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VOLTERRA-STIELTJES INTEGRAL EQUATION IN REFLEXIVE BANACH SPACES

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Abstract. Volterra-Stieltjes integral equations have been studied in the space of continuous functions in many papers for example, (see [3]-[7]). Our aim here is to studing the existence of weak solutions to a nonlinear integral equation of Volterra-Stieltjes type in a reflexive Banach space. A special case will be considered.

1. Introduction and Preliminaries

Let E be a reflexive Banach space with norm $\| \ . \ \|$ and dual E^*. Denote by $C[I, E]$ the Banach space of strongly continuous functions $x : I \to E$ with sup-norm.
Consider the nonlinear Riemann-Stieltjes integral equation

$$x(t) = p(t) + \int_0^t f(s, x(s)) \ d_s g(t, s), \ t \in I = [0, T], \tag{1}$$

This type of equations have been studied by Banaś (see [1]-[6]) and also by some other authors, for example (see [7], [9] and [15]-[17]).
Here, we study the existence of a weak solution $x \in C[I, E]$ in the reflexive Banach space E for the nonlinear Volterra-Stieltjes integral equation (1) where f is assumed to be weakly-weakly continuous.
For the properties of the Stieltjes integral (see Banaś [1]).
Now, we shall present some auxiliary results that will be need in this work. Let E be a Banach space (need not be reflexive) and let $x : [a, b] \to E$, then

(1-) $x(.)$ is said to be weakly continuous (measurable) at $t_0 \in [a, b]$ if for every $\phi \in E^*$, $\phi(x(.))$ is continuous (measurable) at t_0.
(2-) A function $h : E \to E$ is said to be weakly sequentially continuous if h maps weakly convergent sequences in E to weakly convergent sequences in E.

If x is weakly continuous on I, then x is strongly measurable and hence weakly measurable (see [14] and [11]). It is evident that in reflexive Banach spaces, if x is

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weakly continuous function on $[a, b]$, then x is weakly Riemann integrable (see [14]). Since the space of all weakly Riemann-Stieltjes integrable functions is not complete, we will restrict our attention to the existence of weak solutions of equation (1) in the space $C[I, E]$.

Definition 1.

Let $f : I \times E \to E$. Then $f(t, u)$ is said to be weakly-weakly continuous at (t_0, u_0) if given $\epsilon > 0$, $\phi \in E^*$ there exists $\delta > 0$ and a weakly open set U containing u_0 such that

$$| \phi(f(t,u) - f(t_0, u_0)) | < \epsilon$$

whenever

$$| t - t_0 | < \delta \text{ and } u \in U.$$

Now, we have the following fixed point theorem, due to O'Regan, in the reflexive Banach space (see [18]) and some propositions which will be used in the sequel (see [12]).

Theorem 1. *Let E be a Banach space and let Q be a nonempty, bounded, closed and convex subset of $C[I, E]$ and let $F : Q \to Q$ be a weakly sequentially continuous and assume that $FQ(t)$ is relatively weakly compact in E for each $t \in I$. Then, F has a fixed point in the set Q.*

Proposition 1. *A convex subset of a normed space E is closed if and only if it is weakly closed.*

Proposition 2. *A subset of a reflexive Banach space is weakly compact if and only if it is closed in the weak topology and bounded in the norm topology.*

Proposition 3. *Let E be a normed space with $y \in E$ and $y \neq 0$. Then there exists a $\phi \in E^*$ with $\| \phi \| = 1$ and $\| y \| = \phi(y)$.*

2. Solvability of Volterra-Stieltjes operator

In this section we discuss the existence of weak solutions of the equation (1) in the reflexive Banach space E.

Let $f : I \times E \to E$, $g : I \times I \to R$ be functions such that:

(i) $p \in C[I, E]$.
(ii) $f : I \times E \to E$ is weakly-weakly continuous function.
(iii) There exists a constant M such that $\| f(t, x) \| \leq M$.
(iv) The functions $t \to g(t, t)$ and $t \to g(t, 0)$ are continuous on I.
(v) For all $t_1, t_2 \in I$ such that $t_1 < t_2$ the function $s \to g(t_2, s) - g(t_1, s)$ is nondecreasing on I.
(vi) $g(0, s) = 0$ for any $s \in I$.

Remark 1. *Observe that Assumptions (v) and (vi) imply that the function $s \to g(t, s)$ is nondecreasing on the interval I, for any fixed $t \in I$ (Remark 1 in [6]). Indeed, putting $t_2 = t$, $t_1 = 0$ in (v) and keeping in mind (vi), we obtain the desired conclusion. From this observation, it follows immediately that, for every $t \in I$, the function $s \to g(t, s)$ is of bounded variation on I.*

Definition 2. *By a weak solution to (1) we mean a function $x \in C[I, E]$ which satisfies the integral equation (1). This is equivalent to finding $x \in C[I, E]$ with*

$$\phi(x(t)) \ = \ \phi(p(t) + \int_0^t f(s, x(s)) \ d_s g(t, s)), \ t \in I \ \forall \ \phi \in E^*.$$

Now we can prove the following theorem.

Theorem 2. *Under the assumptions (i)-(vi), the Volterra-Stieltjes integral equation (1) has at least one weak solution* $x \in C[I, E]$.

Proof. Define the nonlinear Volterra-Stieltjes integral operator A by

$$Ax(t) = p(t) + \int_0^t f(s, x(s))\, d_s g(t, s),\ t \in I.$$

For every $x \in C[I, E]$, $f(., x(.))$ is weakly continuous ([19]). To see this we equip E and $I \times E$ with weak topology and note that $t \mapsto (t, x(t))$ is continuous as a mapping from I into $I \times E$, then $f(., x(.))$ is a composition of this mapping with f and thus for each weakly continuous $x : I \to E$, $f(., x(.)) : I \to E$ is weakly continuous, means that $\phi(f(., x(.)))$ is continuous, for every $\phi \in E^*$, g is of bounded variation. Hence $f(., x(.))$ is weakly Riemann-Stieltjes integrable on I with respect to $s \to g(t, s)$. Thus A makes sense.
Now, define the set Q by

$$Q = \{x \in C[I, E] : \parallel x \parallel_0 \leq M_0,\ \parallel x(t_2) - x(t_1) \parallel \leq \parallel p(t_2) - p(t_1) \parallel + M[\mid g(t_2, t_2) - g(t_1, t_1) \mid$$

$$+ \mid g(t_2, 0) - g(t_1, 0) \mid],\ for\ all\ t_1, t_2 \in I\}.$$

For notational purposes $\parallel x \parallel_0 = \sup_{t \in I} \parallel x(t) \parallel$.
The remainder of the proof will be given in four steps.
Step 1 : The operator A maps $C[I, E]$ into $C[I, E]$.
Let $t_1,\ t_2 \in I,\ t_2 > t_1$, without loss of generality, assume $Ax(t_2) - Ax(t_1) \neq 0$

$$\begin{aligned}
\parallel Ax(t_2) - Ax(t_1) \parallel \quad &\leq \quad \mid \phi(p(t_2) - p(t_1)) \mid \\
&+ \quad \mid \int_0^{t_2} \phi(f(s, x(s)))\, d_s g(t_2, s) - \int_0^{t_1} \phi(f(s, x(s)))\, d_s g(t_1, s) \mid \\
&\leq \quad \parallel p(t_2) - p(t_1) \parallel + \mid \int_0^{t_1} \phi(f(s, x(s)))\, d_s g(t_2, s) \\
&+ \quad \int_{t_1}^{t_2} \phi(f(s, x(s)))\, d_s g(t_2, s) - \int_0^{t_1} \phi(f(s, x(s)))\, d_s g(t_1, s) \mid \\
&\leq \quad \parallel p(t_2) - p(t_1) \parallel + \mid \int_0^{t_1} \phi(f(s, x(s)))\, d_s [g(t_2, s) - g(t_1, s)] \mid \\
&+ \quad \mid \int_{t_1}^{t_2} \phi(f(s, x(s)))\, d_s g(t_2, s) \mid \\
&\leq \quad \parallel p(t_2) - p(t_1) \parallel + \int_0^{t_1} \mid \phi(f(s, x(s))) \mid\ d_s [\bigvee_{z=0}^{s} (g(t_2, z) - g(t_1, z))] \\
&+ \quad \int_{t_1}^{t_2} \mid \phi(f(s, x(s))) \mid\ d_s [\bigvee_{z=0}^{s} g(t_2, z)] \\
&\leq \quad \parallel p(t_2) - p(t_1) \parallel + \int_0^{t_1} \mid \phi(f(s, x(s))) \mid\ d_s [g(t_2, s) - g(t_1, s)]
\end{aligned}$$

$$
\begin{aligned}
+ &\quad \int_{t_1}^{t_2} \mid \phi(f(s,x(s))) \mid \; d_s g(t_2,s) \\
\leq &\quad \| p(t_2)-p(t_1) \| + M\int_0^{t_1} d_s[g(t_2,s)-g(t_1,s)] + M\int_{t_1}^{t_2} d_s g(t_2,s) \\
\leq &\quad \| p(t_2)-p(t_1) \| + M\{\int_0^{t_1} d_s[g(t_2,s)-g(t_1,s)] + \int_{t_1}^{t_2} d_s g(t_2,s)\} \\
\leq &\quad \| p(t_2)-p(t_1) \| + \\
+ &\quad M\{[g(t_2,t_1)-g(t_1,t_1)]-[g(t_2,0)-g(t_1,0)]+[g(t_2,t_2)-g(t_2,t_1)]\} \\
\leq &\quad \| p(t_2)-p(t_1) \| + \\
+ &\quad M\{g(t_2,t_1)-g(t_1,t_1)-g(t_2,0)+g(t_1,0)+g(t_2,t_2)-g(t_2,t_1)\} \\
\leq &\quad \| p(t_2)-p(t_1) \| + M\{[g(t_2,t_2)-g(t_1,t_1)]-[g(t_2,0)-g(t_1,0)]\} \\
\leq &\quad \| p(t_2)-p(t_1) \| + M\{\mid g(t_2,t_2)-g(t_1,t_1) \mid + \mid g(t_2,0)-g(t_1,0) \mid\}.
\end{aligned}
$$

Hence

$$
\| Ax(t_2)-Ax(t_1) \| \leq \| p(t_2)-p(t_1) \| + M\{\mid g(t_2,t_2)-g(t_1,t_1) \mid + \mid g(t_2,0)-g(t_1,0) \mid\}, \tag{2}
$$

and so $Ax \in C[I,E]$.

Step 2 : The operator A maps Q into Q.

Take $x \in Q$, note that the inequality (2) shows that AQ is norm continuous. Then by using Proposition 3 we get

$$
\begin{aligned}
\| Ax(t) \| &= \phi(Ax(t)) \leq \mid \phi(p(t)) \mid + \mid \phi(\int_0^t f(s,x(s)) \; d_s g(t,s)) \mid \\
&\leq \| p \|_0 + \int_0^t \mid \phi(f(s,x(s))) \mid \; d_s(\bigvee_{z=0}^{s} g(t,z)) \\
&\leq \| p \|_0 + M\int_0^t \; d_s(\bigvee_{z=0}^{s} g(t,z)) \\
&\leq \| p \|_0 + M\int_0^t \; d_s g(t,s) \\
&\leq \| p \|_0 + M[\; g(t,t)-g(t,0)\;] \\
&\leq \| p \|_0 + M[\; \mid g(t,t) \mid + \mid g(t,0) \mid \;] \\
&\leq \| p \|_0 + M[\; \sup_{t\in I} \mid g(t,t) \mid + \sup_{t\in I} \mid g(t,0) \mid \;] \\
&\leq \| p \|_0 + M[\; k_1 + k_2 \;] = M_0,
\end{aligned}
$$

where $k_1 = \sup_{t\in I} \mid g(t,t) \mid$; $k_2 = \sup_{t\in I} \mid g(t,0) \mid$.

Then

$$\| Ax \|_0 = \sup_{t\in I} \| Ax(t) \| \leq M_0.$$

Hence, $Ax \in Q$ and $AQ \subset Q$ which prove that $A : Q \to Q$, and AQ is bounded in $C[I,E]$.

Step 3 : $AQ(t)$ is relatively weakly compact in E.

Note that Q is nonempty, closed, convex and uniformly bounded subset of $C[I,E]$ and AQ is bounded in norm. According to propositions 1 and 2, AQ is relatively weakly compact in $C[I,E]$ implies $AQ(t)$ is relatively weakly compact in E, for each $t \in I$.

Step 4 : The operator A is weakly sequentially continuous.
Let $\{x_n(t)\}$ be sequence in Q weakly convergent to $x(t)$ in E, since Q is closed we have $x \in Q$. Fix $t \in I$, since f satisfies (ii), then we have $f(t, x_n(t))$ converges weakly to $f(t, x(t))$. By the Lebesgue dominated convergence theorem (see assumption (iii)) for Pettis integral ([13]), we have for each $\phi \in E^*$. $s \in I$

$$\phi(\int_0^t f(s, x_n(s))\ d_s h(t, s)) = \int_0^t \phi(f(s, x_n(s)))\ d_s g(t, s)$$

$$\to \int_0^t \phi(f(s, x(s)))\ d_s g(t, s),\ \forall \phi \in E^*,\ t \in I.$$

i.e. $\phi(Ax_n(t)) \to \phi(Ax(t)),\ \forall\ t \in I,\ Ax_n(t)$ converging weakly to $Ax(t)$ in E.
Thus, A is weakly sequentially continuous on Q.

Since all conditions of Theorem 1 are satisfied, then the operator A has at least one fixed point $x \in Q$ and the nonlinear Stieltjes integral equation (1) has at least one weak solution.■

Corollary 1. *Under the assumptions of Theorem 2 (with $g(t, s) = g(s)$), the Volterra-Stieltjes integral equation*

$$x(t) = p(t) + \int_0^t f(s, x(s))\ dg(s),$$

has a weak solution $x \in C[I, E]$.

Now, let $r > 0$ be given and define the set

$$B_r = \{x \in C[I, E],\ x(t) \in E : \parallel x \parallel_0 \leq r\}.$$

Lemma 1.

Let $f : I \times B_r \to E$ be weakly-weakly continuous, then

- For each $t \in I,\ f(t, .)$ is weakly continuous, hence weakly sequentially continuous (see [8]),
- For each weakly continuous $x : I \to B_r,\ f(., x(.))$ is weakly continuous on I (see [21]),
- f is norm bounded, i.e.,there exists an M_r such that $\parallel f(t, x) \parallel \leq M_r$ for all $(t, x) \in I \times B_r$ (see [20]).

Now we have the following Theorem.

Theorem 3. *Under the assumptions (i) and (iv)-(vi), if $f : I \times B_r \to E$ is weakly-weakly continuous and $M_r < r$, where M_r is defined as in Lemma 1, then the Volterra-Stieltjes integral equation (1) has at least one weak solution $x \in C[I, E]$.*

Proof. Define the nonlinear Volterra-Stieltjes integral operator A by

$$Ax(t) = p(t) + \int_0^t f(s, x(s))\ d_s g(t, s),\ t \in I.$$

For any $x \in C[I, E]$, we have $f(., x(.))$ is weakly continuous on I (Lemma 1), then $\phi(f(., x(.)))$ is continuous on I for every $\phi \in E^*$ and hence $\phi(f(., x(.)))$ is Riemann-Stieltjes integrable on I with respect to $s \to g(t, s)$. Thus A makes sense.

Now, define the set Q by

$$Q = \{x \in B_r,\ \| x(t_2) - x(t_1) \| \leq \| p(t_2) - p(t_1) \| +$$

$$+ M_r\{| g(t_2, t_2) - g(t_1, t_1) | + | g(t_2, 0) - g(t_1, 0) |\},\ for\ all\ t_1, t_2 \in I\}.$$

For notational purposes $\| x \|_0 = \sup_{t \in I} \| x(t) \|$.
The rest of proof runs as in proof of Theorem2.

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