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ON A CLASS OF N-NORMED DOUBLE SEQUENCES RELATED TO p-SUMMABLE DOUBLE SEQUENCE SPACE $l_p^{(2)}$

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ABSTRACT. In this work we introduce the $m^2(\phi)$- class of n-normed double sequences related to p-absolute convergence double sequence space. We study some properties like solidity, simetricity, convergence-free of $m^2(\phi)$ and obtain some inclusion relations involved $m^2(\phi)$.

1. Introduction

Throughout this work, $\mathbb{N}$ and $\mathbb{R}$ denote the set of positive integers and real numbers, respectively. Let $n \in \mathbb{N}$ and X be a $\mathbb{R}$-linear space. A n-norm is a function satisfying following four properties on X^n(see, [5],[7],[10]): For all $z_1, ..., z_n \in X$

1. $\|(z_1, ..., z_n)\|_n = 0$ if and only if $z_1, ..., z_n$ are linear depended,
2. $\|(z_1, ..., z_n)\|_n$ is constant under permutation,
3. $\|(z_1, ..., \alpha z_n)\|_n = |\alpha| \|(z_1, ..., z_n)\|_n$ for any $\alpha \in R$,
4. $\|(z_1, ..., z_{n-1}, x + y)\|_n \leq \|(z_1, ..., z_{n-1}, x)\|_n + \|(z_1, ..., z_{n-1}, y)\|_n$.

In this case a double $(X, \|.\|_n)$ is called a n-normed space. If every Cauchy sequence is convergent, then this space is called a n-Banach space. A double sequence on a normed linear space X is a function x from $\mathbb{N} \times \mathbb{N}$ into X and briefly denoted by $\{x_{k,l}\}$. Throughout this work, w and w^2 denote the spaces of single sequences and double sequences, respectively. If, for all $\varepsilon > 0$, there is a $n_\varepsilon \in \mathbb{N}$ such that $\|x_{k,l} - a\|_X < \varepsilon$ whenever $k > n_\varepsilon$ and $l > n_\varepsilon$, then a double sequence $\{x_{k,l}\}$ is said to be converge (in Pringsheim's sense) to a $a \in X$. If, for all $\varepsilon > 0$, there is a $n_\varepsilon \in \mathbb{N}$ such that $\|x_{k,l} - x_{p,q}\|_X < \varepsilon$ whenever $k, l, p, q > n_\varepsilon$, then a double sequence $\{x_{k,l}\}$ is said to be a double Cauchy sequence in X. A double series is infinity sum $\sum_{k,l=1}^{\infty} x_{k,l}$ and its convergence implies the convergence by $\|.\|_X$ of partial sums sequence $\{S_{n,m}\}$, where $S_{n,m} = \sum_{k=1}^{n} \sum_{l=1}^{m} x_{k,l}$(see [2],[8],[9]).

Throughout this work, we will use the convergence in Pringsheim's sense of the double sequences.

If each double Cauchy sequence in X converge an element of X according to n-norm, then X is said to be a double complete space according to n-norm. A double complete n-normed space is said to be a double n- Banach space.

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A n-normed double sequence space E is said to be solid if $\{\alpha_{k,l}x_{k,l}\} \in E$ whenever $\{x_{k,l}\} \in E$ for all double sequences $\{\alpha_{k,l}\}$ of scalars with $|\alpha_{k,l}| \leq 1$ for all $k, l \in \mathbb{N}$(see [3],[4]).

Let $x = \{x_{k,l}\}$ be a double sequence. A set $S(x)$ is defined by

$$S(x) = \left\{\left\{x_{\pi_1(k),\pi_2(k)}\right\} : \pi_1 \text{ and } \pi_2 \text{ are permutations of } \mathbb{N}\right\}.$$

If $S(x) \subseteq E$ for all $x \in E$, then E is said to be symmetric.

If $\{x_{k,l}\} \in E$, whenever $\{y_{k,l}\} \in E$ and $y_{k,l} = 0$ implies $x_{k,l} = 0$, then a double sequence space E is said to be convergence-free.

Throughout this work $\{\phi_{k,l}\}$ is taken as a non-decreasing double sequence of the positive real numbers such that

$$k\phi_{k+1,l} \leq (k+1)\,\phi_{k,l} \text{ and } l\phi_{k,l+1} \leq (l+1)\,\phi_{k,l}$$

for all $(k, l) \in \mathbb{N} \times \mathbb{N}$.

Now let $\wp_s$ be a family of subsets σ having most elements s in $\mathbb{N}$. The space $m(\phi)$, introduced by Sargent in [11], is in the form

$$m(\phi) = \left\{x = \{x_k\} : \|x\|_{m(\phi)} = \sup_{s\geq 1,\ \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} |x_k| < \infty\right\}.$$

Tripathy and Borgshain in [12] expanded to n-normed spaces it. They introduced this new space as follows:

$$(m(\phi), \|.\|_n) = \left\{x = \{x_k\} : \|x\|_{n,m(\phi)} = \sup_{s\geq 1,\ \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{k\in\sigma} \|(z_1, ..., z_{n-1}, x_k)\|_n < \infty\right\}.$$

for all $z_1, ..., z_{n-1} \in X$. The spaces in this form for single sequences was studied by many authors(see, [1],[12],[13]).

Let $\wp_{s,t}$ be the class of subsets $\sigma = \sigma_1 \times \sigma_2$ in $\mathbb{N} \times \mathbb{N}$ such that element numbers of σ_1 and σ_2 are most s and t respectively.

The aim of this work is to introduce the space $m^2(\phi)$ and investigate various properties of it. This space is defined by

$$\left(m^2(\phi), \|.\|_n\right) = \left\{x = \{x_{k,l}\} : \|x\|_{n,m^2(\phi)} = \sup_{\substack{(s,t)\geq(1,1) \\ \sigma_1\times\sigma_2\in\wp_{s,t}}} \frac{1}{\phi_{s,t}} \sum_{k\in\sigma_1} \sum_{l\in\sigma_2} \|(z_1, ..., z_{n-1}, x_{k,l})\|_n < \infty\right\}$$

for all $z_1, ..., z_{n-1} \in X$.

2. MAIN RESULTS

Definition 1. *A double sequence space E is said to be monotone if $x = (x_{kl}u_{kl}) \in E$ for all $x = (x_{kl})$ and $u = (u_{kl}) \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$ (see [14]).*

The following lemma is an easy result of the definitions:

Lemma 1. *If a double sequence space E is solid, then E is monotone.*

Proposition 1. *Let X be a n-Banach space. Then $\left(m^2(\phi), \|.\|_n\right)$ is also a n-Banach space with the norm $\|.\|_{n,m^2(\phi)}$.*

Proof. Let $\left\{x^{(i)}\right\}$ be a double Cauchy sequence in $\left(m^2(\phi), \|.\|_n\right)$ such that $x^{(i)} = \left\{x^{(i)}_{k,l}\right\}_{k,l=1}^{\infty}$ for all $i \in N$. Then for arbitrary $\varepsilon > 0$ there is a $n_\varepsilon \in N$ such that

$$\left\|x^{(i)} - x^{(j)}\right\|_{n,m^2(\phi)} < \frac{\varepsilon}{\phi_{1,1}}$$

for each $i, j \geq n_\varepsilon$. Then the inequality

$$\left\|\left(z_1, ..., z_{n-1}, \left(x^{(i)}_{k,l} - x^{(j)}_{k,l}\right)\right)\right\|_n < \varepsilon$$

holds for all $i, j \geq n_\varepsilon$ and $(k,l) \in \mathbb{N} \times \mathbb{N}$, since $(\phi_{k,l})$ is a non-decreasing double sequence of the positive real numbers. So $\left\{x^{(i)}_{k,l}\right\}$ is a double Cauchy sequence in X. Since X is a n-Banach space, $\left\{x^{(i)}_{k,l}\right\}$ is convergent in X. We say

$$\lim_{i\to\infty} x^{(i)}_{k,l} = x_{k,l}$$

for each $(k,l) \in \mathbb{N} \times \mathbb{N}$. Now, for this double sequence $x = \{x_{k,l}\}$ we have to show that $\lim_{i\to\infty} x^{(i)} = x$ and $x \in \left(m^2(\phi), \|.\|_n\right)$. Since $\left\{x^{(i)}\right\}$ is a double Cauchy sequence in $\left(m^2(\phi), \|.\|_n\right)$, there is a number $n_\varepsilon \epsilon N$ such that

$$\sup_{\substack{(s,t)\geq(1,1)\\ \sigma_1\times\sigma_2\in\wp_{s,t}}} \frac{1}{\phi_{s,t}} \sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \left\|\left(z_1, ..., z_{n-1}, \left(x^{(i)}_{k,l} - x^{(j)}_{k,l}\right)\right)\right\|_n < \varepsilon$$

where $i,\ j \geq n_\varepsilon$.

Taking limit as $j \to \infty$, we have

$$\sup_{\substack{(s,t)\geq(1,1)\\ \sigma_1\times\sigma_2\in\wp_{s,t}}} \frac{1}{\phi_{s,t}} \sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \left\|\left(z_1, ..., z_{n-1}, \left(x^{(i)}_{k,l} - x_{k,l}\right)\right)\right\|_n < \varepsilon$$

for all $i \geq n_\varepsilon$. This implies that

$$\left\|x^{(i)} - x\right\|_{n,m^2(\phi)} < \varepsilon$$

for all $i \geq n_\varepsilon$ and so $\lim_{i\to\infty} x^{(i)} = x$. We also have that

$$\|x\|_{n,m^2(\phi)} \leq \left\|x^{(i)} - x\right\|_{n,m^2(\phi)} + \left\|x^{(i)}\right\|_{n,m^2(\phi)} < \varepsilon + \left\|x^{(i)}\right\|_{n,m^2(\phi)} < \infty$$

for a fixed $i \geq n_\varepsilon$, and hence $x \in \left(m^2(\phi), \|.\|_n\right)$. Thus $\left(m^2(\phi), \|.\|_n\right)$ is a n-Banach space with the norm $\|.\|_{n,m^2(\phi)}$. □

Proposition 2. *The class* $\left(m^2(\phi), \|.\|_n\right)$ *of double sequences is solid.*

Proof. Let $\{x_{k,l}\} \in \left(m^2(\phi), \|.\|_n\right)$ and let $\{\alpha_{k,l}\} \in w^2$ be any double sequence of scalars with $|\alpha_{k,l}| \leq 1$. Then we can write

$$\begin{aligned} & \sup_{\substack{(s,t)\geq(1,1)\\ \sigma_1\times\sigma_2\in\wp_{s,t}}} \frac{1}{\phi_{s,t}} \sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \left\|(z_1,...,z_{n-1},\alpha_{k,l}x_{k,l})\right\|_n \\ = & \sup_{\substack{(s,t)\geq(1,1)\\ \sigma_1\times\sigma_2\in\wp_{s,t}}} \frac{1}{\phi_{s,t}} \sum_{k\in\sigma_1}\sum_{l\in\sigma_2} |\alpha_{k,l}| \left\|(z_1,...,z_{n-1},x_{k,l})\right\|_n \\ \leq & \sup_{\substack{(s,t)\geq(1,1)\\ \sigma_1\times\sigma_2\in\wp_{s,t}}} \frac{1}{\phi_{s,t}} \sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \left\|(z_1,...,z_{n-1},x_{k,l})\right\|_n . \end{aligned}$$

Thus we obtain

$$\left\|\{\alpha_{k,l}x_{k,l}\}\right\|_{n,m^2(\phi)} \leq \left\|\{x_{k,l}\}\right\|_{n,m^2(\phi)} .$$

This implies that $\{\alpha_{k,l}x_{k,l}\} \in \left(m^2(\phi), \|.\|_n\right)$, and hence the class $\left(m^2(\phi), \|.\|_n\right)$ is solid.

Corollary 1. *The space* $\left(m^2(\phi), \|.\|_n\right)$ *is monotone.*

□

Proposition 3. *The class* $\left(m^2(\phi), \|.\|_n\right)$ *of double sequences is symmetric.*

Proof. Let $\{x_{k,l}\} \in \left(m^2(\phi), \|.\|_n\right)$ and let $\{y_{k,l}\} \in w^2$ be any permutation of it. Then there exists a $(p_k, q_k) \in \mathbb{N}\times\mathbb{N}$ such that $y_{k,l} = x_{p_k,q_l}$ for all $(k,l) \in \mathbb{N}\times\mathbb{N}$. Hence we have

$$\begin{aligned} \left\|\{x_{k,l}\}\right\|_{n,m^2(\phi)} &= \sup_{\substack{(s,t)\geq(1,1)\\ \sigma_1\times\sigma_2\in\wp_{s,t}}} \frac{1}{\phi_{s,t}} \sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \left\|(z_1,...,z_{n-1},x_{k,l})\right\|_n \\ &= \sup_{\substack{(s,t)\geq(1,1)\\ \sigma_1\times\sigma_2\in\wp_{s,t}}} \frac{1}{\phi_{s,t}} \sum_{p_k\in\sigma_1}\sum_{q_l\in\sigma_2} \left\|(z_1,...,z_{n-1},x_{p_k,q_l})\right\|_n \\ &= \left\|\{x_{p_k,m_{ql}}\}\right\|_{n,m^2(\phi)} = \left\|\{y_{k,l}\}\right\|_{n,m^2(\phi)} . \end{aligned}$$

□

Example 1. *Let $n = 2$ and a double sequence ϕ be given by $\phi(s,t) = s.t$. Also we take a double sequence $\{x_{k,l}\}$ such that $x_{k,l} = \frac{1}{k} + \frac{1}{l}$ and define 2-norm on $R\times R$ such that*

$$\left\|(z, x_{k,l})\right\|_2 = z.x_{k,l}.$$

Then we obtain

$$\begin{aligned} \|x\|_{n,m^2(\phi)} &= \sup_{\substack{(s,t)\geq(1,1)\\ \sigma_1\times\sigma_2\in\wp_{s,t}}} \frac{1}{s.t} \sum_{k\in\sigma_1}\sum_{l\in\sigma_2} z.\left(\frac{1}{k}+\frac{1}{l}\right) \\ &\leq \sup\left\{ \frac{1}{s.t} \sum_{k=1}^{s}\sum_{l=1}^{t} z.\left(\frac{1}{k}+\frac{1}{l}\right) : (s,t)\geq(1,1) \right\} \end{aligned}$$

$$\leq \sup\left\{2\frac{1}{s.t}\ z.s.t : (s,t) \geq (1,1)\right\} = 2z < \infty$$

for all $z \in \mathbb{R}$.

Remark 1. *The class* $\left(m^2(\phi), \|.\|_n\right)$ *of double sequences have not to convergence-free. This can be immediately observed from the example above.*

Theorem 1. *Let* ψ *be an other double sequence like* ϕ. *Then* $\left(m^2(\phi), \|.\|_n\right) \subseteq \left(m^2(\psi), \|.\|_n\right)$ *if and only if* $\sup\limits_{(s,t)\geq(1,1)} \left(\frac{\phi_{s,t}}{\psi_{s,t}}\right) < \infty$.

Proof. Let $K = \sup\limits_{(s,t)\geq(1,1)} \left(\frac{\phi_{s,t}}{\psi_{s,t}}\right) < \infty$. Then, $\phi_{s,t} \leq K.\psi_{s,t}$ for all $(s,t) \geq (1,1)$. If $\{x_{k,l}\} \in \left(m^2(\phi), \|.\|_n\right)$, then

$$\sup_{\substack{(s,t)\geq(1,1)\\ \sigma_1\times\sigma_2\in\wp_{s,t}}} \frac{1}{\phi_{s,t}} \sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \|(z_1,...,z_{n-1},x_{k,l})\|_n < \infty.$$

Thus

$$\sup_{\substack{(s,t)\geq(1,1)\\ \sigma_1\times\sigma_2\in\wp_{s,t}}} \frac{1}{K\psi_{s,t}} \sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \|(z_1,...,z_{n-1},x_{k,l})\|_n < \infty$$

and hence $\|\{x_{k,l}\}\|_{n,m^2(\psi)} < \infty$. This shows that $\left(m^2(\phi), \|.\|_n\right) \subseteq \left(m^2(\psi), \|.\|_n\right)$.

Conversely, let $\left(m^2(\phi), \|.\|_n\right) \subseteq \left(m^2(\psi), \|.\|_n\right)$ and $\alpha_{s,t} = \frac{\phi_{s,t}}{\psi_{s,t}}$ for all $(s,t) \geq (1,1)$. Suppose that $\sup\limits_{(s,t)\geq(1,1)} \alpha_{s,t} = \infty$. Then there exists a subsequence $\{\alpha_{s_i,t_i}\}$ of $\{\alpha_{s,t}\}$ such that $\lim\limits_{i\to\infty} \alpha_{s_i,t_i} = \infty$. Let us take a non-zero arbitrary sequence $\{x_{k,l}\}$ in $\left(m^2(\phi), \|.\|_n\right)$. Using the hypothesis, we have

$$\begin{aligned}
&\sup_{\substack{(s,t)\geq(1,1)\\ \sigma_1\times\sigma_2\in\wp_{s,t}}} \frac{1}{\psi_{s,t}} \sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \|(z_1,...,z_{n-1},x_{k,l})\|_n \\
= &\sup_{\substack{(s,t)\geq(1,1)\\ \sigma_1\times\sigma_2\in\wp_{s,t}}} \frac{\alpha_{s,t}}{\phi_{s,t}} \sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \|(z_1,...,z_{n-1},x_{k,l})\|_n \\
\geq &\sup_{\substack{i\geq 1\\ \sigma_1\times\sigma_2\in\wp_{s_i,t_i}}} \alpha_{s_i,t_i}\frac{1}{\phi_{s_i,t_i}} \sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \|(z_1,...,z_{n-1},x_{k,l})\|_n = \infty.
\end{aligned}$$

This is a contradiction as $\{x_{k,l}\} \notin \left(m^2(\psi), \|.\|_n\right)$. The proof is completed.

Corollary 2. $\left(m^2(\phi), \|.\|_n\right) = \left(m^2(\psi), \|.\|_n\right)$ *if and only if* $\sup\limits_{(s,t)\geq(1,1)} \alpha_{s,t} < \infty$ *and* $\sup\limits_{(s,t)\geq(1,1)} \alpha_{s,t}^{-1} < \infty$, *where* $\alpha_{s,t} = \frac{\phi_{s,t}}{\psi_{s,t}}$ *for all* $(s,t) \geq (1,1)$.

□

Theorem 2. *(a)* $\left(l_1^{(2)}, \|.\|_n\right) \subseteq \left(m^2(\phi), \|.\|_n\right) \subseteq \left(l_\infty^{(2)}, \|.\|_n\right)$.

(b) $\left(m^2(\phi), \|.\|_n\right) = \left(l_1^{(2)}, \|.\|_n\right)$ if and only if $\sup_{(s,t)\geq(1,1)} \phi_{s,t} = \phi < \infty$.

(c) $\left(m^2(\phi), \|.\|_n\right) = \left(l_\infty^{(2)}, \|.\|_n\right)$ if and only if $\sup_{(s,t)\geq(1,1)} \frac{\phi_{s,t}}{s.t} = \phi < \infty$ and $\sup_{(s,t)\geq(1,1)} \frac{s.t}{\phi_{s,t}} = \phi < \infty$.

Proof. Firstly we write clearly this topic spaces:

$$\left(l_1^{(2)}, \|.\|_n\right) = \left\{ x = \{x_{k,l}\} : \sum_{k,l=1}^{\infty} \|(z_1, ..., z_{n-1}, x_{k,l})\|_n < \infty \right\},$$

$$\left(m^2(\phi), \|.\|_n\right) = \left\{ x = \{x_{k,l}\} : \|x\|_{n,m^2(\phi)} = \sup_{\substack{(s,t)\geq(1,1) \\ \sigma_1\times\sigma_2\in\wp_{s,t}}} \frac{1}{\phi_{s,t}} \sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \|(z_1, ..., z_{n-1}, x_{k,l})\|_n < \infty \right\}$$

and

$$\left(l_\infty^{(2)}, \|.\|_n\right) = \left\{ x = \{x_{k,l}\} : \sup_{(k,l)\in N\times N} \|(z_1, ..., z_{n-1}, x_{k,l})\|_n < \infty \right\}$$

for all $z_1, ..., z_{n-1} \in X$.

(a) Take $x = \{x_{k,l}\} \in l_1^{(2)}$ and let a set A be defined as follows:

$$A = \left\{ \frac{1}{\phi_{s,t}} \sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \|(z_1, ..., z_{n-1}, x_{k,l})\|_n : (s,t) \geq (1,1),\ \sigma_1\times\sigma_2 \in \wp_{s,t} \right\}$$

for all $z_1, ..., z_{n-1} \in X$. Then we can write $\|x\|_{n,m^2(\phi)} = \sup A$. Since $\{\phi_{s,t}\}$ is a non-decreasing double sequence, $\left\{\frac{1}{\phi_{s,t}}\right\}$ is a non-increasing double sequence. So we obtain

$$\frac{1}{\phi_{1,1}} \sum_{k,l=1}^{\infty} \|(z_1, ..., z_{n-1}, x_{k,l})\|_n \geq a$$

for all $a \in A$ and hence

$$\|x\|_{l_1^{(2)}} = \sum_{k,l=1}^{\infty} \|(z_1, ..., z_{n-1}, x_{k,l})\|_n \geq \phi_{1,1}. \sup A = \phi_{1,1}. \|x\|_{n,m^2(\phi)}.$$

Therefore $x \in \left(m^2(\phi), \|.\|_n\right)$. Thus we have $\left(l_1^{(2)}, \|.\|_n\right) \subseteq \left(m^2(\phi), \|.\|_n\right)$.

It is clear that

$$\sup A \geq \frac{1}{\phi_{1,1}} \|(z_1, ..., z_{n-1}, x_{k,l})\|_n$$

for all $(k,l) \in N\times N$, and hence

$$\|x\|_{n,m^2(\phi)} \geq \frac{1}{\phi_{1,1}} \sup_{(k,l)\in N\times N} \|(z_1, ..., z_{n-1}, x_{k,l})\|_n = \frac{1}{\phi_{1,1}} \|x\|_{n,l_\infty^{(2)}}.$$

This shows that if $\{x_{k,l}\} \in \left(m^2(\phi), \|.\|_n\right)$, then $\{x_{k,l}\} \in \left(l_\infty^2, \|.\|_n\right)$. Thus we have $\left(m^2(\phi), \|.\|_n\right) \subseteq \left(l_\infty^{(2)}, \|.\|_n\right)$.

(b) Let $\sup_{(s,t)\geq(1,1)} \phi_{s,t} < \infty$. It is clear that $\left(m^2(\psi), \|.\|_n\right) = \left(l_1^{(2)}, \|.\|_n\right)$ if $\psi_{s,t} = 1$ for all $(s,t) \geq (1,1)$. Then we can write $\sup_{(s,t)\geq(1,1)} \phi_{s,t} = \sup_{(s,t)\geq(1,1)} \frac{\phi_{s,t}}{\psi_{s,t}} < \infty$. By Theorem 1, we have $\left(m^2(\phi), \|.\|_n\right) \subset \left(m^2(\psi), \|.\|_n\right)$, and

$$\left(m^2(\phi), \|.\|_n\right) = \left(l_1^{(2)}, \|.\|_n\right)$$

according to (a). We can see just the opposite of this from Theorem 1 again.

(c) Firstly we show that $\left(l_\infty^{(2)}, \|.\|_n\right) = \left(m^2(\psi), \|.\|_n\right)$ if $\psi(s,t) = s.t$ for all $(s,t) \in \mathbb{N}\times\mathbb{N}$. Let $\{x_{k,l}\} \in \left(l_\infty^{(2)}, \|.\|_n\right)$. Then we have

$$\frac{1}{st}\sum_{k\in\sigma_1}\sum_{l\in\sigma_2} \|(z_1, ..., z_{n-1}, x_{k,l})\|_n \leq \frac{1}{st} st \sup_{(k,l)\in\mathbb{N}\times\mathbb{N}} \|(z_1, ..., z_{n-1}, x_{k,l})\|_n < \infty.$$

This gives the inclusion $\left(l_\infty^{(2)}, \|.\|_n\right) \subset \left(m^2(\psi), \|.\|_n\right)$. The reverse inclusion is a result of the alternative (a). Thus we have $\left(m^2(\psi), \|.\|_n\right) = \left(l_\infty^{(2)}, \|.\|_n\right)$. By Theorem 1, the proof is completed.

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□

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