0^T \int_0^1 \int_0^1 \Big(\tau(t) z_k' + (1-\tau'(t)\rho)\frac{\partial}{\partial\rho} z_k\Big)\,w\,dx\,d\rho\,dt \\ &\to \textstyle\int_0^T \int_0^1 \int_0^1 \Big(\tau(t) z' + (1-\tau'(t)\rho)\frac{\partial}{\partial\rho} z\Big)\,w\,dx\,d\rho\,dt\end{aligned}$$

as $k \to +\infty$. Hence

$$\int_0^T \int_0^1 (\rho_1 \varphi'' - K(\varphi_x + \psi)_x) u\,dx\,dt = 0$$

$$\begin{aligned}&\textstyle\int_0^T \int_0^1 (\rho_2 \psi'' - b\psi_{xx} + K(\varphi_x + \psi) + \int_0^t h(t-s)\psi_{xx}(x,s)\,ds + \mu_1(t) g_1(\psi') \\ &+\mu_2(t) g_2(z)) v\,dx\,dt = 0\end{aligned}$$

and

$$\int_0^T \int_0^1 \int_0^1 (\tau(t) z' + (1-\tau'(t)\rho)\frac{\partial}{\partial\rho} z) w\,dx\,d\rho\,dt = 0.$$

Thus the problem (P) admits a global weak solution (φ, ψ).

4. Asymptotic behavior

The method of the proof of decay rates heavily relies on the introduction of suitable Lyapunov functionals (see [6]) and ideas introduced in [19] and [2], where convex analysis was exploited to obtain a precise description of the decay rates corresponding to the energy of the wave equations with dissipation that is not quantified at the origin. We construct a Lyapunov functional $\mathcal{L}$ equivalent to E. For this, we define several functionals which allow us to obtain the estimates needed.

First, we have the following estimate.

Lemma 4 Let (φ,ψ,z) be the solution of (15). Then the functional F_1 defined by

$$F_1(t)=-\int_0^1(\rho_1\varphi_t\varphi+\rho_2\psi_t\psi)\,dx \tag{74}$$

satisfies, along the solution, the estimate

$$\begin{aligned}\tfrac{dF_1(t)}{dt}&\le-\textstyle\int_0^1(\rho_1\varphi_t^2+\rho_2\psi_t^2)dx+K\int_0^1|\varphi_x+\psi|^2dx\\ &\quad+c\textstyle\int_0^1\psi_x^2dx+ch\circ\psi_x+c\int_0^1g_1^2(\psi_t)dx+c|\mu_2(t)|\int_0^1g_2^2(z(x,1,t))dx.\end{aligned} \tag{75}$$

Proof. By taking the time derivative of (74)

$$\frac{dF_1(t)}{dt}=-\int_0^1(\rho_1\varphi_t^2+\rho_2\psi_t^2)dx-\int_0^1(\rho_1\varphi_{tt}\varphi+\rho_2\psi_{tt}\psi)dx.$$

Therefore, by using the first and the second equations in (15) and some integrations by parts, we obtain from the above inequality

$$\begin{aligned}\tfrac{dF_1(t)}{dt}&=-\textstyle\int_0^1(\rho_1\varphi_t^2+\rho_2\psi_t^2)dx+K\int_0^1|\varphi_x+\psi|^2dx\\ &\quad-\textstyle\int_0^1\psi_x(x,t)\int_0^th(t-s)\psi_x(x,s)ds\,dx+b\int_0^1\psi_x^2dx\\ &\quad+\mu_1(t)\textstyle\int_0^1\psi g_1(\psi_t)dx+\mu_2(t)\int_0^1\psi g_2(z(x,1,t))dx.\end{aligned} \tag{76}$$

By exploiting Young's inequality and Poincaré's inequality, then (75) holds.

Lemma 5 Let (φ,ψ,z) be the solution of (15). Assume that

$$\frac{\rho_1}{K}=\frac{\rho_2}{b}. \tag{77}$$

Then the functional F_2 defined by

$$F_2(t)=\rho_2\int_0^1\psi_t(\varphi_x+\psi)dx+\rho_2\int_0^1\psi_x\varphi_t\,dx-\frac{\rho_1}{k}\int_0^1\varphi_t\int_0^th(t-s)\psi_x(s)\,ds\,dx. \tag{78}$$

satisfies, along the solution, the estimate

$$\begin{aligned}\tfrac{dF_2(t)}{dt}&\le[(b\varphi_x-\textstyle\int_0^th(t-s)\psi_x(s)\,ds)\varphi_x]_{x=0}^{x=1}-(K-\varepsilon)\int_0^1(\varphi_x+\psi)^2dx\\ &\quad+\varepsilon\textstyle\int_0^1\varphi_t^2dx-\frac{c}{\varepsilon}h'\circ\psi_x+\frac{c}{\varepsilon}\int_0^1\psi_x^2dx\\ &\quad+\rho_2\textstyle\int_0^1\psi_t^2dx+\frac{c}{\varepsilon}\int_0^1g_1^2(\psi_t)dx+\frac{c}{\varepsilon}|\mu_2(t)|\int_0^1g_2^2(z(x,1,t))dx\end{aligned} \tag{79}$$

for any $0<\varepsilon<1$.

Proof. Differentiating $F_2(t)$, with respect to t, we obtain

$$\begin{aligned}&\tfrac{dF_2(t)}{dt}=\textstyle\int_0^1\rho_2\psi_{tt}(\varphi_x+\psi)dx+\int_0^1\rho_2\psi_t(\varphi_x+\psi)_tdx+\rho_2\int_0^1\psi_x\varphi_{tt}dx\\ &+\rho_2\textstyle\int_0^1\psi_{tx}\varphi_tdx-\frac{\rho_1}{k}\int_0^1\varphi_{tt}\int_0^th(t-s)\psi_x(s)\,ds\,dx-\frac{\rho_1}{k}\int_0^1\varphi_t(\int_0^th(t-s)\psi_x(s)\,ds)'dx.\\ &=\textstyle\int_0^1(\varphi_x+\psi)[b\psi_{xx}-k(\varphi_x+\psi)-\int_0^th(t-s)\psi_{xx}(s)\,ds-\mu_1(t)g_1(\psi_t)\\ &-\mu_2(t)g_2(z(x,1,t)]dx+\rho_2\textstyle\int_0^1\psi_t^2\,dx+\frac{\rho_2}{\rho_1}\int_0^1k(\varphi_x+\psi)_x\psi_x\,dx\\ &-\textstyle\int_0^1(\varphi_x+\psi)_x\int_0^th(t-s)\psi_x(s)\,ds\,dx-\frac{\rho_1}{k}\int_0^1\varphi_t(\int_0^th(t-s)\psi_x(s)\,ds)'dx.\end{aligned}$$

Then, by using Eqs.(15) and (77) we find

$$\frac{dF_2(t)}{dt} = [(b\psi_x - \int_0^t h(t-s)\psi_x(s)\,ds)\varphi_x]_{x=0}^{x=1} - K\int_0^1(\varphi_x+\psi)^2dx + \rho_2\int_0^1 \psi_t^2dx$$
$$-\mu_1(t)\int_0^1(\varphi_x+\psi)g_1(\psi_t)dx - \mu_2(t)\int_0^1(\varphi_x+\psi)g_2(z(x,1,t))dx$$
$$-\frac{\rho_1}{k}h(t)\int_0^1 \varphi_t\psi_x\,dx - \frac{\rho_1}{k}\int_0^1\varphi_t\int_0^t h'(t-s)\psi_x(s)\,ds\,dx.$$

By the Young inequality (79) is established.

Lemma 6 Let $m \in C^1([0,1])$ be a function satisfying $m(0) = -m(1) = 2$. Then there exists $c > 0$ such that, for any $0 < \varepsilon < 1$, the functional F_3 defined by

$$F_3(t) = \frac{1}{4\varepsilon}\int_0^1 \rho_2 m(x)\psi_t(b\psi_x - \int_0^t h(t-s)\psi_x(s)\,ds)\,dx + \frac{\varepsilon}{k}\int_0^1 \rho_1 m(x)\varphi_t\varphi_x\,dx$$

satisfies, along the solution, the estimate

$$F_3'(t) \le -\frac{1}{4\varepsilon}((b\psi_x(1,t) - \int_0^t h(t-s)\psi_x(1,s)\,ds)^2 + (b\psi_x(0,t)$$
$$-\int_0^t h(t-s)\psi_x(0,s)\,ds)^2) - \varepsilon((\varphi_x(1,t))^2 + (\varphi_x(0,t))^2) + (\frac{k}{4} + \frac{c}{k}\varepsilon)\int_0^1(\psi+\varphi_x)^2\,dx$$
$$+c\varepsilon\rho_1\int_0^1\varphi_t^2\,dx + \frac{c}{\varepsilon^2}h\circ\psi_x + \frac{c}{\varepsilon^2}\int_0^1\psi_x^2\,dx$$
$$+\frac{c}{\varepsilon}\int_0^1\psi_t^2\,dx + c\int_0^1 g_1^2(\psi_t)\,dx + c|\mu_2(t)|\int_0^1 g_2^2(z(x,1,t))\,dx$$
$$+c\int_0^1 g_2^2(z(x,1,t))\,dx - \frac{c}{\varepsilon}h'\circ\psi_x \tag{80}$$

Proof. Using Eqs. (15) and integrating by parts, obtain

$$F_3'(t) = \frac{1}{4\varepsilon}\Big[-((b\psi_x(1,t) - \int_0^t h(t-s)\psi_x(1,s)\,ds)^2 + (b\psi_x(0,t)$$
$$-\int_0^t h(t-s)\psi_x(0,s)\,ds)^2) - \int_0^1\frac{1}{2}m'(x)(b\psi_x - \int_0^t h(t-s)\psi_x(s)\,ds)^2\,dx$$
$$-k\int_0^1 m(x)(\varphi_x+\psi)(b\psi_x - \int_0^t h(t-s)\psi_x(s)\,ds)\,dx$$
$$-\int_0^1 m(x)\mu_1(t)g_1(\psi_t)(b\psi_x - \int_0^t h(t-s)\psi_x(s)\,ds)\,dx$$
$$-\int_0^1 m(x)\mu_2(t)g_2(z(x,1,t))(b\psi_x - \int_0^t h(t-s)\psi_x(s)\,ds)\,dx - \int_0^1\frac{b\rho_2}{2}m'(x)(\psi_t)^2\,dx$$
$$+\rho_2\int_0^1 m(x)\psi_t\int_0^t h'(t-s)(\psi_x(t)-\psi_x(s))ds\,dx - \rho_2 h(t)\int_0^1 m(x)\psi_t\psi_x dx\Big]$$
$$\frac{\varepsilon}{k}\Big[-k((\varphi_x(1,t))^2 + (\varphi_x(0,t))^2) - \int_0^1\frac{k}{2}m'(x)\varphi_x^2\,dx + \int_0^1 km(x)\psi_x\varphi_x\,dx$$
$$-\int_0^1\frac{\rho_1}{2}m'(x)(\varphi_t)^2\,dx\Big]$$

Then by the Young and Poincaré inequalities and the fact that

$$\varphi_x^2 \le 2(\psi+\varphi_x)^2 + 2\psi^2$$

we obtain

$$F_3'(t) \le \frac{1}{4\varepsilon}\Big[(b\psi_x(1,t) - \int_0^t h(t-s)\psi_x(1,s)\,ds)^2$$
$$+(b\psi_x(0,t) - \int_0^t h(t-s)\psi_x(0,s)\,ds)^2)$$
$$+\frac{c}{\varepsilon}\int_0^1\psi_x^2\,dx + \frac{c}{\varepsilon}h\circ\psi_x + \varepsilon\frac{k}{b}\int_0^1(\psi+\varphi_x)^2\,dx + \varepsilon\int_0^1 g_1^2(\psi_t)\,dx$$
$$+\varepsilon|\mu_2(t)|\int_0^1 g_2^2(z(x,1,t))\,dx + c\int_0^1\psi_t^2\,dx - ch'\circ\psi_x\Big]$$
$$+\frac{\varepsilon}{k}\Big[-k((\varphi_x(1,t))^2 + (\varphi_x(0,t))^2) + c\int_0^1\psi_x^2\,dx + c\int_0^1(\psi+\varphi_x)^2\,dx + c\int_0^1\varphi_t^2\,dx\Big]$$

This gives (80).

Lemma 7 Assume that $(H1)$ hold. Then, for sufficiently small ε, the functional F defined by

$$F(t) = 2c\varepsilon F_1(t) + F_2(t) + F_3(t)$$

satisfies, along the solution, the estimate

$$\begin{array}{l} F'(t) \leq -\frac{k}{2}\int_0^1(\psi+\varphi_x)^2\,dx - \tau\int_0^1 \varphi_t^2\,dx + c\int_0^1 \psi_t^2\,dx + +c\int_0^1 \psi_x^2\,dx \\ +c\int_0^1 g_1^2(\psi_t)\,dx + c|\mu_2(t)|\int_0^1 g_2^2(z(x,1,t))\,dx + ch\circ\psi_x - ch'\circ\psi_x, \end{array} \tag{81}$$

where $\tilde{\tau} = c\varepsilon\rho_1$.

Proof. Using Lemmas 4, 5, 6 and the fact that

$$\begin{array}{l} [(b\varphi_x - \int_0^t h(t-s)\psi_x(s)\,ds)\varphi_x]_{x=0}^{x=1} \leq \frac{1}{4\varepsilon}\Big[(b\psi_x(1,t) - \int_0^t h(t-s)\psi_x(1,s)\,ds)^2 \\ +(b\psi_x(0,t) - \int_0^t h(t-s)\psi_x(0,s)\,ds)^2) + \varepsilon[\psi_x^2(1)+\psi_x^2(0)]\Big], \end{array} \tag{82}$$

for any $0<\varepsilon<1$, we obtain (81).

Next, we introduce the following functional

$$I(t) = \int_0^1 (\rho_2\psi_t\psi + \rho_1\varphi_t\omega)dx, \tag{83}$$

where w is the solution of

$$-\omega_{xx} = \psi_x, \qquad \omega(0)=\omega(1)=0. \tag{84}$$

Then we have the following estimate.

Lemma 8 Let (φ,ψ,z) be the solution of (15), then for any $\delta>0$, we have the following estimate

$$\begin{array}{l} \frac{dI(t)(t)}{dt} \leq \frac{-1}{2}(b - \int_0^\infty h(s)ds)\int_0^1 \psi_x^2(x,t)dx + \frac{c}{\delta}\int_0^1 \psi_t^2(x,t)dx \\ +\delta\int_0^1 \varphi_t^2(x,t)dx + c\int_0^1 g_1^2(\psi_t)dx + c|\mu_2(t)|\int_0^1 g_2^2(z(x,1,t))dx + ch\circ\psi_x. \end{array} \tag{85}$$

Proof. Using Eqs. (15), we have

$$\begin{array}{l} \frac{dI(t)}{dt} = (-b + \int_0^\infty h(s)ds)\int_0^1 \psi_x^2 dx + \rho_2\int_0^1 \psi_t^2 dx - K\int_0^1 \psi^2 dx \\ +\int_0^1(\int_0^t h(t-s)(\psi_x(t)-\psi_x(t))ds)\psi_x dx + K\int_0^1 \omega_x^2 dx + \rho_1\int_0^1 \psi_t\omega_t dx \\ -\mu_1(t)\int_0^1 \psi g_1(\psi_t)dx - \mu_2(t)\int_0^1 \psi g_2(z(x,1,t))dx. \end{array} \tag{86}$$

It is clear that, from (84), we have

$$\begin{array}{c} \int_0^1 \omega_x^2 dx \leq \int_0^1 \psi^2 dx \leq \int_0^1 \psi_x^2 dx \\ \int_0^1 \omega_t^2 dx \leq \int_0^1 \omega_{tx}^2 dx \leq \int_0^1 \psi_t^2 dx \end{array} \tag{87}$$

By using Young's inequality and Poincaré's inequality, the last two terms in (86) can be estimated as

$$\begin{array}{l} \mu_1(t)\int_0^1 \psi g_1(\psi_t)dx + \mu_2(t)\int_0^1 \psi g_2(z(x,1,t))dx \\ +\int_0^1(\int_0^t h(t-s)(\psi_x(t)-\psi_x(t))ds)\psi_x dx \\ \leq \frac{1}{2}(b - \int_0^\infty h(s)ds)\int_0^1 \psi_x^2 dx + c\int_0^1 g_1^2(\psi_t)dx + c|\mu_2(t)|\int_0^1 g_2^2(z(x,1,t))dx. \end{array} \tag{88}$$

Consequently, from (86)-(88), we obtain (85).

Now, let us introduce the following functional

$$I_3(t) = \xi(t)\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}G_2(z(x,\rho,t))\,d\rho\,dx. \tag{89}$$

Then the following result holds.
Lemma 9 Let (φ, ψ, z) be the solution of (15). Then it holds

$$\begin{array}{rcl} \frac{d}{dt}I_3(t) & \leq & -2I_3(t) - \xi(t)(1-\tau'(t))e^{-2\tau(t)}\int_0^1 G_2(z(x,1,t))dx \\ & + & \xi(t)\int_0^1 G_2(\psi_t(x,t))\,dx. \end{array} \tag{90}$$

Proof. Differentiating (89) with respect to t and using the third equation in (15), we have

$$\begin{array}{l} \frac{d}{dt}I_3(t) = (\xi(t)\tau(t))'\int_0^1\int_0^1 e^{-2\tau(t)\rho}G_2(z(x,\rho,t))\,d\rho\,dx \\ -2\xi(t)\tau(t)\tau'(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}\rho G_2(z(x,\rho,t))d\rho\,dx \\ +\xi(t)\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}z_t g_2(z)\,d\rho\,dx \end{array} \tag{91}$$

By using the third equation in (15), the last term in (91) can be rewritten as follows

$$\begin{array}{l} \xi(t)\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}z_t g_2(z)\,d\rho\,dx \\ = \xi(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}(\tau'(t)\rho-1)z_\rho g_2(z)\,d\rho\,dx \end{array} \tag{92}$$

Also, one can see that

$$\begin{array}{l} \xi(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}(\tau'(t)\rho-1)z_\rho g_2(z)\,d\rho\,dx \\ = \xi(t)\int_0^1\int_0^1 \frac{d}{d\rho}\left(e^{-2\tau(t)\rho}(\tau'(t)\rho-1)G_2(z)\right)\,d\rho\,dx \\ \\ +2\xi(t)\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}(\tau'(t)\rho-1)G_2(z)\,d\rho\,dx \\ -\xi(t)\tau'(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}G_2(z)\,d\rho\,dx \end{array} \tag{93}$$

Using (93) and (92), Eq. (91) takes the form

$$\begin{array}{l} \frac{d}{dt}I_3(t) = -2I_3(t) + \xi'(t)\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}G_2(z(x,\rho,t))dx \\ -\xi(t)(1-\tau'(t))e^{-2\tau(t)}\int_0^1 G_2(z(x,1,t))dx + \xi(t)\int_0^1 G_2(\psi_t(x,t))\,dx. \end{array} \tag{94}$$

For $N_1, N_2 > 0$, let

$$L(t) = N_1E(t) + N_2I(t) + F(t) + I_3(t). \tag{95}$$

By combining (20), (81), (85), (90), we obtain

$$\begin{array}{l} \frac{d}{dt}L(t) \leq -\mu_1(t)\left(N_1a_1 - \overline{\xi}\alpha_2\right)\int_0^1 \psi_t g_1(\psi_t(x,t))\,dx \\ -\mu_1(t)\left(N_1a_2 + \alpha_1(1-d)e^{-2\tau_1} - \beta(N_2c+c')\right)\int_0^1 z(x,1,t)g_2(z(x,1,t))\,dx \\ -\left(\frac{N_2}{2}(b-\int_0^\infty h(s)ds) + N_{1h(t)} - c\right)\int_0^1 \psi_x^2\,dx \\ -\left(\tilde{\tau}(t) - N_2\delta\right)\int_0^1 \varphi_t^2\,dx + \left(N_2\frac{c}{\delta} + c\right)\int_0^1 \psi_t^2\,dx - \frac{k}{2}\int_0^1(\psi+\varphi_x)^2\,dx \\ +(N_2c+c)\int_0^1 g_1^2(\psi_t)\,dx + (N_2c+c)h\circ\psi_x + (\frac{N_1}{2}-c)h'\circ\psi_x - 2I_3. \end{array} \tag{96}$$

At this point, we have to choose our constants very carefully. First, let us choose N_2 sufficiently large so that

$$\left(\frac{N_2}{2}(b - \int_0^\infty h(s)ds) - c\right) > 0.$$

Next, we choose δ sufficiently small such that

$$(\tilde{\tau} - N_2\delta) > 0.$$

Then, we pick the constant $N_1 > 0$ sufficiently large such that

$$\left(N_1a_1 - \overline{\xi}\alpha_2\right)$$

and

$$\left(N_1a_2 + \alpha_1(1-d)e^{-2\tau_1} - \beta(N_2c+c)c_3\right).$$

and

$$(\frac{N_1}{2}-c)>0.$$

Thus, (96) becomes

$$\begin{aligned}&\tfrac{d}{dt}L(t)\le -d_1\int_0^1\psi_x^2\,dx-d_2\int_0^1\varphi_t^2\,dx-\tfrac{k}{2}\int_0^1(\psi+\varphi_x)^2\,dx\\&+d_3h\circ\psi_x+d_4h'\circ\psi_x-2I_3+c\int_0^1((\psi_t)^2+g_1^2(\psi_t))\,dx\\&\le -dE(t)+c\int_0^1((\psi_t)^2+g_1^2(\psi_t))\,dx+ch\circ\psi_x.\end{aligned}\tag{97}$$

At this stage, we are in position to compare $L(t)$ with $E(t)$.
We have the following Lemma.
Lemma 10 For N_1 large enough, there exist two positive constants β_1 and β_2 depending on N_1, N_2 and ϵ, such that

$$\beta_1E(t)\le L(t)\le\beta_2E(t)\qquad\forall t\ge 0.\tag{98}$$

Proof. We consider the functional

$$\mathcal{H}(t)=N_2I(t)+F(t)+I_3(t)$$

and show that

$$|\mathcal{H}(t)|\le\hat{C}E(t),\qquad C>0.$$

from (74),(83),(78) and (89), we obtain

$$\begin{aligned}&|\mathcal{H}(t)|\le N_2\left|\int_0^1\rho_2\psi_t\psi+\rho_1\varphi_t\omega)(x,t)dx\right|+\left|-\int_0^1(\rho_1\varphi_t\varphi+\rho_2\psi_t\psi)dx\right|+\\&\left|\rho_2\int_0^1\psi_t(\varphi_x+\psi)dx+\rho_2\int_0^t\psi_x\varphi_t\,dx-\tfrac{\rho_1}{k}\int_0^1\varphi_t\int_0^t h(t-s)\psi_x(s)dsdx\right|\\&+\left|\tfrac{1}{4\varepsilon}\int_0^1\rho_2m(x)\psi_t(b\psi_x-\int_0^t h(t-s)\psi_x(s)ds)\,dx+\tfrac{\varepsilon}{k}\int_0^1\rho_1m(x)\varphi_t\varphi_x\,dx\right|\\&+\left|\int_0^1\int_0^1e^{-2\tau(t)\rho}G_2(z(x,\rho,t))\,d\rho\,dx\right|.\end{aligned}\tag{99}$$

By using (87),(84), the trivial relation

$$\int_0^1\varphi^2(x,t)dx\le 2\int_0^1(\varphi_x+\psi)^2(x,t)dx+2\int_0^1\psi_x^2(x,t)dx,$$

Young's and Poincaré's inequalities, we get

$$\begin{aligned}|\mathcal{H}(t)|\le\alpha_1\int_0^1\varphi_t^2(x,t)dx&+\alpha_2\int_0^1\psi_t^2(x,t)dx\\&+\alpha_3\int_0^1(\varphi_x+\psi)^2(x,t)dx+\alpha_4\int_0^1\psi_x^2(x,t)dx\\&+\alpha_5h\circ\psi_x+\int_0^1\int_0^1G_2(z(x,\rho,t))dx\ d\rho\end{aligned}\tag{100}$$

where $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ are positive constants.

According to (100) , we have

$$|H(t)|\le\hat{C}E(t)$$

for

$$\hat{C}=2\max\left\{\frac{\alpha_1}{\rho_1},\frac{\alpha_2}{\rho_2},\frac{\alpha_3}{k},\frac{\alpha_4}{b},\frac{1}{2\xi}\right\}.$$

Therefore, we obtain

$$|L(t)-N_1E(t)|\le\hat{C}E(t).$$

So, we can choose N_1 large enough so that $\beta_1=N_1-\hat{C}>0,\beta_2=N_1+\hat{C}>0$. Then (98) holds true.
Therefore, (97) takes the form

$$\frac{d}{dt}L(t)\le -C_3E(t)+C_4h\circ\psi_x+C_5(\|\psi'\|_2^2+\|g_1(\psi_t)\|_2^2),\tag{101}$$

where C_3, C_4 and C_5 are three positive constants.

Now, we estimate the last term in the right hand side of (101). We define

$$\Omega^+ = \{x \in (0,1) : \, |\psi'| \geq \varepsilon'\}, \quad \Omega^- = \{x \in (0,1) : \, |\psi'| \leq \varepsilon'\}.$$

From (5) and (6), it follows that

$$\begin{aligned}\mu_1(t) \int_{\Omega^+} (|\psi'|^2 + |g_1(\psi')|^2)\, dx &\leq -d\mu_1(t)E(t) + c_1\mu_1(t) \int_{\Omega^+} \psi' g_1(\psi')\, dx \\ &\leq -d\mu_1(t)E(t) - c_1 E'(t).\end{aligned} \tag{102}$$

Case 1: H is linear on $[0, \varepsilon']$. In this case one can easily check that there exists $c_1' > 0$, such that $|g_1(s)| \leq c_1'|s|$ for all $|s| \leq \varepsilon'$, and thus

$$\begin{aligned}\mu_1(t) \int_{\Omega^+} (|\psi'|^2 + |g_1(\psi')|^2)\, dx &\leq -d\mu_1(t)E(t) + c_1\mu_1(t) \int_{\Omega^+} \psi' g_1(\psi')\, dx \\ &\leq -d\mu_1(t)E(t) - c_1 E'(t).\end{aligned} \tag{103}$$

Substitution of (102) and (103) into (101) gives

$$(\mu_1(t)L(t) + cE(t))'(t) \leq -d\mu_1(t)H_2(E(t)) + C_4 h \circ \psi_x \tag{104}$$

where $c = C_5(c_1 + c_1')$ and here and in the sequel we take C_i to be a generic positive constant.

Case 2: $H'(0) = 0$ and $H'' > 0$ on $]0, \varepsilon']$.

Since H is convex and increasing, H^{-1} is concave and increasing. By the virtue of (5), the reversed Jensen's inequality for concave function, and (20), it follows that

$$\begin{aligned}\mu_1(t) \int_{\Omega^-} (|\psi'|^2 + |g_1(\psi')|^2)\, dx &\leq \mu_1(t) \int_{\Omega^-} H^{-1}(\psi' g_1(\psi'))\, dx \\ &\leq \mu_1(t)|\Omega| H^{-1}\left(\frac{1}{|\Omega|}\int_{\Omega^-} \psi' g_1(\psi')\, dx\right) \\ &\leq C\mu_1(t)H^{-1}(-C'E'(t)).\end{aligned} \tag{105}$$

A combination of (101), (102) and (105) yields

$$\begin{aligned}(\mu_1(t)L(t) + C_5c_1E(t))'(t) \leq &-C_3\mu_1(t)E(t) + C_4(h \circ \psi_x)(t) \\ &+\tilde{C}_5\mu_1(t)H^{-1}(-C'E'(t)), \quad t \geq 0.\end{aligned} \tag{106}$$

Let us denote by H^* the conjugate function of the convex function H, i.e.,

$$H^*(s) = \sup_{t \in R_+} (st - H(t)).$$

Then H^* is the Legendre transform of H, which is given by

$$H^*(s) = s(H')^{-1}(s) - H[(H')^{-1}(s)], \quad \forall s \geq 0 \tag{107}$$

and which satisfies the following inequality

$$st \leq H^*(s) + H(t), \quad \forall s, t \geq 0. \tag{108}$$

The relation (107) and the fact that $H'(0) = 0$ and $(H')^{-1}, H$ are increasing functions yield

$$H^*(s) \leq s(H')^{-1}(s), \quad \forall s \geq 0. \tag{109}$$

Making use of $E'(t) \leq 0, H''(t) \geq 0$, (106) and (109) we derive for $\varepsilon_0 > 0$ small enough

$$
\begin{aligned}
& [H'(\varepsilon_0 E(t))\{\mu_1(t)L(t) + C_5 c_1 E(t)\} + \tilde{C}_5 C' E(t)]' \\
= \; & \varepsilon_0 E'(t) H''(\varepsilon_0 E(t))(\mu_1(t)L(t) + C_5 c_1 E(t)) + H'(\varepsilon_0 E(t))(L'(t)\mu_1(t) \\
& + \mu_1'(t)L(t) + C_5 c_1 E'(t)) + \tilde{C}_5 C' E'(t) \\
\leq \; & -C_3 \mu_1(t) H'(\varepsilon_0 E(t)) E(t) + C_4 H'(\varepsilon_0 E(t))(h \circ \psi_x)(t) \\
& + \tilde{C}_5 \mu_1(t) H'(\varepsilon_0 E(t)) H^{-1}(-C' E'(t)) + \tilde{C}_5 C' E'(t) \\
\leq \; & -C_3 \mu_1(t) H'(\varepsilon_0 E(t)) E(t) + \tilde{C}_5 \mu_1(t) H^*(H'(\varepsilon_0 E(t))) \\
& + C_4 H'(\varepsilon_0 E(0))(h \circ \psi_x)(t) \\
\leq \; & -C_3 \mu_1(t) H'(\varepsilon_0 E(t)) E(t) + \tilde{C}_5 \mu_1(t) H'(\varepsilon_0 E(t)) \varepsilon_0 E(t) \\
& + C_4 H'(\varepsilon_0 E(0))(h \circ \psi_x)(t) \\
\leq \; & -\tilde{C}_3 \mu_1(t) H'(\varepsilon_0 E(t)) E(t) + C_4 H'(\varepsilon_0 E(0))(h \circ \psi_x)(t) \\
= \; & -\tilde{C}_3 \mu_1(t) H_2(E(t)) + C_4 H'(\varepsilon_0 E(0))(h \circ \psi_x)(t).
\end{aligned}
\tag{110}
$$

We note that in the second inequality, we have used (108) and
$0 \leq H'(\varepsilon_0 E(t)) \leq H'(\varepsilon_0 E(0))$.

Let

$$
\tilde{L}(t) = \begin{cases} \mu_1(t)L(t) + cE(t) & \text{if } H \text{ is linear on } [0, \varepsilon'] \\ H'(\varepsilon_0 E(t))\{\mu_1(t)L(t) & \\ \qquad + C_5 c_1 E(t)\} + \tilde{C}_5 C' E(t) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0, \varepsilon']. \end{cases}
\tag{111}
$$

From (104) and (110), it follows

$$
\tilde{L}'(t) \leq -c_4 \mu_1(t) H_2(E(t)) + c_5 (h \circ \psi_x)(t), \quad \forall t \geq 0. \tag{112}
$$

On the other hand, by choosing $M > 0$ larger if needed, we can observe from Lemma 10 that $L(t)$ is equivalent to $E(t)$. So, $\tilde{L}(t)$ is also equivalent to $E(t)$. Moreover, because the fact that $\zeta'(t) \leq 0$, there exists $\overline{\varepsilon} > 0$, such that

$$
\zeta(t)\tilde{L}(t) + 2c_5 E(t) \leq \overline{\varepsilon} E(t), \quad \forall t \geq t_0. \tag{113}
$$

Finally, let

$$
\mathcal{L}(t) = \varepsilon(\zeta(t)\tilde{L}(t) + 2c_5 E(t)), \qquad \text{for } 0 < \varepsilon < \frac{1}{\overline{\varepsilon}},
$$

then we observe, from (112), (H1), (20) and (113), that

$$
\begin{aligned}
\mathcal{L}'(t) \; = \; & \varepsilon(\zeta'(t)\tilde{L}(t) + \zeta(t)\tilde{L}'(t) + 2c_5 E'(t)) \\
\leq \; & -c_4 \varepsilon \mu_1(t) \zeta(t) H_2(E(t)) + c_5 \varepsilon \zeta(t)(h \circ \psi_x)(t) + 2c_5 \varepsilon E'(t) \\
\leq \; & -c_4 \varepsilon \mu_1(t) \zeta(t) H_2(E(t)) - c_5 \varepsilon (h' \circ \psi_x)(t) + 2c_5 \varepsilon E'(t) \\
\leq \; & -c_4 \varepsilon \mu_1(t) \zeta(t) H_2(E(t)) \\
\leq \; & -c_4 \varepsilon \mu_1(t) \zeta(t) H_2\left(\tfrac{1}{\overline{\varepsilon}}\left(\zeta(t)\tilde{L}(t) + 2c_5 E(t)\right)\right) \\
\leq \; & -c_4 \varepsilon \mu_1(t) \zeta(t) H_2(\varepsilon(\zeta(t)\tilde{L}(t) + 2c_5 E(t))) \\
= \; & -c_4 \varepsilon \mu_1(t) \zeta(t) H_2(\mathcal{L}(t)).
\end{aligned}
\tag{114}
$$

We have used the fact H_2 is increasing in the last two inequalities. Noting that $H_1' = -1/H_2$ (see (19)), we infer from (114)

$$
\mathcal{L}'(t) H_1'(\mathcal{L}(t)) \geq c_4 \varepsilon \mu_1(t) \zeta(t), \quad \forall t \geq t_0.
$$

A simple Integration over (t_0, t) then yields

$$H_1(\mathcal{L}(t)) \geq H_1(\mathcal{L}(t_0)) + c_4\varepsilon \int_0^t \mu_1(t)\zeta(s)\,ds - c_4\varepsilon \int_0^{t_0} \mu_1(t)\zeta(s)\,ds.$$

Choose $\varepsilon > 0$ sufficiently small so that $H_1(\mathcal{L}(t_0)) - c_4\varepsilon \int_0^{t_0} \alpha(t)\zeta(s)\,ds > 0$, then, thanks to the fact H_1^{-1} is decreasing, we infer

$$\begin{aligned} \mathcal{L}(t) &\leq H_1^{-1}\left(H_1(\mathcal{L}(t_0)) - c_4\varepsilon \int_0^{t_0} \mu_1(t)\zeta(s)\,ds + c_4\varepsilon \int_0^t \mu_1(t)\zeta(s)\,ds\right) \\ &\leq H_1^{-1}\left(c_4\varepsilon \int_0^t \mu_1(t)\zeta(s)\,ds\right). \end{aligned}$$

Consequently, the equivalence of $\mathcal{L}, \tilde{L}, L$ and E, yield

$$E(t) \leq C_0 H_1^{-1}\left(\omega \int_0^t \mu_1(t)\zeta(s)\,ds\right).$$

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