

COUPLED AND TRIPLED FIXED POINT THEOREMS FOR WEAK CONTRACTIONS IN WEAK PARTIAL METRIC SPACES

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ABSTRACT. In the present paper, we prove coupled and tripled fixed point theorems for (ψ, ϕ) contractions on complete weak partial metric spaces which generalize certain corresponding results of Aydi et al. besides some other ones.

1. INTRODUCTION

The notion of partial metric space was introduced by Matthews [19] as a part of his study of denotational semantics of data flow network. By now, it remains an established fact that partial metric spaces play an important role in developing models in the theory of computations. In partial metric spaces, the distance of a point from itself need not be zero. Besides initiating the definition of a partial metric space, Matthews [19] also proved a partial metric space version of Banach contraction principle. In recent years, Valero [23], Oltra and Valero [22], Altun et al [4], Altun and Sola [3] and some others proved some generalizations of partial metric space version of Banach contraction principle proved in Matthews [19]. For the work of this kind, one can be eferred to [6, 7, 8, 9, 16, 19].

On the other hand, the idea of coupled fixed point was initiated by Guo and Lakshmikantham [13] which was also utilized by Bhaskar and Lakshmikantham [14] wherein authors introduced the notion of mixed monotone property for a weakly linear contractive mapping $F : X \times X \rightarrow X$, (wherein X is a partially ordered metric space) and utilized the same to prove some theorems on existence and uniqueness of coupled fixed points, which can be viewed as coupled formulation of certain results of Nieto and Lopez [20]. Recently, many authors obtained important coupled fixed point theorems and their generalizations (e.g.[2, 17, 21]). Also, Berinde and Borcut [12] introduced the concept of tripled fixed point. For further details, we refer the readers to [1, 10].

In this paper, we prove coupled as well as tripled fixed point theorems for weak

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contractions on weak partial metric spaces which generalize the corresponding results of Aydi et al. [11] besides some other ones.

Definition 1 [14]. Let (X, \preceq) be a partially ordered set and $F : X \times X \rightarrow X$ a mapping. The mapping F is said to have the mixed monotone property if F is monotone non-decreasing in its first argument and monotone non-increasing in its second argument i.e., for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$$

and

$$y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).$$

Definition 2 [14]. Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \rightarrow X$ if $x = F(x, y)$ and $y = F(y, x)$.

Example 1 Let $X = [0, \infty)$ and $F : X \times X \rightarrow X$ be defined by $F(x, y) = x + y$ for all $x, y \in X$. Clearly, F has a unique coupled fixed point namely: $(0, 0)$.

Definition 3 [12]. Let (X, \preceq) be a partially ordered set and $F : X \times X \times X \rightarrow X$. The mapping F is said to have the mixed monotone property if F is monotone non-decreasing in its first and third argument and monotone non-increasing in its second argument i.e., for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \preceq x_2 \Rightarrow F(x_1, y, z) \preceq F(x_2, y, z),$$

$$y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow F(x, y_1, z) \succeq F(x, y_2, z),$$

and

$$z_1, z_2 \in X, z_1 \preceq z_2 \Rightarrow F(x, y, z_1) \preceq F(x, y, z_2).$$

Definition 4 [12]. Let X be a nonempty set. An element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of the mapping $F : X \times X \times X \rightarrow X$ if $x = F(x, y, z)$, $y = F(y, x, z)$ and $z = F(z, y, x)$.

Definition 5 [19]. A partial metric on a nonempty set X is a function $p : X \times X \rightarrow [0, \infty)$ such that for all $x, y, z \in X$:

$$(p^1) \quad x = y \iff p(x, x) = p(x, y) = p(y, y),$$

$$(p^2) \quad p(x, x) \leq p(x, y),$$

$$(p^3) \quad p(x, y) = p(y, x),$$

$$(p^4) \quad p(x, y) \leq p(x, z) + p(z, y) - p(z, z).$$

A partial metric space (abbreviated as PMS) is a pair (X, p) such that X is a nonempty set and p a partial metric on X .

Definition 6 [15]. A weak partial metric space (abbreviated as WPMS) on a nonempty set X is a function $p^w : X \times X \rightarrow [0, \infty)$ such that for all $x, y, z \in X$:

$$(p^w_1) \quad x = y \iff p^w(x, x) = p^w(x, y) = p^w(y, y) \text{ (} T_0\text{-separation axiom),}$$

$$(p^w_2) \quad p^w(x, y) = p^w(y, x) \text{ (symmetry),}$$

$(p_3^w) p^w(x, y) \leq p^w(x, z) + p^w(z, y) - p^w(z, z)$ (modified triangular inequality).

Recall that Heckmann [15] has shown that, if p^w is a weak partial metric on X , then for all $x, y \in X$, we have the following weak small self-distance property:

$$p^w(x, y) \geq \frac{p^w(x, x) + p^w(y, y)}{2}.$$

Weak small self-distance property reflects that WPMS are not far from small self-distance axiom. Clearly, every PMS is a WPMS, but not conversely.

Remark 1 [5]. If p is partial metric on X , then the functions $d_p, d_w : X \times X \rightarrow [0, \infty)$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y) \quad (1)$$

and

$$\begin{aligned} d_w(x, y) &= \max\{p(x, y) - p(x, x), p(x, y) - p(y, y)\} \\ &= p(x, y) - \min\{p(x, x), p(y, y)\} \end{aligned} \quad (2)$$

are ordinary metrics on X .

Proposition 1 [5]. If $a, b, c \in \mathbb{R}_+$, then we have

$$\min\{a, c\} + \min\{b, c\} \leq \min\{a, b\} + c.$$

Proposition 2 [5]. If (X, p^w) is a WPMS, then $d_w : X \times X \rightarrow [0, \infty)$, defined by (2) is an ordinary metric on X .

In a WPMS, the convergence of a sequence, Cauchy sequence, completeness and continuity of a function are defined in the same manner as in PMS. To give some fixed point results on a WPMS, we need the following lemma:

Lemma 1 [5]. Let (X, p^w) be WPMS. Then

- (a) $\{x_n\}$ is a Cauchy sequence in (X, p^w) if and only if it is a Cauchy sequence in the metric space (X, d_w) ,
- (b) (X, p^w) is complete if and only if (X, d_w) is complete.

Definition 7 [5]. Let (X, p^w) be a WPMS. A sequence $\{x_n\}$ is called a p^w convergent to $x \in X$ if $\lim_{n, m \rightarrow \infty} p^w(x_n, x_m) = p^w(x, x)$. Such a point $x \in X$ is called the limit of the sequence $\{x_n\}$ and is denoted by $x_n \rightarrow x$.

Thus if $x_n \rightarrow x$ in a WPMS (X, p^w) , then for any $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $|p^w(x_n, x_m) - p^w(x, x)| < \epsilon$, for all $n, m > n_\epsilon$.

Definition 8 [18]. Let Ψ be the set of all functions $\psi : [0, +\infty) \rightarrow [0, +\infty)$ which satisfy

- (i) ψ is continuous and non-decreasing,

(ii) $\psi(t) = 0$ if and only if $t = 0$,

(iii) $\psi(t + s) \leq \psi(t) + \psi(s)$, $\forall t, s \in [0, +\infty)$.

Again, let Φ be the set all functions $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ which satisfy $\lim_{t \rightarrow r} \varphi(t) > 0$ for all $r > 0$ and $\lim_{t \rightarrow 0^+} \varphi(t) = 0$.

Remark 2 Let Ψ be the set of all functions $\psi : [0, +\infty) \rightarrow [0, +\infty)$ which satisfy (iii). Then for any $t \in [0, +\infty)$, we have

$$\frac{1}{2}\psi(t) \leq \psi\left(\frac{t}{2}\right).$$

Hassen Aydi et al. [11] proved the following coupled fixed point theorem employing a relatively more general contraction condition which generalizes some relevant results due to Luong and Thuan[18].

Theorem 1 Let (X, \leq) be partially ordered set equipped with a partial metric p on X such that (X, p) is a complete PMS. Let $F : X \times X \rightarrow X$ be a mapping enjoying the mixed monotone property on X . Assume that there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\psi(p(F(x, y), F(u, v))) \leq \psi\left(\frac{p(x, u) + p(y, v)}{2}\right) - \varphi\left(\frac{p(x, u) + p(y, v)}{2}\right)$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$. Suppose either F is continuous or X has the following properties:-

(i) if a non-decreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all n ,

(ii) if a non-increasing sequence $x_n \rightarrow x$, then $x_n \geq x$ for all n .

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$ i.e., F has a coupled fixed point.

In this paper, we employ a relatively new weak contraction condition to prove some coupled and tripled fixed point theorems in weak partial metric spaces.

2. RESULTS ON COUPLED FIXED POINTS

Now, we are equipped to prove our main result as follows.

Theorem 2 Let (X, \leq) be partially ordered set equipped with a a weak partial metric p^w on X such that (X, p^w) is a complete WPMS. Let $F : X \times X \rightarrow X$ be a mapping enjoying the mixed monotone property on X . Assume that there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\begin{aligned} \psi(p^w(F(x, y), F(u, v))) &\leq \psi\left(\frac{M(x, u) + M(y, v)}{2}\right) \\ &\quad - \varphi\left(\frac{M(x, u) + M(y, v)}{2}\right), \end{aligned} \quad (3)$$

where

$$\begin{aligned} M(x, u) &= \max\{p^w(x, u), p^w(x, F(x, y)), p^w(u, F(u, v)), \\ &\quad \frac{1}{2}[p^w(u, F(x, y)) + p^w(x, F(u, v))]\}, \\ M(y, v) &= \max\{p^w(y, v), p^w(y, F(y, x)), p^w(v, F(v, u)), \\ &\quad \frac{1}{2}[p^w(v, F(y, x)) + p^w(y, F(v, u))]\}, \end{aligned}$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$. Suppose either F is continuous or X has the following properties:-

(i) if a non-decreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all n ,

(ii) if a non-increasing sequence $x_n \rightarrow x$, then $x_n \geq x$ for all n .

If there exist $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$ i.e., F has a coupled fixed point. Furthermore, $p^w(x, x) = p^w(y, y) = 0$.

Proof. Since $x_0 \leq F(x_0, y_0) = x_1$ (say) and $y_0 \geq F(y_0, x_0) = y_1$ (say), writing $x_2 = F(x_1, y_1)$ and $y_2 = F(y_1, x_1)$, we can have

$$\begin{aligned} F^2(x_0, y_0) &= F(F(x_0, y_0), F(y_0, x_0)) = F(x_1, y_1) = x_2 \\ F^2(y_0, x_0) &= F(F(y_0, x_0), F(x_0, y_0)) = F(y_1, x_1) = y_2. \end{aligned}$$

Owing to the mixed monotone property of F , we have,

$x_2 = F(x_1, y_1) \geq F(x_0, y_0) = x_1$, $y_2 = F(y_1, x_1) \leq F(y_0, x_0) = y_1$ and hence in general (for $n = 1, 2, 3, \dots$)

$$\begin{aligned} x_{n+1} &= F^{n+1}(x_0, y_0) = F(F^n(x_0, y_0), F^n(y_0, x_0)), \\ y_{n+1} &= F^{n+1}(y_0, x_0) = F(F^n(y_0, x_0), F^n(x_0, y_0)). \end{aligned}$$

One can easily verify that

$$\begin{aligned} x_0 &\leq F(x_0, y_0) = x_1 \leq F(x_1, y_1) = x_2 \leq \dots \leq F^{n+1}(x_0, y_0) = x_{n+1}, \\ y_0 &\geq F(y_0, x_0) = y_1 \geq F(y_1, x_1) = y_2 \geq \dots \geq F^{n+1}(y_0, x_0) = y_{n+1}. \end{aligned}$$

As $x_n \geq x_{n-1}$ and $y_n \leq y_{n-1}$, using (3), we have

$$\begin{aligned} \psi(p^w(x_n, x_{n+1})) &= \psi(p^w(F(x_{n-1}, y_{n-1}), F(x_n, y_n))) \\ &\leq \psi\left(\frac{M(x_{n-1}, x_n) + M(y_{n-1}, y_n)}{2}\right) \\ &\quad - \varphi\left(\frac{M(x_{n-1}, x_n) + M(y_{n-1}, y_n)}{2}\right) \\ &\leq \psi\left(\frac{M(x_{n-1}, x_n) + M(y_{n-1}, y_n)}{2}\right). \end{aligned}$$

Similarly, as $y_{n-1} \geq y_n$ and $x_n \leq x_{n+1}$, using (3), we have

$$\begin{aligned} \psi(p^w(y_n, y_{n+1})) &= \psi(p^w(F(y_{n-1}, x_{n-1}), F(y_n, x_n))) \\ &\leq \psi\left(\frac{M(y_{n-1}, y_n) + M(x_{n-1}, x_n)}{2}\right) - \\ &\quad \varphi\left(\frac{M(y_{n-1}, y_n) + M(x_{n-1}, x_n)}{2}\right) \\ &\leq \psi\left(\frac{M(y_{n-1}, y_n) + (M(x_{n-1}, x_n))}{2}\right). \end{aligned}$$

Observe that

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{p^w(x_{n-1}, x_n), p^w(x_{n-1}, F(x_{n-1}, y_{n-1})), p^w(x_n, F(x_n, y_n)), \\ &\quad \frac{1}{2}[p^w(x_{n-1}, F(x_n, y_n)) + p^w(x_n, F(x_{n-1}, y_{n-1}))]\} \\ &= \max\{p^w(x_{n-1}, x_n), p^w(x_{n-1}, x_n), p^w(x_n, x_{n+1}), \\ &\quad \frac{1}{2}[p^w(x_{n-1}, x_{n+1}) + p^w(x_n, x_n)]\} \\ &\leq \max\{p^w(x_{n-1}, x_n), p^w(x_n, x_{n+1})\}. \end{aligned} \quad (4)$$

Similarly,

$$M(y_{n-1}, y_n) \leq \max\{p^w(y_{n-1}, y_n), p^w(y_n, y_{n+1})\}. \quad (5)$$

Now, we distinguish the following cases:-

Case 1. If $M(x_{n-1}, x_n) = p^w(x_n, x_{n+1})$ and $M(y_{n-1}, y_n) = p^w(y_n, y_{n+1})$, then using (4), (5) and non-decreasing property of ψ , we have

$$p^w(x_n, x_{n+1}) \leq \left(\frac{p^w(x_n, x_{n+1}) + p^w(y_n, y_{n+1})}{2}\right), \quad (6)$$

and

$$p^w(y_n, y_{n+1}) \leq \left(\frac{p^w(y_n, y_{n+1}) + p^w(x_n, x_{n+1})}{2}\right). \quad (7)$$

Since

$$\begin{aligned} \min\{p^w(x_n, x_{n+1}), p^w(y_n, y_{n+1})\} &\leq \frac{p^w(x_n, x_{n+1}) + p^w(y_n, y_{n+1})}{2} \\ &\leq \max\{p^w(x_n, x_{n+1}), p^w(y_n, y_{n+1})\} \end{aligned} \quad (8)$$

using (6), (7), and (8), we have

$$p^w(x_n, x_{n+1}) = p^w(y_n, y_{n+1}).$$

Making use of the condition (3), we get:

$$\begin{aligned} \psi(p^w(x_n, x_{n+1})) &\leq \psi\left(\frac{M(x_{n-1}, x_n) + M(y_{n-1}, y_n)}{2}\right) \\ &\quad - \varphi\left(\frac{M(x_{n-1}, x_n) + M(y_{n-1}, y_n)}{2}\right) \\ &\leq \psi(p^w(x_n, x_{n+1})) - \varphi(p^w(x_n, x_{n+1})), \end{aligned}$$

which is a contradiction to the fact that $\lim_{t \rightarrow r} \varphi(t) > 0, \forall r > 0$.

Case 2. If $M(x_{n-1}, x_n) = p^w(x_{n-1}, x_n)$ and $M(y_{n-1}, y_n) = p^w(y_{n-1}, y_n)$, then using (4), (5) and non-decreasing property of ψ , we have

$$p^w(x_n, x_{n+1}) \leq \left(\frac{p^w(x_{n-1}, x_n) + p^w(y_{n-1}, y_n)}{2} \right), \quad (9)$$

and

$$p^w(y_n, y_{n+1}) \leq \left(\frac{p^w(y_{n-1}, y_n) + p^w(x_{n-1}, x_n)}{2} \right). \quad (10)$$

On adding (9) and (10), we have

$$p^w(x_n, x_{n+1}) + p^w(y_n, y_{n+1}) \leq p^w(x_{n-1}, x_n) + p^w(y_{n-1}, y_n).$$

Write $t_n = p^w(x_{n+1}, x_n) + p^w(y_n, y_{n+1})$. Then the sequence t_n is non-increasing and bounded below, therefore there is some $t \geq 0$ such that

$$\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} [p^w(x_n, x_{n+1}) + p^w(y_n, y_{n+1})] = t. \quad (11)$$

Now, we show that $t = 0$. Assume that $t > 0$. By putting $M_1 = p^w(x_n, x_{n+1})$, and $M_2 = p^w(y_n, y_{n+1})$, we have

$$\begin{aligned} \psi\left(\frac{M_1 + M_2}{2}\right) &\leq \psi(\max\{M_1, M_2\}) \\ &= \max\{\psi(M_1), \psi(M_2)\} \\ &\leq \psi\left(\frac{M_1 + M_2}{2}\right) - \varphi\left(\frac{M_1 + M_2}{2}\right). \end{aligned}$$

On taking the limit as $n \rightarrow \infty$ besides using (11), the fact $\lim_{y \rightarrow r} \varphi(y) > 0$ (for all $r > 0$) and continuity of ψ , we have

$$\begin{aligned} \psi\left(\frac{t}{2}\right) &= \lim_{n \rightarrow \infty} \psi\left(\frac{t_n}{2}\right) \\ &\leq \lim_{n \rightarrow \infty} \left[\psi\left(\frac{t_{n-1}}{2}\right) - \varphi\left(\frac{t_{n-1}}{2}\right) \right] \\ &= \psi\left(\frac{t}{2}\right) - \lim_{t_{n-1} \rightarrow t} \varphi\left(\frac{t_{n-1}}{2}\right) \end{aligned}$$

which is a contradiction to $\lim_{t \rightarrow r} \varphi(t) > 0, \forall r > 0$ so that $t = 0$ and henceforth

$$\begin{aligned} \lim_{n \rightarrow \infty} p^w(x_n, x_{n+1}) &= 0, \\ \lim_{n \rightarrow \infty} p^w(y_n, y_{n+1}) &= 0. \end{aligned}$$

Case 3. If $M(x_{n-1}, x_n) = p^w(x_{n-1}, x_n)$ and $M(y_{n-1}, y_n) = p^w(y_n, y_{n+1})$, then using the non-decreasing property of function ψ along with equations (4) and (5), we have

$$p^w(x_n, x_{n+1}) \leq \left(\frac{p^w(x_{n-1}, x_n) + p^w(y_n, y_{n+1})}{2} \right), \quad (12)$$

and

$$p^w(y_n, y_{n+1}) \leq \left(\frac{p^w(x_{n-1}, x_n) + p^w(y_n, y_{n+1})}{2} \right). \quad (13)$$

On using (12) and (13), we have

$$\begin{aligned} p^w(y_n, y_{n+1}) &\leq \left(\frac{p^w(y_n, y_{n+1}) + p^w(x_n, x_{n-1})}{2} \right) \\ &\leq p^w(x_n, x_{n-1}), \end{aligned}$$

so that in view of the condition (3), we have

$$\begin{aligned} \psi(p^w(x_n, x_{n+1})) &\leq \psi\left(\frac{M(y_{n-1}, y_n) + M(x_{n-1}, x_n)}{2}\right) \\ &\leq \psi\left(\frac{p^w(y_{n+1}, y_n) + p^w(x_{n-1}, x_n)}{2}\right) \\ &\leq \psi\left(\frac{p^w(x_{n-1}, x_n) + p^w(x_{n-1}, x_n)}{2}\right) \\ &\leq \psi(p^w(x_{n-1}, x_n)). \end{aligned}$$

Following the steps involved in proving $t = 0$ in Case 2, let us put $t_n = p^w(x_n, x_{n+1})$. Then the sequence t_n is non-increasing and bounded below, therefore there is some $t \geq 0$, such that

$$\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} [p^w(x_n, x_{n+1})] = 0.$$

Similarly, $0 \leq p^w(y_n, y_{n+1}) \leq p^w(x_{n-1}, x_n)$. By taking the limit as $n \rightarrow \infty$, we have

$$0 \leq \lim_{n \rightarrow \infty} [p^w(y_n, y_{n+1})] \leq \lim_{n \rightarrow \infty} [p^w(x_{n-1}, x_n)] = 0, \text{ so that } \lim_{n \rightarrow \infty} [p^w(y_n, y_{n+1})] = 0.$$

Case 4: If $M(x_{n-1}, x_n) = p^w(x_{n+1}, x_n)$ and $M(y_{n-1}, y_n) = p^w(y_{n-1}, y_n)$, then the proof is similar to the proof of Case 3.

Using (11) and the fact $\lim_{y \rightarrow r} \varphi(y) > 0$ for all $r > 0$ along with continuity of ψ , we have

$$\begin{aligned} \psi\left(\frac{t}{2}\right) &= \lim_{n \rightarrow \infty} \psi\left(\frac{t_n}{2}\right) \\ &\leq \lim_{n \rightarrow \infty} \left[\psi\left(\frac{t_{n-1}}{2}\right) - \varphi\left(\frac{t_{n-1}}{2}\right) \right] \\ &= \psi\left(\frac{t}{2}\right) - \lim_{t_{n-1} \rightarrow t} \varphi\left(\frac{t_{n-1}}{2}\right) \end{aligned}$$

which is a contradiction to the fact $\lim_{t \rightarrow r} \varphi(t) > 0, \forall r > 0$ so that $t = 0$. Thus, we conclude that

$$\begin{aligned} \lim_{n \rightarrow \infty} p^w(x_n, x_{n+1}) &= 0, \\ \lim_{n \rightarrow \infty} p^w(y_n, y_{n+1}) &= 0. \end{aligned} \tag{14}$$

As

$$p^w(x_n, x_n) + p^w(x_{n+1}, x_{n+1}) \leq 2p^w(x_n, x_{n+1}),$$

using (14), we have

$$\lim_{n \rightarrow \infty} p^w(x_n, x_n) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} p^w(y_n, y_n) = 0.$$

Therefore, in view of the definition of d_w , we have

$$p^w(x_n, x_m) = d_w(x_n, x_m) + \min\{p^w(x_n, x_n), p^w(x_m, x_m)\}.$$

so that $\lim_{n,m \rightarrow \infty} d_w(x_n, x_m) = 0$. Thus, using (14), we have

$$p^w(x, x) = \lim_{n \rightarrow \infty} p^w(x_n, x) = \lim_{n \rightarrow \infty} p^w(x_n, x_m) = 0,$$

$$p^w(y, y) = \lim_{n \rightarrow \infty} p^w(y_n, y) = \lim_{n \rightarrow \infty} p^w(y_n, y_m) = 0.$$

Since (X, p^w) is complete, therefore (X, d_w) is also complete so that there exist $x \in X$ such that

$$\lim_{n \rightarrow \infty} d_w(x_n, x) = 0.$$

Now, we show that $x = F(x, y)$ and $y = F(y, x)$. To accomplish this, assume that X satisfies conditions (i) and (ii) (of Theorem 2). Since x_n is a non-decreasing sequence with $x_n \rightarrow x$ and as y_n is a non-increasing sequence with $y_n \rightarrow y$, we have $x_n \leq x$ and $y_n \geq y$ for all n while making use of the condition (p^3) , we have

$$\begin{aligned} p^w(x, F(x, y)) &\leq p^w(x, x_{n+1}) + p^w(x_{n+1}, F(x, y)) \\ &= p^w(x, x_{n+1}) + p^w(F(x_n, y_n), F(x, y)). \end{aligned}$$

Therefore,

$$\begin{aligned} \psi(p^w(x, F(x, y))) &\leq \psi(p^w(x, x_{n+1})) + \psi(p^w(F(x_n, y_n), F(x, y))) \\ &\leq \psi(p^w(x, x_{n+1})) + \psi\left(\frac{M(x_n, x) + M(y_n, y)}{2}\right) \\ &\quad - \varphi\left(\frac{M(x_n, x) + M(y_n, y)}{2}\right) \\ &\leq \psi(p^w(x, x_{n+1})) + \psi\left(\frac{p^w(x_n, x) + p^w(y_n, y)}{2}\right) \\ &\quad - \varphi\left(\frac{p^w(x_n, x) + p^w(y_n, y)}{2}\right). \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ and using

$$\lim_{n \rightarrow \infty} p^w(x_n, x) = \lim_{n \rightarrow \infty} p^w(y_n, y) = 0,$$

together with the properties of ψ and φ , we get $\psi(p^w(x, F(x, y))) = 0$ so that $p^w(x, F(x, y)) = 0$ or $x = F(x, y)$. Similarly, one can also show that $y = F(y, x)$. Thus we have shown that F has a coupled fixed point. This concludes the proof.

Using $\frac{1}{2}\psi(t) \leq \psi(\frac{t}{2})$ in Theorem 2.1, we obtain the following:

Corollary 1 Let (X, \leq) be partially ordered set and there is a weak partial metric p^w on X such that (X, p^w) is a complete WPMS. Let $F : X \times X \rightarrow X$ be a mapping having the mixed monotone property on X . Assume that there exist $\psi \in \Psi$ and $\varphi \in \Phi$, such that

$$\begin{aligned} \psi(p^w(F(x, y), F(u, v))) &\leq \frac{1}{2}\psi(M(x, u) + M(y, v)) \\ &\quad - \varphi\left(\frac{M(x, u) + M(y, v)}{2}\right), \end{aligned}$$

where

$$M(x, u) = \max\{p^w(x, u), p^w(x, F(x, y)), p^w(u, F(u, v)), \frac{1}{2}[p^w(u, F(x, y)) + p^w(x, F(u, v))]\},$$

$$M(y, v) = \max\{p^w(y, v), p^w(y, F(y, x)), p^w(v, F(v, u)), \frac{1}{2}[p^w(v, F(y, x)) + p^w(y, F(v, u))]\}.$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$. Suppose either F is continuous or X has the following properties:-

(i) if a non-decreasing $x_n \rightarrow x$, then $x_n \leq x \forall n$.

(ii) if a non-increasing $x_n \rightarrow x$, then $x_n \geq x \forall n$.

if there exist $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$ i.e., F has a coupled fixed point. Furthermore,

$$p^w(x, x) = p^w(y, y) = 0.$$

Choosing $\varphi(t) = \frac{1-k}{2}t$ in Corollary 1, we deduce the following:

Corollary 2 Let (X, \leq) be a partially ordered set and suppose there is a weak partial metric p^w on X such that (X, p^w) is a complete metric space. Let $F : X \times X \rightarrow X$ be a mapping having the mixed monotone property on X . Assume that there exists a real number $k \in [0, 1)$ such that,

$$p^w(F(x, y), F(u, v)) \leq \frac{k}{2}(M(x, u) + M(y, v)),$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$. Suppose either F is continuous or X has the following properties:

(i) if a non-decreasing sequence $x_n \rightarrow x$, then $x_n \leq x, \forall n$,

(ii) if a non-increasing sequence $x_n \rightarrow x$, then $x \leq x_n, \forall n$.

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$ i.e., f has a coupled fixed point. Also $p^w(x, x) = p^w(y, y) = 0$.

Theorem 3 If in addition to the hypotheses of Theorem 2, x_0 and y_0 are comparable, then $x = F(x, y) = F(y, x) = y$ i.e., (x, y) a coupled fixed point.

Proof. In view of Theorem 2, F has a coupled fixed point (x, y) . We are merely required to show that $x = y$. Since x_0 and y_0 are comparable, we may assume that $x_0 \geq y_0$. Using the mathematical induction, one can show that $x_n \geq y_n$ for any $n \in N$. Notice that (by (p_3^w))

$$\begin{aligned} p^w(x, y) &\leq p^w(x, x_{n+1}) + p^w(x_{n+1}, y_{n+1}) + p^w(y_{n+1}, y) \\ &= p^w(x, x_{n+1}) + p^w(y_{n+1}, y) + p^w(F(x_n, y_n), F(y_n, x_n)). \end{aligned}$$

Therefore, using conditions (p_2^w) , (p_1^w) and a property of ψ , we have

$$\begin{aligned}\psi(p^w(x, y)) &\leq \psi(p^w(x, x_{n+1}) + p^w(y_{n+1}, y)) + \psi(p^w(F(x_n, y_n), F(y_n, x_n))) \\ &\leq \psi(p^w(x, x_{n+1}) + p^w(y_{n+1}, y)) + \psi(p^w(x_n, y_n)) \\ &\quad - \varphi(p^w(x_n, y_n)),\end{aligned}\tag{15}$$

(16)

which together with $\lim_{n \rightarrow \infty} p^w(x_n, x) = 0$ gives rise

$$\lim_{n \rightarrow \infty} p^w(x_n, y_n) = p^w(x, y).$$

Assume that $p^w(x, y) \neq 0$. Letting $n \rightarrow \infty$ in (15), we get

$$\begin{aligned}\psi(p^w(x, y)) &\leq \psi(0 + \psi(p^w(x, y))) - \lim_{n \rightarrow \infty} \varphi(p^w(x_n, y_n)) \\ &= \psi(p^w(x, y)) - \lim_{p^w(x_n, y_n) \rightarrow p^w(x, y)} \varphi(p^w(x_n, y_n)),\end{aligned}$$

i.e.,

$$\lim_{p^w(x_n, y_n) \rightarrow p^w(x, y)} \varphi(p^w(x_n, y_n)) \leq 0,$$

a contradiction. Thus $p(x, y) = 0$, so that $x = y$.

Setting $\varphi(t) = \frac{1-k}{2}t$ in Corollary 1, we deduce the following:

Corollary 3 Let (X, \leq) be a partially ordered set and suppose there is a weak partial metric p^w on X such that (X, p^w) is a complete WPMS. Let $F : X \times X \rightarrow X$ be a mapping having the mixed monotone property on X . Assume that there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\psi(p^w(F(x, y), F(u, v))) \leq \frac{1}{2}\psi(M(x, u) + M(y, v)) - \varphi\left(\frac{M(x, u) + M(y, v)}{2}\right),$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$. Suppose either F is continuous or X has the following properties:

- (i) if a non-decreasing sequence $x_n \rightarrow x$, then $x_n \leq x, \forall n$,
- (ii) if a non-increasing sequence $x_n \rightarrow x$, then $x \leq x_n, \forall n$.

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$ then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$ i.e., F has a coupled fixed point. Also $p^w(x, x) = p^w(y, y) = 0$.

Now, we furnish an example to illustrate Theorem 2.

Example 2 Let $X = [0, \infty)$ and $p^w(x, y) = \frac{x+y}{2}$, then $d_w(x, y) = \frac{1}{2}|x - y|$ and in view of Lemma 1 (X, p^w) is a complete WPMS. Define $F : X \times X \rightarrow X$,

$$F(x, y) = \begin{cases} \frac{2x-y}{2} & ; \quad x \geq 3y, \\ 0 & ; \quad x < 3y. \end{cases}$$

Firstly, we show that the condition (3) of Theorem 2 is satisfied with $\psi(t) = \frac{9}{10}t$ and $\varphi(t) = \frac{3}{10}$. To accomplish this, we distinguish the following cases.

Case 1. If $x \geq 3y$, then $F(x, y) = \frac{2x-y}{2}$ and $F(y, x) = 0$.

Now, we have

$$\begin{aligned}\psi(p^w(F(x, y), F(y, x))) &= \frac{9}{10}p^w(F(x, y), F(y, x)) \\ &= \frac{9}{20}\left(x - \frac{y}{2}\right),\end{aligned}$$

Also,

$$\begin{aligned}M(x, y) &= \max\left\{\frac{x+y}{2}, x - \frac{y}{4}, \frac{y}{2}, \frac{1}{2}\left(x + \frac{y}{4}\right)\right\} \\ &= x - \frac{y}{4},\end{aligned}$$

and

$$\begin{aligned}M(y, x) &= \max\left\{\frac{x+y}{2}, \frac{y}{2}, x - \frac{y}{4}, \frac{1}{2}\left(x + \frac{y}{4}\right)\right\} \\ &= x - \frac{y}{4},\end{aligned}$$

therefore

$$\begin{aligned}\psi(p^w(F(x, y), F(y, x))) &= \frac{9}{20}\left(x - \frac{y}{2}\right) \\ &\leq \frac{9}{10}\left(x - \frac{y}{4}\right) - \frac{3}{10}\left(x - \frac{y}{4}\right) \\ &= \frac{6}{10}\left(x - \frac{y}{4}\right).\end{aligned}$$

Thus, equation (3) is satisfied.

Case 2. If $x < 3y$, then we have two subcases:

(a) If $x < 3y < 9x$ then, $F(x, y) = F(y, x) = 0$.

Hence,

$$\psi(p^w(F(x, y), F(y, x))) = 0$$

Also,

$$M(x, y) = M(y, x) = \frac{x+y}{2},$$

therefore

$$\begin{aligned}\psi(p^w(F(x, y), F(y, x))) &= 0 \\ &\leq \frac{6}{20}(x+y).\end{aligned}$$

Thus, equation (3) is satisfied.

(b) If $3y > 9x$, then
 $F(x, y) = 0$, and $F(y, x) = \frac{2y-x}{2}$.

$$\begin{aligned}\psi(p^w(F(x, y), F(y, x))) &= \frac{9}{20}p^w(F(x, y), F(y, x)) \\ &= \frac{9}{20}\left(y - \frac{x}{2}\right).\end{aligned}$$

Also,

$$\begin{aligned}M(x, y) &= \max\left\{\frac{x+y}{2}, y - \frac{x}{4}, \frac{x}{2}, \frac{1}{2}\left(y + \frac{x}{4}\right)\right\} \\ &= y - \frac{x}{4},\end{aligned}$$

and similarly,

$$M(y, x) = y - \frac{x}{4},$$

therefore

$$\begin{aligned}\psi(p^w(F(x, y), F(y, x))) &= \frac{9}{20}\left(y - \frac{x}{2}\right) \\ &\leq \frac{9}{10}\left(y - \frac{x}{4}\right) - \frac{3}{10}\left(y - \frac{x}{4}\right) \\ &= \frac{6}{10}\left(y - \frac{x}{4}\right).\end{aligned}$$

Thus, equation (3) is verified.

By a routine calculation, one can verify other conditions of Theorem 2. Observe that F has a coupled fixed point $(1, 0) \in X \times X$.

3. A TRIPLED FIXED POINT RESULT IN WPMS.

Lastly, we prove the following tripled fixed point theorem in WPMS.

Theorem 4 Let (X, \leq) be partially ordered set and there is a weak partial metric p^w on X such that (X, p^w) is a complete WPMS. Let $F : X \times X \times X \rightarrow X$ be a mapping enjoying the mixed monotone property on X . Assume that there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that,

$$\begin{aligned}\psi(p^w(F(x, y, z), F(u, v, w))) &\leq \psi\left(\frac{M_1 + M_2 + M_3}{3}\right) \\ &\quad - \varphi\left(\frac{M_1 + M_2 + M_3}{3}\right),\end{aligned}\quad (17)$$

where

$$\begin{aligned} M_1 &= \max\{p^w(x, u), p^w(x, F(x, y, z)), p^w(u, F(u, v, w)), \\ &\quad \frac{1}{2}[p^w(u, F(x, y, z)) + p^w(x, F(u, v, w))]\}, \\ M_2 &= \max\{p^w(y, v), p^w(y, F(y, x, z)), p^w(v, F(v, u, w)), \\ &\quad \frac{1}{2}[p^w(v, F(y, x, z)) + p^w(y, F(v, u, w))]\}, \\ M_3 &= \max\{p^w(z, w), p^w(z, F(z, x, y)), p^w(w, F(w, u, v)), \\ &\quad \frac{1}{2}[p^w(v, F(z, x, y)) + p^w(z, F(w, u, v))]\}. \end{aligned}$$

for all $x, y, z, u, v, w \in X$ with $x \geq u$, $y \leq v$ and $z \leq w$. Suppose either F is continuous or X has the following properties:-

(i) if a non-decreasing $x_n \rightarrow x$, then $x_n \leq x \forall n$.

(ii) if a non-increasing $x_n \rightarrow x$, then $x_n \geq x \forall n$.

If there exist $x_0 \leq F(x_0, y_0, z_0)$, $y_0 \geq F(y_0, x_0, z_0)$ and $z_0 \geq F(z_0, x_0, y_0)$, then there exist $x, y, z \in X$ such that $x = F(x, y, z)$, $y = F(y, x, z)$ and $z = F(z, x, y)$ i.e., F has a Tripled fixed point. Furthermore, $p^w(x, x) = p^w(y, y) = p^w(z, z) = 0$.

Proof. As $x_0 \leq F(x_0, y_0, z_0) = x_1$ (say), $y_0 \geq F(y_0, x_0, z_0) = y_1$ (say) and $z_0 \geq F(z_0, x_0, y_0) = z_1$, putting $x_2 = F(x_1, y_1, z_1)$, $y_2 = F(y_1, x_1, z_1)$ and $z_2 = F(z_1, x_1, y_1)$, we have

$$\begin{aligned} F^2(x_0, y_0, z_0) &= F(F(x_0, y_0, z_0), F(y_0, x_0, z_0), F(z_0, x_0, y_0)) = F(x_1, y_1, z_1) = x_2, \\ F^2(y_0, x_0, z_0) &= F(F(y_0, x_0, z_0), F(x_0, y_0, z_0), F(z_0, x_0, y_0)) = F(y_1, x_1, z_1) = y_2, \\ F^2(z_0, x_0, y_0) &= F(F(z_0, x_0, y_0), F(x_0, y_0, z_0), F(y_0, x_0, z_0)) = F(z_1, x_1, y_1) = z_2. \end{aligned}$$

Then in general, $x_n \geq x_{n-1}$, $y_n \leq y_{n-1}$ and $z_n \leq z_{n-1}$. From (17), we have

$$\begin{aligned} \psi(p^w(x_n, x_{n+1})) &= \psi(p^w(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n))) \\ &\leq \psi\left(\frac{M_1 + M_2 + M_3}{3}\right) - \varphi\left(\frac{M_1 + M_2 + M_3}{3}\right) \\ &\leq \psi\left(\frac{M_1 + M_2 + M_3}{3}\right). \end{aligned} \quad (18)$$

Since, $M_1 = M(x_{n-1}, x_n)$, $M_2 = M(y_{n-1}, y_n)$ and $M_3 = M(z_{n-1}, z_n)$, from (17), we have

$$\begin{aligned} \psi(p^w(y_n, y_{n+1})) &= \psi(p^w(F(y_{n-1}, x_{n-1}, z_{n-1}), F(y_n, x_n, z_n))) \\ &\leq \psi\left(\frac{M(y_{n-1}, y_n) + M(x_{n-1}, x_n) + M(z_{n-1}, z_n)}{3}\right). \end{aligned} \quad (19)$$

Also,

$$\begin{aligned} \psi(p^w(z_n, z_{n+1})) &= \psi(p^w(F(z_{n-1}, x_{n-1}, y_{n-1}), F(z_n, x_n, y_n))) \\ &\leq \psi\left(\frac{M(z_{n-1}, z_n) + M(x_{n-1}, x_n) + M(y_{n-1}, y_n)}{3}\right). \end{aligned} \quad (20)$$

Observe that

$$\begin{aligned}
 M(x_{n-1}, x_n) &= \max\{p^w(x_{n-1}, x_n), p^w(x_{n-1}, F(x_{n-1}, y_{n-1}, z_{n-1})), \\
 &\quad p^w(x_n, F(x_n, y_n, z_n)), \frac{1}{2}[p^w(x_{n-1}, F(x_n, y_n, z_n)) + \\
 &\quad p^w(x_n, F(x_{n-1}, y_{n-1}, z_{n-1}))]\} \\
 &= \max\{p^w(x_{n-1}, x_n), p^w(x_{n-1}, x_n), p^w(x_n, x_{n+1}), \\
 &\quad \frac{1}{2}[p^w(x_{n-1}, x_{n+1}) + p^w(x_n, x_n)]\} \\
 &\leq \max\{p^w(x_{n-1}, x_n), p^w(x_n, x_{n+1}), \\
 &\quad \frac{1}{2}[p^w(x_{n-1}, x_n) + p^w(x_n, x_{n+1})]\} \\
 &\leq \max\{p^w(x_{n-1}, x_n), p^w(x_n, x_{n+1})\}.
 \end{aligned}$$

Similarly, we can show that

$$M(y_{n-1}, y_n) \leq \max\{p^w(y_{n-1}, y_n), p^w(y_n, y_{n+1})\},$$

and

$$M(z_{n-1}, z_n) \leq \max\{p^w(z_{n-1}, z_n), p^w(z_n, z_{n+1})\}.$$

Now, we distinguish the following cases:-

Case 1: If $M(a_{n-1}, a_n) = p^w(a_n, a_{n+1}), \forall a_n \in \{x_n, y_n, z_n\}$, then using (18), (19) and (20), we obtain

$$p^w(x_n, x_{n+1}) \leq \left(\frac{p^w(x_n, x_{n+1}) + p^w(y_n, y_{n+1}) + p^w(z_n, z_{n+1})}{3} \right), \quad (21)$$

$$p^w(y_n, y_{n+1}) \leq \left(\frac{p^w(y_n, y_{n+1}) + p^w(x_n, x_{n+1}) + p^w(z_n, z_{n+1})}{3} \right), \quad (22)$$

and

$$p^w(z_n, z_{n+1}) \leq \left(\frac{p^w(z_n, z_{n+1}) + p^w(x_n, x_{n+1}) + p^w(y_n, y_{n+1})}{3} \right). \quad (23)$$

Write $r = p^w(x_n, x_{n+1})$, $s = p^w(y_n, y_{n+1})$ and $w = p^w(z_n, z_{n+1})$, then

$$\begin{aligned}
 \min\{r, s, w\} &\leq \frac{r + s + w}{3} \\
 &\leq \max\{r, s, w\}.
 \end{aligned} \quad (24)$$

From (21), (22), (23) and (24), we have

$$p^w(x_n, x_{n+1}) = p^w(y_n, y_{n+1}) = p^w(z_n, z_{n+1}).$$

On using the condition (17) along with non-decreasing property of ψ , we have:

$$\begin{aligned}
 \psi(p^w(x_n, x_{n+1})) &\leq \psi\left(\frac{M(x_{n-1}, x_n) + M(y_{n-1}, y_n) + M(z_{n-1}, z_n)}{3}\right) \\
 &\quad - \varphi\left(\frac{M(x_{n-1}, x_n) + M(y_{n-1}, y_n) + M(z_{n-1}, z_n)}{3}\right) \\
 &\leq \psi(p^w(x_n, x_{n+1})) - \varphi(p^w(x_n, x_{n+1})),
 \end{aligned}$$

which is a contradiction to $\varphi(t) \geq 0$.

Case 2: If $M(a_{n-1}, a_n) = p^w(a_{n-1}, a_n), \forall a_n \in \{x_n, y_n, z_n\}$, then using (18), (19)

and (20) along with non-decreasing property of ψ , we have

$$p^w(x_n, x_{n+1}) \leq \left(\frac{p^w(x_{n-1}, x_n) + p^w(y_{n-1}, y_n) + p^w(z_{n-1}, z_n)}{3} \right), \quad (25)$$

$$p^w(y_n, y_{n+1}) \leq \left(\frac{p^w(y_{n-1}, y_n) + p^w(x_{n-1}, x_n) + p^w(z_{n-1}, z_n)}{3} \right), \quad (26)$$

$$p^w(z_n, z_{n+1}) \leq \left(\frac{p^w(z_{n-1}, z_n) + p^w(x_{n-1}, x_n) + p^w(y_{n-1}, y_n)}{3} \right). \quad (27)$$

By adding (25), (26) and (27), we have

$$p^w(x_n, x_{n+1}) + p^w(y_n, y_{n+1}) + p^w(z_n, z_{n+1}) \leq p^w(x_{n-1}, x_n) + p^w(y_{n-1}, y_n) + p^w(z_{n-1}, z_n).$$

If we put $t_n = p^w(x_n, x_{n+1}) + p^w(y_n, y_{n+1}) + p^w(z_n, z_{n+1})$, then the sequence t_n is non-increasing and bounded below, Therefore there is some $t \geq 0$ such that,

$$\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} [p^w(x_n, x_{n+1}) + p^w(y_n, y_{n+1}) + p^w(z_n, z_{n+1})] = t. \quad (28)$$

Now, we show that $t = 0$, Assume that $t > 0$. As ψ is non-decreasing, therefore for positive numbers a, b and c , we have $\psi(\max\{a, b, c\}) \geq \psi(\frac{a+b+c}{3})$. Using (18) and (19), we have

$$\begin{aligned} & \psi \left(\frac{p^w(x_n, x_{n+1}) + p^w(y_n, y_{n+1}) + p^w(z_n, z_{n+1})}{3} \right) \\ & \leq \psi(\max\{p^w(x_n, x_{n+1}), p^w(y_n, y_{n+1}) + p^w(z_n, z_{n+1})\}) \\ & = \max\{\psi(p^w(x_n, x_{n+1})), \psi(p^w(y_n, y_{n+1})), \psi(p^w(z_n, z_{n+1}))\} \\ & \leq \psi \left(\frac{p^w(x_{n-1}, x_n) + p^w(y_{n-1}, y_n) + p^w(z_{n-1}, z_n)}{3} \right) \\ & \quad - \varphi \left(\frac{p^w(x_{n-1}, x_n) + p^w(y_{n-1}, y_n) + p^w(z_{n-1}, z_n)}{3} \right). \end{aligned}$$

Then, taking the limit as $n \rightarrow \infty$ besides using (28) and keeping in mind $\lim_{y \rightarrow r} \varphi(y) > 0$ for all $r > 0$ along with continuity of φ , we have

$$\begin{aligned} \psi \left(\frac{t}{3} \right) & = \lim_{n \rightarrow \infty} \psi \left(\frac{t_n}{3} \right) \\ & \leq \lim_{n \rightarrow \infty} \left[\psi \left(\frac{t_{n-1}}{3} \right) - \varphi \left(\frac{t_{n-1}}{3} \right) \right] \\ & = \psi \left(\frac{t}{3} \right) - \lim_{t_{n-1} \rightarrow t} \varphi \left(\frac{t_{n-1}}{3} \right), \end{aligned}$$

which is a contradiction to

$$\lim_{t \rightarrow r} \varphi(t) > 0, \forall r > 0$$

so that $t = 0$, and

$$\begin{aligned} \lim_{n \rightarrow \infty} p^w(x_n, x_{n+1}) & = 0, \\ \lim_{n \rightarrow \infty} p^w(y_n, y_{n+1}) & = 0, \\ \lim_{n \rightarrow \infty} p^w(z_n, z_{n+1}) & = 0. \end{aligned} \quad (29)$$

Case 3: If $M(a_{n-1}, a_n) = p^w(a_{n-1}, a_n)$, and $M(b_{n-1}, b_n) = p^w(b_n, b_{n+1})$, where $a_n \in \{x_n, y_n, z_n\}$ and $b_n \in \{x_n, y_n, z_n\} \setminus \{a_n\}$, then owing to (18), (19) and (20) along with non-decreasing property of ψ , we have

$$p^w(x_n, x_{n+1}) \leq \left(\frac{p^w(x_{n-1}, x_n) + p^w(y_n, y_{n+1}) + p^w(z_n, z_{n+1})}{3} \right), \quad (30)$$

$$p^w(y_n, y_{n+1}) \leq \left(\frac{p^w(x_{n-1}, x_n) + p^w(y_n, y_{n+1}) + p^w(z_n, z_{n+1})}{3} \right), \quad (31)$$

and

$$p^w(x_n, x_{n+1}) \leq \left(\frac{p^w(x_{n-1}, x_n) + p^w(y_n, y_{n+1}) + p^w(z_n, z_{n+1})}{3} \right). \quad (32)$$

From (30), (31) and (32), we have

$$\begin{aligned} p^w(y_n, y_{n+1}) &\leq \left(\frac{p^w(y_n, y_{n+1}) + p^w(z_n, z_{n+1}) + p^w(x_n, x_{n-1})}{3} \right) \\ &\leq p^w(x_n, x_{n-1}). \end{aligned}$$

Similarly $p^w(z_n, z_{n+1}) \leq p^w(x_n, x_{n-1})$. Now, from condition (17), we have

$$\begin{aligned} \psi(p^w(x_n, x_{n+1})) &\leq \psi \left(\frac{M(x_{n-1}, x_n) + M(y_{n-1}, y_n) + M(z_{n-1}, z_n)}{3} \right) \\ &\leq \psi \left(\frac{p^w(x_{n-1}, x_n) + p^w(y_{n+1}, y_n) + p^w(z_{n+1}, z_n)}{3} \right) \\ &\leq \psi \left(\frac{p^w(x_{n-1}, x_n) + p^w(x_{n-1}, x_n) + p^w(x_{n-1}, x_n)}{3} \right) \\ &\leq \psi(p^w(x_{n-1}, x_n)). \end{aligned}$$

Following the steps of proving $t = 0$ in Case 2, put $t_n = p^w(x_n, x_{n+1})$. Then the sequence t_n is non-increasing and bounded below, therefore there is some $t \geq 0$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} t_n &= \lim_{n \rightarrow \infty} [p^w(x_n, x_{n+1})] = 0, \\ \lim_{n \rightarrow \infty} t_n &= \lim_{n \rightarrow \infty} [p^w(y_n, y_{n+1})] = 0, \end{aligned} \quad (33)$$

$$\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} [p^w(z_n, z_{n+1})] = 0. \quad (34)$$

On the other hand,

$$\begin{aligned} d_w(x_{n+1}, x_n) &= p^w(x_{n+1}, x_n) - \min\{p^w(x_n, x_n), p^w(x_{n+1}, x_{n+1})\} \\ &\leq p^w(x_{n+1}, x_n). \end{aligned}$$

Taking the limit of both the sides of the preceding inequality as $n \rightarrow \infty$ and making use of (29), we have $\lim_{n \rightarrow \infty} d_w(x_{n+1}, x_n) = 0$.

Similarly, $\lim_{n \rightarrow \infty} d_w(y_{n+1}, y_n) = 0$, and $\lim_{n \rightarrow \infty} d_w(z_{n+1}, z_n) = 0$.

Therefore, for $k = 1, 2, \dots$, we have

$$d_w(x_{n+k}, x_n) \leq d_w(x_{n+k}, x_{n+k-1}) + d_w(x_{n+k-1}, x_{n+k-2}) + \dots + d_w(x_{n+1}, x_n).$$

By taking the limit of both of sides of the above inequality as $n \rightarrow \infty$ and using (29), we have

$$\begin{aligned}\lim_{n \rightarrow \infty} d_w(x_{n+k}, x_n) &= 0, \\ \lim_{n \rightarrow \infty} d_w(y_{n+k}, y_n) &= 0, \\ \lim_{n \rightarrow \infty} d_w(z_{n+k}, z_n) &= 0.\end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_w) . Since (X, p^w) is complete, therefore the sequence $\{x_n\}$ converges in the metric space (X, d_w) , i.e., $\lim_{n \rightarrow \infty} d_w(x_n, x) = 0$ for some x in X . Again, we have

$$p^w(x, x) = \lim_{n \rightarrow \infty} p^w(x_n, x) = \lim_{n, m \rightarrow \infty} p^w(x_n, x_m), \quad (35)$$

Also,

$$p^w(y, y) = \lim_{n \rightarrow \infty} p^w(y_n, y) = \lim_{n, m \rightarrow \infty} p^w(y_n, y_m), \quad (36)$$

and

$$p^w(z, z) = \lim_{n \rightarrow \infty} p^w(z_n, z) = \lim_{n, m \rightarrow \infty} p^w(z_n, z_m). \quad (37)$$

Moreover, as $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_w) , we have $\lim_{n, m \rightarrow \infty} d_w(x_n, x_m) = 0$. On the other hand

$$p^w(x_n, x_n) + p^w(x_{n+1}, x_{n+1}) \leq 2p^w(x_n, x_{n+1}),$$

which together with (29) gives rise

$$\begin{aligned}\lim_{n \rightarrow \infty} p^w(x_n, x_n) &= 0, \\ \lim_{n \rightarrow \infty} p^w(y_n, y_n) &= 0, \\ \lim_{n \rightarrow \infty} p^w(z_n, z_n) &= 0.\end{aligned}$$

Therefore, in view of the definition of d_w , we have

$$p^w(x_n, x_m) = d_w(x_n, x_m) + \min\{p^w(x_n, x_n), p^w(x_m, x_m)\}.$$

and so $\lim_{n, m \rightarrow \infty} d_w(x_n, x_m) = 0$. Thus from (35), we have

$$p^w(x, x) = \lim_{n \rightarrow \infty} p^w(x_n, x) = \lim_{n \rightarrow \infty} p^w(x_n, x_n) = 0,$$

Also,

$$p^w(y, y) = \lim_{n \rightarrow \infty} p^w(y_n, y) = \lim_{n \rightarrow \infty} p^w(y_n, y_n) = 0,$$

and

$$p^w(z, z) = \lim_{n \rightarrow \infty} p^w(z_n, z) = \lim_{n \rightarrow \infty} p^w(z_n, z_n) = 0.$$

Since (X, p^w) is complete, so is (X, d_w) and hence there exists $x \in X$ such that

$$\lim_{n \rightarrow \infty} d_w(x_n, x) = 0.$$

Now, we proceed to show that $x = F(x, y, z)$, $y = F(y, x, z)$ and $z = F(z, x, y)$. As X satisfies the conditions (i) and (ii), therefore, x_n is a non-decreasing sequence with $x_n \rightarrow x$ then $x_n \leq x$ for all n . Also, y_n, z_n are a non-increasing sequence with

$y_n \rightarrow y$ and $z_n \rightarrow z$ and hence we have $y_n \geq y$ and $z_n \geq z$ for all n .
Now, we have

$$\begin{aligned} p^w(x, F(x, y, z)) &\leq p^w(x, x_{n+1}) + p^w(x_{n+1}, F(x, y, z)) \\ &= p^w(x, x_{n+1}) + p^w(F(x_n, y_n, z_n), F(x, y, z)). \end{aligned}$$

Therefore,

$$\begin{aligned} \psi(p^w(x, F(x, y, z))) &\leq \psi(p^w(x, x_{n+1})) + \psi(p^w(F(x_n, y_n, z_n), F(x, y, z))) \\ &\leq \psi(p^w(x, x_{n+1})) + \psi\left(\frac{M(x_n, x) + M(y_n, y)}{2}\right) \\ &\quad - \varphi\left(\frac{M(x_n, x) + M(y_n, y)}{2}\right) \\ &\leq \psi(p^w(x, x_{n+1})) + \psi\left(\frac{p^w(x_n, x) + p^w(y_n, y)}{2}\right) \\ &\quad - \varphi\left(\frac{p^w(x_n, x) + p^w(y_n, y)}{2}\right). \end{aligned}$$

Taking the limit of the preceding inequality as $n \rightarrow \infty$, and using

$$\lim_{n \rightarrow \infty} p^w(x_n, x) = \lim_{n \rightarrow \infty} p^w(y_n, y) = \lim_{n \rightarrow \infty} p^w(z_n, z) = 0,$$

and the properties of ψ and φ , we get

$$\psi(p^w(x, F(x, y, z))) = 0, \quad \text{thus} \quad p^w(x, F(x, y, z)) = 0.$$

Hence $x = F(x, y, z)$. Similarly, we can show that $y = F(y, x, z)$ and $z = F(z, x, y)$.

This completes the proof.

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