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NEW GENERAL INTEGRAL INEQUALITIES FOR (α, m)-GA-CONVEX FUNCTIONS VIA HADAMARD FRACTIONAL INTEGRALS

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Abstract. In this paper, the authors give a new identity for Hadamard fractional integrals. By using of this identity, the authors obtain new estimates on generalization of Hadamard, Ostrowski and Simpson type inequalities for (α, m)-GA-convex functions via Hadamard fractional integrals.

1. Introduction

Let a real function f be defined on some nonempty interval I of real line $\mathbb{R}$. The function f is said to be convex on I if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Following inequalities are well known in the literature as Hermite-Hadamard inequality, Ostrowski inequality and Simpson inequality respectively:

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Theorem 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping differentiable in I°, the interior of I, and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$, $x \in [a,b]$, then we the following inequality holds:*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq M(b-a)\left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right]$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller one.

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Theorem 3. *Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $\left\| f^{(4)} \right\|_{\infty} = \sup\limits_{x \in (a,b)} \left| f^{(4)}(x) \right| < \infty$. Then the following inequality holds:*

$$\left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2} \right) \right] - \frac{1}{b-a} \int\limits_a^b f(x)dx \right| \leq \frac{\left\| f^{(4)} \right\|_{\infty}}{2880} (b-a)^4 .$$

The following definitions are well known in the literature.

Definition 1. [11, 12]. *A function $f : I \subseteq (0, \infty) \to \mathbb{R}$ is said to be GA-convex (geometric-arithmatically convex) if*

$$f(x^t y^{1-t}) \leq t f(x) + (1-t) f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Definition 2. [9]. *Let $f : (0, b] \to \mathbb{R}, b > 0,$ and $(\alpha, m) \in (0, 1]^2$. If*

$$f(x^t y^{m(1-t)}) \leq t^{\alpha} f(x) + m (1 - t^{\alpha}) f(y)$$

for all $x, y \in (0, b]$ and $t \in [0, 1]$, then f is said to be a (α, m)-GA-convex function .

Note that $(\alpha, m) \in \{(1, m), (1, 1), (\alpha, 1)\}$ one obtains the following classes of functions: m-GA-convex, GA-convex, α-GA-convex (or GA-s-convex in the first sense, if we take s instead of α (see [7])).

We will now give definitions of the right hand side and left hand side of Hadamard fractional integrals which are used throughout this paper.

Definition 3. [10]. *Let $f \in L[a, b]$. The right hand side and left hand side of Hadamard fractional integrals $J^{\theta}_{a+} f$ and $J^{\theta}_{b-} f$ of order $\theta > 0$ with $b > a \geq 0$ are defined by*

$$J^{\theta}_{a+} f(x) = \frac{1}{\Gamma(\theta)} \int\limits_a^x \left(\ln \frac{x}{t} \right)^{\theta - 1} f(t) \frac{dt}{t}, \ a < x < b$$

and

$$J^{\theta}_{b-} f(x) = \frac{1}{\Gamma(\theta)} \int\limits_x^b \left(\ln \frac{t}{x} \right)^{\theta - 1} f(t) \frac{dt}{t}, \ a < x < b$$

respectively, where $\Gamma(\theta)$ is the Gamma function defined by $\Gamma(\theta) = \int\limits_0^{\infty} e^{-t} t^{\theta-1} dt$.

In [8], İşcan present Hermite-Hadamard's inequalities for GA-convex functions in fractional integral forms as follows:

Theorem 4. *Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a function such that $f \in L[a, b]$, where $a, b \in I$ with $a < b$. If f is a GA-convex function on $[a, b]$, then the following inequalities for fractional integrals holds:*

$$f\left(\sqrt{ab} \right) \leq \frac{\Gamma(\theta + 1)}{2 \left(\ln \frac{b}{a} \right)^{\theta}} \left\{ J^{\theta}_{a+} f(b) + J^{\theta}_{b-} f(a) \right\} \leq \frac{f(a) + f(b)}{2}$$

with $\alpha > 0$.

In [8], İşcan give the following identity for differentiable functions..

Lemma 1. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a,b]$, where $a,b \in I$ with $a < b$. Then for all $x \in [a,b]$, $\lambda \in [0,1]$ and $\alpha > 0$ we have:*

$$I_f(x,\lambda,\theta,a,b) = \begin{bmatrix} (1-\lambda)\left[\ln^\theta \frac{x}{a} + \ln^\theta \frac{b}{x}\right] f(x) \\ +\lambda\left[f(a)\ln^\theta \frac{x}{a} + f(b)\ln^\theta \frac{b}{x}\right] \\ -\Gamma(\theta+1)\left[J^\theta_{x-}f(a) + J^\theta_{x+}f(b)\right] \end{bmatrix}$$

$$= \begin{bmatrix} a\left(\ln\frac{x}{a}\right)^{\theta+1}\int_0^1 \left(t^\theta-\lambda\right)\left(\frac{x}{a}\right)^t f'\left(x^t a^{1-t}\right)dt \\ -b\left(\ln\frac{b}{x}\right)^{\theta+1}\int_0^1 \left(t^\theta-\lambda\right)\left(\frac{x}{b}\right)^t f'\left(x^t b^{1-t}\right)dt. \end{bmatrix}$$

In recent years, many authors have studied errors estimations for Hermite-Hadamard, Ostrowski and Simpson inequalities; for refinements, counterparts, generalization see [1, 2, 3, 4, 5, 6, 13, 14, 15, 16, 17, 18].

In this paper, we define new identity for fractional integrals. By using of this identity, we obtain a generalization of Hadamard, Ostrowski and Simpson type inequalities for (α,m)-GA-convex functions via Hadamard fractional integrals.

2. Main results

Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, the interior of I, throughout this section we will take

$$K_f(\lambda,\theta,x^m,a^m,b^m) = \begin{bmatrix} (1-\lambda)m^\theta\left[\ln^\theta \frac{x}{a} + \ln^\theta \frac{b}{x}\right] f(x^m) \\ +\lambda m^\theta\left[f(a^m)\ln^\theta \frac{x}{a} + f(b^m)\ln^\theta \frac{b}{x}\right] \\ -\Gamma(\theta+1)\left[J^\theta_{x^m-}f(a^m) + J^\theta_{x^m+}f(b^m)\right] \end{bmatrix}$$

where $a,b \in I$ with $a < b$, $x \in [a,b]$, $\lambda \in [0,1]$, $\theta > 0$ and Γ is Euler Gamma function.

Similarly to Lemma 1, we can prove the following lemma.

Lemma 2. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a^m,b^m]$, where $a^m, b \in I$ with $a < b$ and $m \in (0,1]$. Then for all $x \in [a,b]$, $\lambda \in [0,1]$ and $\theta > 0$ we have:*

$$K_f(\lambda,\theta,x^m,a^m,b^m)$$
$$= \begin{bmatrix} m^{\theta+1}a^m\left(\ln\frac{x}{a}\right)^{\theta+1}\int_0^1 \left(t^\theta-\lambda\right)\left(\frac{x}{a}\right)^{mt} f'\left(x^{mt} a^{m(1-t)}\right)dt \\ -m^{\theta+1}b^m\left(\ln\frac{b}{x}\right)^{\theta+1}\int_0^1 \left(t^\theta-\lambda\right)\left(\frac{x}{b}\right)^{mt} f'\left(x^{mt} b^{m(1-t)}\right)dt \end{bmatrix}.$$

Theorem 5. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a^m,b^m]$, where $a^m, b \in I^\circ$ with $a < b$ and $m \in (0,1]$. If $|f'|^q$ is (α,m)-GA-convex on $[a^m,b]$ for some fixed $q \geq 1$, $x \in [a,b]$, $\lambda \in [0,1]$ and $\theta > 0$ then the following inequality for fractional integrals holds:*

$$|K_f(\lambda,\theta,x^m,a^m,b^m)| \leq m^{\theta+1}C_0(\theta,\lambda)^{1-\frac{1}{q}} \tag{2.1}$$

$$\times\left[\begin{array}{c} a^{m}\left(\ln\frac{x}{a}\right)^{\theta+1}\left(\begin{array}{c}\left|f'\left(x^{m}\right)\right|^{q}C_{1}\left(x,\theta,\lambda,q,m,\alpha\right)\\ +m\left|f'\left(a\right)\right|^{q}C_{2}\left(x,\theta,\lambda,q,m,\alpha\right)\end{array}\right)^{\frac{1}{q}}\\ +b^{m}\left(\ln\frac{b}{x}\right)^{\theta+1}\left(\begin{array}{c}\left|f'\left(x^{m}\right)\right|^{q}C_{3}\left(x,\theta,\lambda,q,m,\alpha\right)\\ +m\left|f'\left(b\right)\right|^{q}C_{4}\left(x,\theta,\lambda,q,m,\alpha\right)\end{array}\right)^{\frac{1}{q}}\end{array}\right]$$

where

$$C_0(\theta,\lambda)=\frac{2\theta\lambda^{1+\frac{1}{\theta}}+1}{\theta+1}-\lambda,$$

$$C_1(x,\theta,\lambda,q,m,\alpha)=\int_0^1\left|t^{\theta}-\lambda\right|\left(\frac{x}{a}\right)^{qmt}t^{\alpha}dt,$$

$$C_2(x,\theta,\lambda,q,m,\alpha)=\int_0^1\left|t^{\theta}-\lambda\right|\left(\frac{x}{a}\right)^{qmt}(1-t^{\alpha})\,dt,$$

$$C_3(x,\theta,\lambda,q,m,\alpha)=\int_0^1\left|t^{\theta}-\lambda\right|\left(\frac{x}{b}\right)^{qmt}t^{\alpha}dt,$$

$$C_4(x,\theta,\lambda,q,m,\alpha)=\int_0^1\left|t^{\theta}-\lambda\right|\left(\frac{x}{b}\right)^{qmt}(1-t^{\alpha})\,dt.$$

Proof. From Lemma 2, property of the modulus and using the power-mean inequality we have

$$\left|K_f\left(\lambda,\theta,x^m,a^m,b^m\right)\right|\leq m^{\theta+1}\left(\int_0^1\left|t^{\theta}-\lambda\right|dt\right)^{1-\frac{1}{q}} \tag{2.2}$$

$$\times\left[\begin{array}{c} a^{m}\left(\ln\frac{x}{a}\right)^{\theta+1}\left(\int_0^1\left|t^{\theta}-\lambda\right|\left(\frac{x}{a}\right)^{qmt}\left|f'\left(x^{mt}a^{m(1-t)}\right)\right|^{q}dt\right)^{\frac{1}{q}}\\ +b^{m}\left(\ln\frac{b}{x}\right)^{\theta+1}\left(\int_0^1\left|t^{\theta}-\lambda\right|\left(\frac{x}{b}\right)^{qmt}\left|f'\left(x^{mt}b^{m(1-t)}\right)\right|^{q}dt\right)^{\frac{1}{q}}\end{array}\right].$$

Since $|f'|^q$ is (α,m)-GA-convex on $[a^m,b]$, for all $t\in[0,1]$

$$\left|f'\left(x^{mt}a^{m(1-t)}\right)\right|^{q}\leq t^{\alpha}\left|f'\left(x^{m}\right)\right|^{q}+m\left(1-t^{\alpha}\right)\left|f'\left(a\right)\right|^{q}, \tag{2.3}$$

$$\left|f'\left(x^{mt}b^{m(1-t)}\right)\right|^{q}\leq t^{\alpha}\left|f'\left(x^{m}\right)\right|^{q}+m\left(1-t^{\alpha}\right)\left|f'\left(b\right)\right|^{q}. \tag{2.4}$$

By a simple computation

$$\int_0^1\left|t^{\theta}-\lambda\right|dt=\int_0^{\lambda^{1/\theta}}\left(\lambda-t^{\theta}\right)dt+\int_{\lambda^{1/\theta}}^1\left(t^{\theta}-\lambda\right)dt=\frac{2\theta\lambda^{1+\frac{1}{\theta}}+1}{\theta+1}-\lambda. \tag{2.5}$$

If we use (2.3), (2.4) and (2.5) in (2.2), we obtain (2.1). This completes the proof. □

Corollary 1. *Under the assumptions of Theorem 5 with $q=1$, the inequality* (2.1) *reduced to the following inequality:*

$$\begin{aligned}
&K_f\left(\lambda,\theta,x^m,a^m,b^m\right)\\
&\le m^{\theta+1}\left[\begin{array}{c} a^m\left(\ln\frac{x}{a}\right)^{\theta+1}\left(\begin{array}{c}\left|f'\left(x^m\right)\right|C_1\left(x,\theta,\lambda,1,m,\alpha\right)\\ +m\left|f'\left(a\right)\right|C_2\left(x,\theta,\lambda,1,m,\alpha\right)\end{array}\right)\\ +b^m\left(\ln\frac{b}{x}\right)^{\theta+1}\left(\begin{array}{c}\left|f'\left(x^m\right)\right|C_3\left(x,\theta,\lambda,1,m,\alpha\right)\\ +m\left|f'\left(b\right)\right|C_4\left(x,\theta,\lambda,1,m,\alpha\right)\end{array}\right)\end{array}\right].
\end{aligned}$$

Corollary 2. *Under the assumptions of Theorem 5 with $x=\sqrt{ab}$, $\lambda=\frac{1}{3}$ from the inequality* (2.1) *we get the following Simpson type inequality for fractional integrals:*

$$\begin{aligned}
&\frac{2^{\theta-1}}{\left(m\ln\frac{b}{a}\right)^{\theta}}\left|K_f\left(\frac{1}{3},\theta,\left(\sqrt{ab}\right)^m,a^m,b^m\right)\right|\\
&=\left|\begin{array}{c}\frac{1}{6}\left[f(a^m)+4f\left(\left(\sqrt{ab}\right)^m\right)+f(b^m)\right]\\ -\frac{2^{\theta-1}\Gamma(\theta+1)}{\left(m\ln\frac{b}{a}\right)^{\theta}}\left[J^{\theta}_{\left(\sqrt{ab}\right)^m-}f(a^m)+J^{\theta}_{\left(\sqrt{ab}\right)^m+}f(b^m)\right]\end{array}\right|\\
&\le\frac{m\ln\frac{b}{a}}{4}C_0^{1-\frac{1}{q}}\left(\theta,\frac{1}{3}\right)\\
&\times\left[\begin{array}{c} a^m\left[\begin{array}{c}\left|f'\left(\left(\sqrt{ab}\right)^m\right)\right|^q C_1\left(\sqrt{ab},\theta,\frac{1}{3},q,m,\alpha\right)\\ +m\left|f'\left(a\right)\right|^q C_2\left(\sqrt{ab},\theta,\frac{1}{3},q,m,\alpha\right)\end{array}\right]^{\frac{1}{q}}\\ +b^m\left[\begin{array}{c}\left|f'\left(\left(\sqrt{ab}\right)^m\right)\right|^q C_3\left(\sqrt{ab},\theta,\frac{1}{3},q,m,\alpha\right)\\ +m\left|f'\left(b\right)\right|^q C_4\left(\sqrt{ab},\theta,\frac{1}{3},q,m,\alpha\right)\end{array}\right]^{\frac{1}{q}}\end{array}\right].
\end{aligned}$$

Corollary 3. *Under the assumptions of Theorem 5 with $x=\sqrt{ab}$, $\lambda=0$ from the inequality* (2.1) *we get the following midpoint-type inequality for fractional integrals:*

$$\begin{aligned}
&\frac{2^{\theta-1}}{\left(m\ln\frac{b}{a}\right)^{\theta}}\left|K_f\left(0,\theta,\left(\sqrt{ab}\right)^m,a^m,b^m\right)\right|\\
&=\left|f\left(\left(\sqrt{ab}\right)^m\right)-\frac{2^{\theta-1}\Gamma\left(\theta+1\right)}{\left(m\ln\frac{b}{a}\right)^{\theta}}\left[J^{\theta}_{\left(\sqrt{ab}\right)^m-}f(a^m)+J^{\theta}_{\left(\sqrt{ab}\right)^m+}f(b^m)\right]\right|\\
&\le\frac{m\ln\frac{b}{a}}{4}\left(\frac{1}{\theta+1}\right)^{1-\frac{1}{q}}\\
&\times\left[\begin{array}{c} a^m\left[\begin{array}{c}\left|f'\left(\left(\sqrt{ab}\right)^m\right)\right|^q C_1\left(\sqrt{ab},\theta,0,q,m,\alpha\right)\\ +m\left|f'\left(a\right)\right|^q C_2\left(\sqrt{ab},\theta,0,q,m,\alpha\right)\end{array}\right]^{\frac{1}{q}}\\ +b^m\left[\begin{array}{c}\left|f'\left(\left(\sqrt{ab}\right)^m\right)\right|^q C_3\left(\sqrt{ab},\theta,0,q,m,\alpha\right)\\ +m\left|f'\left(b\right)\right|^q C_4\left(\sqrt{ab},\theta,0,q,m,\alpha\right)\end{array}\right]^{\frac{1}{q}}\end{array}\right].
\end{aligned}$$

Remark 1. *If we take $\theta = 1$, $m = 1$ in Corollary 3 we have the following midpoint-type inequality for α-GA-convex function (or GA-s-convex function in the first sense), which is the same with the inequality* (9) *of Theorem 3.4.b. in* [7],

$$\left| f\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \right|$$

$$\leq \ln \frac{b}{a} \left(\frac{1}{2}\right)^{3-\frac{1}{q}} \left[\begin{array}{c} a \left[\begin{array}{c} \left| f'\left(\sqrt{ab}\right)\right|^q C_1\left(\sqrt{ab}, 1, 0, q, 1, \alpha\right) \\ + \left| f'(a)\right|^q C_2\left(\sqrt{ab}, 1, 0, q, 1, \alpha\right) \end{array} \right]^{\frac{1}{q}} \\ +b \left[\begin{array}{c} \left| f'\left(\sqrt{ab}\right)\right|^q C_3\left(\sqrt{ab}, 1, 0, q, 1, \alpha\right) \\ + \left| f'(b)\right|^q C_4\left(\sqrt{ab}, 1, 0, q, 1, \alpha\right) \end{array} \right]^{\frac{1}{q}} \end{array} \right].$$

Remark 2. *If we take $\theta = 1$, $m = 1$, $\alpha = 1$ in Corollary 3 we have the following midpoint-type inequality for GA-convex function, which is the same with the inequality* (13) *of Corollary 3.5 in* [7]*:*

$$\left| f\left(\sqrt{ab}\right) - \frac{1}{\ln b - \ln a} \int_a^b \frac{f(x)}{x} dx \right|$$

$$\leq \ln \frac{b}{a} \left(\frac{1}{2}\right)^{3-\frac{1}{q}} \left[\begin{array}{c} a \left[\begin{array}{c} \left| f'\left(\sqrt{ab}\right)\right|^q C_1\left(\sqrt{ab}, 1, 0, q, 1, 1\right) \\ + \left| f'(a)\right|^q C_2\left(\sqrt{ab}, 1, 0, q, 1, 1\right) \end{array} \right]^{\frac{1}{q}} \\ +b \left[\begin{array}{c} \left| f'\left(\sqrt{ab}\right)\right|^q C_3\left(\sqrt{ab}, 1, 0, q, 1, 1\right) \\ + \left| f'(b)\right|^q C_4\left(\sqrt{ab}, 1, 0, q, 1, 1\right) \end{array} \right]^{\frac{1}{q}} \end{array} \right].$$

Corollary 4. *Under the assumptions of Theorem 5 with $x = \sqrt{ab}$, $\lambda = 1$ from the inequality* (2.1) *we get the following trapezoid-type inequality for fractional integrals:*

$$\frac{2^{\theta-1}}{\left(m \ln \frac{b}{a}\right)^{\theta}} \left| K_f\left(1, \theta, \left(\sqrt{ab}\right)^m, a^m, b^m\right)\right|$$

$$= \left| \frac{f(a^m) + f(b^m)}{2} - \frac{2^{\theta-1}\Gamma(\theta+1)}{\left(m \ln \frac{b}{a}\right)^{\theta}} \left[J^{\theta}_{\left(\sqrt{ab}\right)^m -} f(a^m) + J^{\theta}_{\left(\sqrt{ab}\right)^m +} f(b^m)\right] \right|$$

$$\leq \frac{m \ln \frac{b}{a}}{4} \left(\frac{\theta}{\theta+1}\right)^{1-\frac{1}{q}}$$

$$\times \left[\begin{array}{c} a^m \left[\begin{array}{c} \left| f'\left(\left(\sqrt{ab}\right)^m\right)\right|^q C_1\left(\sqrt{ab}, \theta, 1, q, m, \alpha\right) \\ +m \left| f'(a)\right|^q C_2\left(\sqrt{ab}, \theta, 1, q, m, \alpha\right) \end{array} \right]^{\frac{1}{q}} \\ +b^m \left[\begin{array}{c} \left| f'\left(\left(\sqrt{ab}\right)^m\right)\right|^q C_3\left(\sqrt{ab}, \theta, 1, q, m, \alpha\right) \\ +m \left| f'(b)\right|^q C_4\left(\sqrt{ab}, \theta, 1, q, m, \alpha\right) \end{array} \right]^{\frac{1}{q}} \end{array} \right].$$

Corollary 5. *Let the assumptions of Theorem 5 hold. If* $|f'(u)| \leq M$ *for all* $u \in [a^m, b]$ *and* $\lambda = 0$*, then from the inequality* (2.1) *we get the following Ostrowski type inequality for fractional integrals*

$$\left| \left[\left(\ln \frac{x}{a} \right)^{\theta} + \left(\ln \frac{b}{x} \right)^{\theta} \right] f(x^m) - \frac{\Gamma(\theta+1)}{m^{\theta}} \left[J^{\theta}_{x^m-} f(a^m) + J^{\theta}_{x^m+} f(b^m) \right] \right|$$

$$\leq \frac{mM}{(\theta+1)^{1-\frac{1}{q}}} \left[\begin{array}{c} a^m \left(\ln \frac{x}{a} \right)^{\theta+1} \left(\begin{array}{c} C_1(x,\theta,0,q,m,\alpha) \\ +mC_2(x,\theta,0,q,m,\alpha) \end{array} \right)^{\frac{1}{q}} \\ +b^m \left(\ln \frac{b}{x} \right)^{\theta+1} \left(\begin{array}{c} C_3(x,\theta,\lambda,q,m,\alpha) \\ +mC_4(x,\theta,\lambda,q,m,\alpha) \end{array} \right)^{\frac{1}{q}} \end{array} \right].$$

for each $x \in [a, b]$.

Theorem 6. *Let* $f : I \subseteq (0, \infty) \to \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[a^m, b^m]$*, where* $a^m, b \in I^{\circ}$ *with* $a < b$ *and* $m \in (0, 1]$*. If* $|f'|^q$ *is* (α, m)*-GA-convex on* $[a^m, b]$ *for some fixed* $q > 1$*,* $x \in [a, b]$*,* $\lambda \in [0, 1]$ *and* $\theta > 0$ *then the following inequality for fractional integrals holds:*

$$|K_f(\lambda,\theta,x^m,a^m,b^m)| \tag{2.6}$$

$$\leq m^{\theta+1} R_0^{\frac{1}{p}}(\theta,\lambda,p) \left[\begin{array}{c} a^m \left(\ln \frac{x}{a} \right)^{\theta+1} \left(\begin{array}{c} |f'(x^m)|^q R_1(x,q,m,\alpha) \\ +m|f'(a)|^q R_2(x,q,m,\alpha) \end{array} \right)^{\frac{1}{q}} \\ +b^m \left(\ln \frac{b}{x} \right)^{\theta+1} \left(\begin{array}{c} |f'(x^m)|^q R_3(x,q,m,\alpha) \\ +m|f'(b)|^q R_4(x,q,m,\alpha) \end{array} \right)^{\frac{1}{q}} \end{array} \right]$$

where

$$R_0(\theta,\lambda,p) = \int_0^1 \left| t^{\theta} - \lambda \right|^p dt$$

$$= \begin{cases} \frac{1}{\theta p+1} & , \quad \lambda = 0 \\ \left\{ \frac{\lambda^{(\theta p+1)/\theta}}{\theta} \beta\left(\frac{1}{\theta}, p+1\right) + \frac{(1-\lambda)^{p+1}}{\theta(p+1)} \right. \\ \left. \times \ {}_2F_1\left(1-\frac{1}{\theta}, 1; p+2; 1-\lambda\right) \right\} & , \quad 0 < \lambda < 1 \\ \frac{1}{\theta}\beta\left(\frac{1}{\theta}, p+1\right) & , \quad \lambda = 1 \end{cases},$$

$$R_1(x,q,m,\alpha) = \int_0^1 \left(\frac{x}{a}\right)^{mqt} t^{\alpha} dt,$$

$$R_2(x,q,m,\alpha) = \int_0^1 \left(\frac{x}{a}\right)^{mqt} (1-t^{\alpha})\, dt,$$

$$R_3(x,q,m,\alpha) = \int_0^1 \left(\frac{x}{b}\right)^{mqt} t^{\alpha} dt,$$

$$R_4(x,q,m,\alpha) = \int_0^1 \left(\frac{x}{b}\right)^{mqt} (1-t^{\alpha})\, dt,$$

$_2F_1$ *is hyper-geometrical function defined by*

$$\begin{aligned} {}_2F_1(a,b;c;z) &= \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt, \\ c &> b>0, |z|<1\,(see\ [10]), \end{aligned}$$

β *is beta function defined by*

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt,\ \ x,y>0,$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2, property of the modulus and using the Hölder inequality we have

$$|K_f(\lambda,\theta,x^m,a^m,b^m)|\le m^{\theta+1}\left(\int_0^1\left|t^\theta-\lambda\right|^p dt\right)^{\frac{1}{p}} \tag{2.7}$$

$$\times\left[\begin{array}{c} a^m\left(\ln\frac{x}{a}\right)^{\theta+1}\left(\int\limits_0^1\left(\frac{x}{a}\right)^{qmt}\left|f'\left(x^{mt}a^{m(1-t)}\right)\right|^q dt\right)^{\frac{1}{q}} \\ +b^m\left(\ln\frac{b}{x}\right)^{\theta+1}\left(\int\limits_0^1\left(\frac{x}{b}\right)^{qmt}\left|f'\left(x^{mt}b^{m(1-t)}\right)\right|^q dt\right)^{\frac{1}{q}}\end{array}\right].$$

By a simple computation

$$R_0(\theta,\lambda,p)=\int_0^1\left|t^\theta-\lambda\right|^p dt \tag{2.8}$$

$$=\begin{cases} \frac{1}{\theta p+1} & ,\quad \lambda=0 \\ \left\{\frac{\lambda^{(\theta p+1)/\theta}}{\theta}\beta\left(\frac{1}{\theta},p+1\right)+\frac{(1-\lambda)^{p+1}}{\theta(p+1)} \times {}_2F_1\left(1-\frac{1}{\theta},1;p+2;1-\lambda\right)\right\} & ,\quad 0<\lambda<1 \\ \frac{1}{\theta}\beta\left(\frac{1}{\theta},p+1\right) & ,\quad \lambda=1\end{cases}.$$

Since $|f'|^q$ is (α,m)-GA-convex on $[a^m,b]$, for all $t\in[0,1]$, if we use (2.3), (2.4) and (2.8) in (2.7), we obtain (2.6). This completes the proof. □

Corollary 6. *Under the assumptions of Theorem 6 with* $x=\sqrt{ab}$, $\lambda=\frac{1}{3}$ *from the inequality* (2.6) *we get the following Simpson type inequality for fractional integrals:*

$$\begin{aligned} &\frac{2^{\theta-1}}{\left(m\ln\frac{b}{a}\right)^\theta}\left|K_f\left(\frac{1}{3},\theta,\left(\sqrt{ab}\right)^m,a^m,b^m\right)\right| \\ &=\left|\begin{array}{c}\frac{1}{6}\left[f(a^m)+4f\left(\left(\sqrt{ab}\right)^m\right)+f(b^m)\right] \\ -\frac{2^{\theta-1}\Gamma(\theta+1)}{\left(m\ln\frac{b}{a}\right)^\theta}\left[J^\theta_{\left(\sqrt{ab}\right)^m-}f(a^m)+J^\theta_{\left(\sqrt{ab}\right)^m+}f(b^m)\right]\end{array}\right| \\ &\le\frac{m\ln\frac{b}{a}}{4}R_0^{\frac{1}{p}}\left(\theta,\frac{1}{3},p\right)\end{aligned}$$

$$\times\left[\begin{array}{c} a^m\left[\begin{array}{c}\left|f'\left(\left(\sqrt{ab}\right)^m\right)\right|^q R_1\left(\sqrt{ab},q,m,\alpha\right)\\ +m\left|f'(a)\right|^q R_2\left(\sqrt{ab},q,m,\alpha\right)\end{array}\right]^{\frac{1}{q}}\\ +b^m\left[\begin{array}{c}\left|f'\left(\left(\sqrt{ab}\right)^m\right)\right|^q R_3\left(\sqrt{ab},q,m,\alpha\right)\\ +m\left|f'(b)\right|^q R_4\left(\sqrt{ab},q,m,\alpha\right)\end{array}\right]^{\frac{1}{q}}\end{array}\right].$$

Corollary 7. *Under the assumptions of Theorem 6 with $x=\sqrt{ab}$, $\lambda=0$ from the inequality (2.6) we get the following midpoint-type inequality for fractional integrals:*

$$\frac{2^{\theta-1}}{\left(m\ln\frac{b}{a}\right)^{\theta}}\left|K_f\left(0,\theta,\left(\sqrt{ab}\right)^m,a^m,b^m\right)\right|$$

$$=\left|f\left(\left(\sqrt{ab}\right)^m\right)-\frac{2^{\theta-1}\Gamma(\theta+1)}{\left(m\ln\frac{b}{a}\right)^{\theta}}\left[J^{\theta}_{\left(\sqrt{ab}\right)^m-}f(a^m)+J^{\theta}_{\left(\sqrt{ab}\right)^m+}f(b^m)\right]\right|$$

$$\leq\frac{m\ln\frac{b}{a}}{4}\left(\frac{1}{\theta p+1}\right)^{\frac{1}{p}}\left[\begin{array}{c} a^m\left[\begin{array}{c}\left|f'\left(\left(\sqrt{ab}\right)^m\right)\right|^q R_1\left(\sqrt{ab},q,m,\alpha\right)\\ +m\left|f'(a)\right|^q R_2\left(\sqrt{ab},q,m,\alpha\right)\end{array}\right]^{\frac{1}{q}}\\ +b^m\left[\begin{array}{c}\left|f'\left(\left(\sqrt{ab}\right)^m\right)\right|^q R_3\left(\sqrt{ab},q,m,\alpha\right)\\ +m\left|f'(b)\right|^q R_4\left(\sqrt{ab},q,m,\alpha\right)\end{array}\right]^{\frac{1}{q}}\end{array}\right].$$

Remark 3. *If we take $\theta=1$, $m=1$, $p=\frac{q}{q-1}$ in Corollary 7 we have the following midpoint-type inequality for α-GA-convex function (or GA-s-convex function in the first sense), which is the same with the inequality (17) of Theorem 3.7.b. in [7]:*

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|$$

$$\leq\frac{\ln\frac{b}{a}}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left[\begin{array}{c} a\left[\begin{array}{c}\left|f'\left(\sqrt{ab}\right)\right|^q R_1\left(\sqrt{ab},q,1,\alpha\right)\\ +\left|f'(a)\right|^q R_2\left(\sqrt{ab},q,1,\alpha\right)\end{array}\right]^{\frac{1}{q}}\\ +b\left[\begin{array}{c}\left|f'\left(\sqrt{ab}\right)\right|^q R_3\left(\sqrt{ab},q,1,\alpha\right)\\ +\left|f'(b)\right|^q R_4\left(\sqrt{ab},q,1,\alpha\right)\end{array}\right]^{\frac{1}{q}}\end{array}\right].$$

Remark 4. *If we take $\theta=1$, $m=1$, $\alpha=1$, $p=\frac{q}{q-1}$ in Corollary 7 we have the following midpoint-type inequality for GA-convex function, which is the same with the inequality (21) of Corollary 3.8 in [7]:*

$$\left|f\left(\sqrt{ab}\right)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}dx\right|$$

$$\leq\frac{\ln\frac{b}{a}}{4}\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\left[\begin{array}{c} a\left[\begin{array}{c}\left|f'\left(\sqrt{ab}\right)\right|^q R_1\left(\sqrt{ab},q,1,1\right)\\ +\left|f'(a)\right|^q R_2\left(\sqrt{ab},q,1,1\right)\end{array}\right]^{\frac{1}{q}}\\ +b\left[\begin{array}{c}\left|f'\left(\sqrt{ab}\right)\right|^q R_3\left(\sqrt{ab},q,1,1\right)\\ +\left|f'(b)\right|^q R_4\left(\sqrt{ab},q,1,1\right)\end{array}\right]^{\frac{1}{q}}\end{array}\right].$$

Corollary 8. *Under the assumptions of Theorem 6 with $x = \sqrt{ab}$, $\lambda = 1$ from the inequality (2.6) we get the following trapezoid-type inequality for fractional integrals:*

$$\frac{2^{\theta-1}}{\left(m\ln\frac{b}{a}\right)^{\theta}}\left|K_f\left(1,\theta,\left(\sqrt{ab}\right)^m,a^m,b^m\right)\right|$$

$$=\left|\frac{f(a^m)+f(b^m)}{2}-\frac{2^{\theta-1}\Gamma\left(\theta+1\right)}{\left(m\ln\frac{b}{a}\right)^{\theta}}\left[J^{\theta}_{\left(\sqrt{ab}\right)^m-}f(a^m)+J^{\theta}_{\left(\sqrt{ab}\right)^m+}f(b^m)\right]\right|$$

$$\leq\frac{m\ln\frac{b}{a}}{4}\left(\frac{1}{\theta}\beta\left(\frac{1}{\theta},p+1\right)\right)^{\frac{1}{p}}$$

$$\times\left[\begin{array}{c} a^m\left[\begin{array}{c}\left|f'\left(\left(\sqrt{ab}\right)^m\right)\right|^q R_1\left(\sqrt{ab},q,m,\alpha\right)\\ +m\left|f'\left(a\right)\right|^q R_2\left(\sqrt{ab},q,m,\alpha\right)\end{array}\right]^{\frac{1}{q}}\\ +b^m\left(\begin{array}{c}\left|f'\left(x^m\right)\right|^q R_3\left(\sqrt{ab},q,m,\alpha\right)\\ +m\left|f'\left(b\right)\right|^q R_4\left(\sqrt{ab},q,m,\alpha\right)\end{array}\right)^{\frac{1}{q}}\end{array}\right].$$

Corollary 9. *Let the assumptions of Theorem 6 hold. If $|f'(u)| \leq M$ for all $u \in [a^m, b]$ and $\lambda = 0$, then from the inequality (2.6) we get the following Ostrowski type inequality for fractional integrals:*

$$\left|\left[\left(\ln\frac{x}{a}\right)^{\theta}+\left(\ln\frac{b}{x}\right)^{\theta}\right]f(x^m)-\frac{\Gamma\left(\theta+1\right)}{m^{\theta}}\left[J^{\theta}_{x^m-}f(a^m)+J^{\theta}_{x^m+}f(b^m)\right]\right|$$

$$\leq\frac{mM}{\left(\theta p+1\right)^{\frac{1}{p}}}\left[\begin{array}{c} a^m\left(\ln\frac{x}{a}\right)^{\theta+1}\left(\begin{array}{c}R_1\left(x,q,m,\alpha\right)\\ +mR_2\left(x,q,m,\alpha\right)\end{array}\right)^{\frac{1}{q}}\\ +b^m\left(\ln\frac{b}{x}\right)^{\theta+1}\left(\begin{array}{c}R_3\left(x,q,m,\alpha\right)\\ +R_4\left(x,q,m,\alpha\right)\end{array}\right)^{\frac{1}{q}}\end{array}\right].$$

for each $x \in [a, b]$

Theorem 7. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a^m, b^m]$, where $a^m, b \in I^{\circ}$ with $a < b$ and $m \in (0, 1]$. If $|f'|^q$ is (α, m)-GA-convex on $[a^m, b]$ for some fixed $q > 1$, $x \in [a, b]$, $\lambda \in [0, 1]$ and $\theta > 0$ then the following inequality for fractional integrals holds:*

$$\left|K_f\left(\lambda,\theta,x^m,a^m,b^m\right)\right| \tag{2.9}$$

$$\leq m^{\theta+1}\left[\begin{array}{c} a^m\left(\ln\frac{x}{a}\right)^{\theta+1}T_1^{\frac{1}{p}}\left(x,\theta,\lambda,p,m\right)\left(\frac{\left|f'(x^m)\right|^q+m\alpha\left|f'(a)\right|^q}{\alpha+1}\right)^{\frac{1}{q}}\\ +b^m\left(\ln\frac{b}{x}\right)^{\theta+1}T_2^{\frac{1}{p}}\left(x,\theta,\lambda,p,m\right)\left(\frac{\left|f'(x^m)\right|^q+m\alpha\left|f'(b)\right|^q}{\alpha+1}\right)^{\frac{1}{q}}\end{array}\right]$$

where

$$T_1\left(x,\theta,\lambda,p,m\right)=\int_0^1\left|t^{\theta}-\lambda\right|^p\left(\frac{x}{a}\right)^{mpt}dt,$$

$$T_2(x,\theta,\lambda,p,m)=\int_0^1 \left|t^\theta-\lambda\right|^p \left(\frac{x}{b}\right)^{mpt} dt,$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Since $|f'|^q$ is (α,m)-GA-convex on $[a^m,b]$, for all $t\in[0,1]$, if we use (2.3), (2.4)

$$\int_0^1 \left|f'\left(x^{mt}a^{m(1-t)}\right)\right|^q dt \le \int_0^1 t^\alpha \left|f'(x^m)\right|^q + m\left(1-t^\alpha\right)\left|f'(a)\right|^q dt \tag{2.10}$$

$$=\frac{\left|f'(x^m)\right|^q + m\alpha\left|f'(a)\right|^q}{\alpha+1},$$

$$\int_0^1 \left|f'\left(x^{mt}b^{m(1-t)}\right)\right|^q dt \le \int_0^1 t^\alpha \left|f'(x^m)\right|^q + m\left(1-t^\alpha\right)\left|f'(b)\right|^q dt \tag{2.11}$$

$$=\frac{\left|f'(x^m)\right|^q + m\alpha\left|f'(b)\right|^q}{\alpha+1}.$$

From Lemma 2, property of the modulus, (2.10), (2.11) and using the Hölder inequality, we have

$$\left|K_f\left(\lambda,\theta,x^m,a^m,b^m\right)\right|$$

$$\le m^{\theta+1}\left[\begin{array}{c} a^m\left(\ln\frac{x}{a}\right)^{\theta+1}\left(\int_0^1\left|t^\theta-\lambda\right|^p\left(\frac{x}{a}\right)^{mpt}dt\right)^{\frac{1}{p}} \\ \times\left(\int_0^1\left|f'\left(x^{mt}a^{m(1-t)}\right)\right|^q dt\right)^{\frac{1}{q}} \\ +b^m\left(\ln\frac{b}{x}\right)^{\theta+1}\left(\int_0^1\left|t^\theta-\lambda\right|^p\left(\frac{x}{b}\right)^{mpt}dt\right)^{\frac{1}{p}} \\ \times\left(\int_0^1\left|f'\left(x^{mt}b^{m(1-t)}\right)\right|^q dt\right)^{\frac{1}{q}} \end{array}\right]$$

$$\le m^{\theta+1}\left[\begin{array}{c} a^m\left(\ln\frac{x}{a}\right)^{\theta+1}\left(\int_0^1\left|t^\theta-\lambda\right|^p\left(\frac{x}{a}\right)^{mpt}dt\right)^{\frac{1}{p}} \\ \times\left(\frac{\left|f'(x^m)\right|^q+m\alpha\left|f'(a)\right|^q}{\alpha+1}\right)^{\frac{1}{q}} \\ +b^m\left(\ln\frac{b}{x}\right)^{\theta+1}\left(\int_0^1\left|t^\theta-\lambda\right|^p\left(\frac{x}{b}\right)^{mpt}dt\right)^{\frac{1}{p}} \\ \times\left(\frac{\left|f'(x^m)\right|^q+m\alpha\left|f'(b)\right|^q}{\alpha+1}\right)^{\frac{1}{q}} \end{array}\right].$$

This completes the proof. □

Corollary 10. *Under the assumptions of Theorem 7 with* $x=\sqrt{ab}$, $\lambda=\frac{1}{3}$ *from the inequality* (2.9) *we get the following Simpson type inequality for fractional integrals:*

$$\frac{2^{\theta-1}}{\left(m\ln\frac{b}{a}\right)^{\theta}}\left|K_f\left(\frac{1}{3},\theta,\left(\sqrt{ab}\right)^m,a^m,b^m\right)\right|$$

$$=\left|\begin{array}{c}\frac{1}{6}\left[f(a^m)+4f\left(\left(\sqrt{ab}\right)^m\right)+f(b^m)\right]\\ -\frac{2^{\theta-1}\Gamma(\theta+1)}{\left(m\ln\frac{b}{a}\right)^{\theta}}\left[J^{\theta}_{\left(\sqrt{ab}\right)^m-}f(a^m)+J^{\theta}_{\left(\sqrt{ab}\right)^m+}f(b^m)\right]\end{array}\right|$$

$$\leq\frac{m\ln\frac{b}{a}}{4}\left[\begin{array}{c}a^mT_1^{\frac{1}{p}}\left(\sqrt{ab},\theta,\frac{1}{3},p,m\right)\left(\frac{\left|f'\left(\left(\sqrt{ab}\right)^m\right)\right|^q+m\alpha\left|f'(a)\right|^q}{\alpha+1}\right)^{\frac{1}{q}}\\ +b^mT_2^{\frac{1}{p}}\left(\sqrt{ab},\theta,\frac{1}{3},p,m\right)\left(\frac{\left|f'\left(\left(\sqrt{ab}\right)^m\right)\right|^q+m\alpha\left|f'(b)\right|^q}{\alpha+1}\right)^{\frac{1}{q}}\end{array}\right].$$

Corollary 11. *Under the assumptions of Theorem 7 with* $x=\sqrt{ab}$, $\lambda=0$ *from the inequality* (2.9) *we get the following midpoint-type inequality for fractional integrals:*

$$\frac{2^{\theta-1}}{\left(m\ln\frac{b}{a}\right)^{\theta}}\left|K_f\left(0,\theta,\left(\sqrt{ab}\right)^m,a^m,b^m\right)\right|$$

$$=\left|f\left(\left(\sqrt{ab}\right)^m\right)-\frac{2^{\theta-1}\Gamma(\theta+1)}{\left(m\ln\frac{b}{a}\right)^{\theta}}\left[J^{\theta}_{\left(\sqrt{ab}\right)^m-}f(a^m)+J^{\theta}_{\left(\sqrt{ab}\right)^m+}f(b^m)\right]\right|$$

$$\leq\frac{m\ln\frac{b}{a}}{4}\left[\begin{array}{c}a^mT_1^{\frac{1}{p}}\left(\sqrt{ab},\theta,0,p,m\right)\left(\frac{\left|f'\left(\left(\sqrt{ab}\right)^m\right)\right|^q+m\alpha\left|f'(a)\right|^q}{\alpha+1}\right)^{\frac{1}{q}}\\ +b^mT_2^{\frac{1}{p}}\left(\sqrt{ab},\theta,0,p,m\right)\left(\frac{\left|f'\left(\left(\sqrt{ab}\right)^m\right)\right|^q+m\alpha\left|f'(b)\right|^q}{\alpha+1}\right)^{\frac{1}{q}}\end{array}\right].$$

Corollary 12. *Under the assumptions of Theorem 7 with* $x=\sqrt{ab}$, $\lambda=1$ *from the inequality* (2.9) *we get the following trapezoid-type inequality for fractional integrals:*

$$\frac{2^{\theta-1}}{\left(m\ln\frac{b}{a}\right)^{\theta}}\left|K_f\left(1,\theta,\left(\sqrt{ab}\right)^m,a^m,b^m\right)\right|$$

$$=\left|\frac{f(a^m)+f(b^m)}{2}-\frac{2^{\theta-1}\Gamma(\theta+1)}{\left(m\ln\frac{b}{a}\right)^{\theta}}\left[J^{\theta}_{\left(\sqrt{ab}\right)^m-}f(a^m)+J^{\theta}_{\left(\sqrt{ab}\right)^m+}f(b^m)\right]\right|$$

$$\leq\frac{m\ln\frac{b}{a}}{4}\left[\begin{array}{c}a^mT_1^{\frac{1}{p}}\left(\sqrt{ab},\theta,1,p,m\right)\left(\frac{\left|f'\left(\left(\sqrt{ab}\right)^m\right)\right|^q+m\alpha\left|f'(a)\right|^q}{\alpha+1}\right)^{\frac{1}{q}}\\ +b^mT_2^{\frac{1}{p}}\left(\sqrt{ab},\theta,1,p,m\right)\left(\frac{\left|f'\left(\left(\sqrt{ab}\right)^m\right)\right|^q+m\alpha\left|f'(b)\right|^q}{\alpha+1}\right)^{\frac{1}{q}}\end{array}\right].$$

Corollary 13. *Let the assumptions of Theorem 7 hold. If* $|f'(u)|\leq M$ *for all* $u\in[a^m,b]$ *and* $\lambda=0$, *then from the inequality* (2.9) *we get the following Ostrowski type inequality for fractional integrals:*

$$\left|\left[\left(\ln\frac{x}{a}\right)^{\theta}+\left(\ln\frac{b}{x}\right)^{\theta}\right]f(x^m)-\frac{\Gamma(\theta+1)}{m^{\theta}}\left[J^{\theta}_{x^m-}f(a^m)+J^{\theta}_{x^m+}f(b^m)\right]\right|$$

$$\leq mM\left(\frac{1+m\alpha}{\alpha+1}\right)^{\frac{1}{q}}\left[\begin{array}{c} a^{m}\left(\ln\frac{x}{a}\right)^{\theta+1}T_{1}^{\frac{1}{p}}\left(x,\theta,0,p,m\right) \\ +b^{m}\left(\ln\frac{b}{x}\right)^{\theta+1}T_{2}^{\frac{1}{p}}\left(x,\theta,0,p,m\right) \end{array}\right]$$

for each $x\in[a,b]$

Theorem 8. *Let* $f:I\subseteq(0,\infty)\to\mathbb{R}$ *be a differentiable function on* I° *such that* $f'\in L[a^m,b^m]$, *where* $a^m,b\in I^{\circ}$ *with* $a<b$ *and* $m\in(0,1]$. *If* $|f'|^q$ *is* (α,m)*-GA-convex on* $[a^m,b]$ *for some fixed* $q>1$, $x\in[a,b]$, $\lambda\in[0,1]$ *and* $\theta>0$ *then the following inequality for fractional integrals holds:*

$$|K_f(\lambda,\theta,x^m,a^m,b^m)| \tag{2.12}$$

$$\leq m^{\theta+1}\left[\begin{array}{c} a^{m}\left(\ln\frac{x}{a}\right)^{\theta+1}V_{3}^{\frac{1}{p}}\left[\begin{array}{c} V_1(\theta,\lambda,\alpha,q)\left|f'(x^m)\right|^q \\ +mV_2(\theta,\lambda,\alpha,q)\left|f'(a)\right|^q \end{array}\right]^{\frac{1}{q}} \\ +b^{m}\left(\ln\frac{b}{x}\right)^{\theta+1}V_{4}^{\frac{1}{p}}\left[\begin{array}{c} V_1(\theta,\lambda,\alpha,q)\left|f'(x^m)\right|^q \\ +mV_2(\theta,\lambda,\alpha,q)\left|f'(b)\right|^q \end{array}\right]^{\frac{1}{q}} \end{array}\right]$$

where

$$V_1(\theta,\lambda,\alpha,q)=\int_0^1\left|t^\theta-\lambda\right|^q t^\alpha dt \tag{2.13}$$

$$=\left\{\begin{array}{cc} \frac{1}{\theta q+\alpha+1} & ,\quad \lambda=0 \\ \left\{\frac{\lambda^{(\theta q+\alpha+1)/\theta}}{\theta}\beta\left(\frac{\alpha+1}{\theta},q+1\right)+\frac{(1-\lambda)^{q+1}}{\theta(q+1)}\right. & \\ \left.\times\ {}_2F_1\left(1-\frac{\alpha+1}{\theta},1;q+2;1-\lambda\right)\right\} & ,\quad 0<\lambda<1 \\ \frac{1}{\theta}\beta\left(\frac{\alpha+1}{\theta},q+1\right) & ,\quad \lambda=1 \end{array}\right.,$$

$$V_2(\theta,\lambda,\alpha,q)=\int_0^1\left|t^\theta-\lambda\right|^q(1-t^\alpha)dt \tag{2.14}$$

$$=\left\{\begin{array}{cc} \frac{1}{\theta q+1}-\frac{1}{\theta q+\alpha+1} & ,\quad \lambda=0 \\ \left\{\frac{\lambda^{(\theta q+1)/\theta}}{\theta}\beta\left(\frac{1}{\theta},q+1\right)-\frac{\lambda^{(\theta q+\alpha+1)/\theta}}{\theta}\beta\left(\frac{\alpha+1}{\theta},q+1\right)\right. & \\ \left.+\frac{(1-\lambda)^{q+1}}{\theta(q+1)}\left(\begin{array}{c} {}_2F_1\left(1-\frac{1}{\theta},1;q+2;1-\lambda\right) \\ -{}_2F_1\left(1-\frac{\alpha+1}{\theta},1;q+2;1-\lambda\right) \end{array}\right)\right\} & ,\quad 0<\lambda<1 \\ \frac{1}{\theta}\beta\left(\frac{1}{\theta},q+1\right)-\frac{1}{\theta}\beta\left(\frac{\alpha+1}{\theta},q+1\right) & ,\quad \lambda=1 \end{array}\right.,$$

$$V_3=\int_0^1\left(\frac{x}{a}\right)^{pmt}dt=\left\{\begin{array}{cc} \frac{\left(\frac{x}{a}\right)^{mp}-1}{\ln\left(\frac{x}{a}\right)^{mp}} & ,\quad x\neq a \\ 1 & ,\quad \text{otherwise} \end{array}\right., \tag{2.15}$$

$$V_4=\int_0^1\left(\frac{x}{b}\right)^{pmt}dt=\left\{\begin{array}{cc} \frac{\left(\frac{x}{b}\right)^{mp}-1}{\ln\left(\frac{x}{b}\right)^{mp}} & ,\quad x\neq b \\ 1 & ,\quad \text{otherwise} \end{array}\right., \tag{2.16}$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2, property of the modulus, the Hölder inequality and by using (2.3), (2.4), (2.15) and (2.16) we have

$$\left|K_f\left(\lambda,\theta,x^m,a^m,b^m\right)\right| \tag{2.17}$$

$$\leq m^{\theta+1}\left[\begin{array}{c} a^m\left(\ln\frac{x}{a}\right)^{\theta+1}\left(\int\limits_0^1\left(\frac{x}{a}\right)^{pmt}dt\right)^{\frac{1}{p}} \\ \times\left(\int\limits_0^1\left|t^\theta-\lambda\right|^q\left|f'\left(x^{mt}a^{m(1-t)}\right)\right|^q dt\right)^{\frac{1}{q}} \\ +b^m\left(\ln\frac{b}{x}\right)^{\theta+1}\left(\int\limits_0^1\left(\frac{x}{b}\right)^{pmt}dt\right)^{\frac{1}{p}} \\ \times\left(\int\limits_0^1\left|t^\theta-\lambda\right|^q\left|f'\left(x^{mt}b^{m(1-t)}\right)\right|^q dt\right)^{\frac{1}{q}} \end{array}\right]$$

$$\leq m^{\theta+1}\left[\begin{array}{c} a^m\left(\ln\frac{x}{a}\right)^{\theta+1}V_3^{\frac{1}{p}} \\ \times\left(\int\limits_0^1\left|t^\theta-\lambda\right|^q\left[\begin{array}{c} t^\alpha\left|f'\left(x^m\right)\right|^q \\ +m\left(1-t^\alpha\right)\left|f'\left(a\right)\right|^q\end{array}\right]dt\right)^{\frac{1}{q}} \\ +b^m\left(\ln\frac{b}{x}\right)^{\theta+1}V_4^{\frac{1}{p}} \\ \times\left(\int\limits_0^1\left|t^\theta-\lambda\right|^q\left[\begin{array}{c} t^\alpha\left|f'\left(x^m\right)\right|^q \\ +m\left(1-t^\alpha\right)\left|f'\left(b\right)\right|^q\end{array}\right]dt\right)^{\frac{1}{q}} \end{array}\right].$$

By a simple computation we verify (2.13) and (2.14). If we use (2.13), (2.14), (2.15) and (2.16) in (2.17) we obtain (2.12). This completes the proof. □

Corollary 14. *Under the assumptions of Theorem 8 with $x=\sqrt{ab}$, $\lambda=\frac{1}{3}$ from the inequality* (2.12) *we get the following Simpson type inequality for fractional integrals:*

$$\frac{2^{\theta-1}}{\left(m\ln\frac{b}{a}\right)^\theta}\left|K_f\left(\frac{1}{3},\theta,\left(\sqrt{ab}\right)^m,a^m,b^m\right)\right|$$

$$=\left|\begin{array}{c}\frac{1}{6}\left[f(a^m)+4f\left(\left(\sqrt{ab}\right)^m\right)+f(b^m)\right] \\ -\frac{2^{\theta-1}\Gamma(\theta+1)}{\left(m\ln\frac{b}{a}\right)^\theta}\left[J^\theta_{\left(\sqrt{ab}\right)^m-}f(a^m)+J^\theta_{\left(\sqrt{ab}\right)^m+}f(b^m)\right]\end{array}\right|$$

$$\leq\frac{m\ln\frac{b}{a}}{4}\left[\begin{array}{c} a^m\left(\frac{\left(\frac{b}{a}\right)^{\frac{mp}{2}}-1}{\ln\left(\frac{b}{a}\right)^{\frac{mp}{2}}}\right)^{\frac{1}{p}}\left[\begin{array}{c}V_1\left(\theta,\frac{1}{3},\alpha,q\right)\left|f'\left(x^m\right)\right|^q \\ +mV_2\left(\theta,\frac{1}{3},\alpha,q\right)\left|f'\left(a\right)\right|^q\end{array}\right]^{\frac{1}{q}} \\ +b^m\left(\frac{\left(\frac{a}{b}\right)^{\frac{mp}{2}}-1}{\ln\left(\frac{a}{b}\right)^{\frac{mp}{2}}}\right)^{\frac{1}{p}}\left[\begin{array}{c}V_1\left(\theta,\frac{1}{3},\alpha,q\right)\left|f'\left(x^m\right)\right|^q \\ +mV_2\left(\theta,\frac{1}{3},\alpha,q\right)\left|f'\left(b\right)\right|^q\end{array}\right]^{\frac{1}{q}}\end{array}\right].$$

Corollary 15. *Under the assumptions of Theorem 8 with $x=\sqrt{ab}$, $\lambda=0$ from the inequality* (2.12) *we get the following midpoint-type inequality for fractional integrals:*

$$\frac{2^{\theta-1}}{\left(m\ln\frac{b}{a}\right)^\theta}\left|K_f\left(0,\theta,\left(\sqrt{ab}\right)^m,a^m,b^m\right)\right|$$

$$=\left|f\left(\left(\sqrt{ab}\right)^m\right)-\frac{2^{\theta-1}\Gamma\left(\theta+1\right)}{\left(m\ln\frac{b}{a}\right)^\theta}\left[J^\theta_{\left(\sqrt{ab}\right)^m-}f(a^m)+J^\theta_{\left(\sqrt{ab}\right)^m+}f(b^m)\right]\right|$$

$$\leq \frac{m\ln\frac{b}{a}}{4}\left[\begin{array}{c} a^m\left(\frac{\left(\frac{b}{a}\right)^{\frac{mp}{2}}-1}{\ln\left(\frac{b}{a}\right)^{\frac{mp}{2}}}\right)^{\frac{1}{p}} \\ \times\left[\begin{array}{c}\frac{1}{\theta q+\alpha+1}\left|f'(x^m)\right|^q \\ +\left(\frac{m}{\theta q+1}-\frac{m}{\theta q+\alpha+1}\right)\left|f'(a)\right|^q\end{array}\right]^{\frac{1}{q}} \\ +b^m\left(\frac{\left(\frac{a}{b}\right)^{\frac{mp}{2}}-1}{\ln\left(\frac{a}{b}\right)^{\frac{mp}{2}}}\right)^{\frac{1}{p}} \\ \times\left[\begin{array}{c}\frac{1}{\theta q+\alpha+1}\left|f'(x^m)\right|^q \\ +\left(\frac{m}{\theta q+1}-\frac{m}{\theta q+\alpha+1}\right)\left|f'(b)\right|^q\end{array}\right]^{\frac{1}{q}}\end{array}\right].$$

Corollary 16. *Under the assumptions of Theorem 8 with $x=\sqrt{ab}$, $\lambda=1$ from the inequality (2.12) we get the following trapezoid-type inequality for fractional integrals*

$$\frac{2^{\theta-1}}{\left(m\ln\frac{b}{a}\right)^{\theta}}\left|K_f\left(1,\theta,\left(\sqrt{ab}\right)^m,a^m,b^m\right)\right|$$

$$=\left|\frac{f(a^m)+f(b^m)}{2}-\frac{2^{\theta-1}\Gamma(\theta+1)}{\left(m\ln\frac{b}{a}\right)^{\theta}}\left[J^{\theta}_{\left(\sqrt{ab}\right)^m-}f(a^m)+J^{\theta}_{\left(\sqrt{ab}\right)^m+}f(b^m)\right]\right|$$

$$\leq \frac{m\ln\frac{b}{a}}{4}\left[\begin{array}{c} a^m\left(\frac{\left(\frac{b}{a}\right)^{\frac{mp}{2}}-1}{\ln\left(\frac{b}{a}\right)^{\frac{mp}{2}}}\right)^{\frac{1}{p}}\left[\begin{array}{c}\frac{1}{\theta}\beta\left(\frac{\alpha+1}{\theta},q+1\right)\left|f'(x^m)\right|^q \\ +\left(\begin{array}{c}\frac{1}{\theta}\beta\left(\frac{1}{\theta},q+1\right) \\ -\frac{1}{\theta}\beta\left(\frac{\alpha+1}{\theta},q+1\right)\end{array}\right)\left|f'(a)\right|^q\end{array}\right]^{\frac{1}{q}} \\ +b^m\left(\frac{\left(\frac{a}{b}\right)^{\frac{mp}{2}}-1}{\ln\left(\frac{a}{b}\right)^{\frac{mp}{2}}}\right)^{\frac{1}{p}}\left[\begin{array}{c}\frac{1}{\theta}\beta\left(\frac{\alpha+1}{\theta},q+1\right)\left|f'(x^m)\right|^q \\ +\left(\begin{array}{c}\frac{1}{\theta}\beta\left(\frac{1}{\theta},q+1\right) \\ -\frac{1}{\theta}\beta\left(\frac{\alpha+1}{\theta},q+1\right)\end{array}\right)\left|f'(b)\right|^q\end{array}\right]^{\frac{1}{q}}\end{array}\right].$$

Corollary 17. *Let the assumptions of Theorem 7 hold. If $|f'(u)|\leq M$ for all $u\in[a^m,b]$ and $\lambda=0$, then from the inequality (2.12) we get the following Ostrowski type inequality for fractional integrals:*

$$\left|\left[\left(\ln\frac{x}{a}\right)^{\theta}+\left(\ln\frac{b}{x}\right)^{\theta}\right]f(x^m)-\frac{\Gamma(\theta+1)}{m^{\theta}}\left[J^{\theta}_{x^m-}f(a^m)+J^{\theta}_{x^m+}f(b^m)\right]\right|$$

$$\leq mM\left[\begin{array}{c}\frac{1}{\theta q+\alpha+1} \\ +\left(\frac{m}{\theta q+1}-\frac{m}{\theta q+\alpha+1}\right)\end{array}\right]^{\frac{1}{q}}\left[\begin{array}{c}a^m\left(\ln\frac{x}{a}\right)^{\theta+1}\left(\frac{\left(\frac{x}{a}\right)^{mp}-1}{\ln\left(\frac{x}{a}\right)^{mp}}\right)^{\frac{1}{p}} \\ +b^m\left(\ln\frac{b}{x}\right)^{\theta+1}\left(\frac{\left(\frac{x}{b}\right)^{mp}-1}{\ln\left(\frac{x}{b}\right)^{mp}}\right)^{\frac{1}{p}}\end{array}\right]$$

for each $x\in[a,b]$.

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