Electronic Journal of Mathematical Analysis and Applications
Vol. 5(2) July 2017, pp. 10-17.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

FIXED POINT THEOREMS IN GENERALIZED TYPES OF b-DISLOCATED METRIC SPACES

MUJEEB UR RAHMAN AND MUHAMMAD SARWAR

ABSTRACT. Two generalized types of b-dislocated metric spaces are introduced and some interesting properties and examples of these new spaces are discussed. Some fixed point results are also investigated in these new spaces. The presented work generalizes some well-known results from the existing literature.

1. Introduction and Preliminaries

In 1906, Frechet introduced the notion of metric space which is one of the cornerstones of not only mathematics but also several quantitative sciences. Due to its importance and application potential, this notion has been extended, improved and generalized in many different ways. An incomplete list of the results of such attempts are the following: quasi metric space, symmetric space, partial metric space, cone metric space, G-metric space, dislocated metric space, dislocated quasi metric space, right and left dislocated metric spaces and so on.

Originally motivated by the experience of computer science as discussed below, we show how a mathematics of non-zero self-distance for metric spaces has been established and is now leading to interesting research into the foundations of topology. The approach of this article is to retrace the steps of a standard introduction to b-metric space seeing how it can be further generalized to accommodate non-zero self distance. Proofs presented here consist of straight forward reasoning about non-zero self distance but it will open many ways for the researchers to extend and validate these results in many fields of science especially in computer science, topology, electronic engineering and so on.

Let us begin with an example of a metric space and why non-zero self-distance is worth considering. Let S^∞ be the set of all infinite sequences $x = (x_0, x_1, ...)$ over a set S. For all such sequences x and y let $d_S(x, y) = 2^{-k}$ where k is the largest number (possibly ∞) such that $x_i = y_i$ for each $i < k$. Thus $d_S(x, y)$ is defined to be 1 over 2 to the power of the length of the longest initial sequence common to both x and y. It can be shown that (S^∞, d_S) is a metric space.

1991 *Mathematics Subject Classification.* 47H10, 54H25, 55M20.
Key words and phrases. Dislocated and b-dislocated metric spaces, right and left dislocated b-metric spaces, Cauchy sequence, contraction, fixed point.
Submitted Sep. 28, 2016.

How might Computer Scientists view this metric space? to be interested in an infinite sequence x they would want to know how to compute it. That is, how to write a computer program to print out (on either a screen or paper) the values x_0, then x_1, then x_2 and so on. As x is an infinite sequence, its values cannot be printed out in any finite amount of time and so Computer Scientists are interested in how the sequence x is formed from its parts. The finite sequences $\{\}, \{x_0\}, \{x_0, x_1\}, \{x_0, x_1, x_2\}$ and so on. After each value x_k is printed. The finite sequence $\{x_0, ..., x_k\}$ represents that part of the infinite sequence produced so far. Each finite sequence is thus thought of in computer science as being a partially computed version of the infinite sequence x, which is totally computed. Suppose now that the above definition of d_S is extended to S^* the set of all finite sequences over S. Then all the axioms of metric space still hold except self-distance axiom does not hold i.e if x is a finite sequence then $d_S(x, x) = 2^{-k}$ for some number $k < \infty$, which is not 0. This raises an intriguing contrast between 20th century mathematics of which the theory of metric spaces is our working example, and the contemporary experience of computer science. The truth of the statement $x = x$ is surely unchallenged in mathematics while in computer science its truth can only be asserted to the extent to which x is computed. This article will show that rather than collapsing, the theory of metric spaces, is actually expanded and enriched by the generalization of b-metric space by dropping the requirements for equality to imply indistancy, symmetric property and changes in triangular inequality. Our another motivation of studying these new spaces is that a generalization of Banach contraction mapping theorem can indeed be established in these newly constructed spaces and to open traces for the researchers to find its applications to Logic Programming Semantics in the near future in connection with some problems concerning with the convergence of non-measurable functions with respect to measure.

In 2000, Hitzler and Seda [1] introduced the idea of dislocated metric (d-metric) space. This idea was not new and has been studied in the context of domain theory [2], where dislocated metrics are given the name of metric domains. This idea of metric plays a key role in the development of Logic Programming Semantics. This notion was further generalized to dislocated quasi-metric (dq-metric) and left and right dislocated metric (ld-metric and rd-metric) spaces by Zeyada et al. (see [3],[4], [5]).

The notion of b-metric space was introduced by Czerwik [6] in connection with some problems concerning with the convergence of non-measurable functions with respect to measure. Shukla [7] generalized the idea of b-metric space and initiated the idea of partial b-metric space. Hussain et al. [8] investigated a new idea of b-dislocated metric space by giving the idea of non zero self-distance which generalized not only the concept of b-metric space but also the idea of dislocate metric space. Recently, Rahman and Sarwar [9] gave the concept of dislocated quasi b-metric (dq b-metric) space. They omit symmetric property along with non zero self-distance. This new notion generalized the concept of b-metric as well as the concept of dislocated quasi-metric space.

Definition 1. [9]. Let X be a non empty set. Let $k \geq 1$ be a real number then a mapping $d : X \times X \to [0, \infty)$ is called b-metric if $\forall\ x, y, z \in X$ the following conditions are satisfied.

1) $d(x, x) = 0$;

2) $d(x, y) = d(y, x) = 0$ implies that $x = y$;

3) $d(x,y) = d(y,x)$;
4) $d(x,y) \leq k[d(x,z) + d(z,y)]$.
And the pair (X,d) is called b-metric space. If d satisfies conditions from $2-4$ then (X,d) is called b-dislocated metric space and if d satisfies conditions 2 and 4 only then (X,d) is called dislocated quasi b-metric space.
Theorem 1.1.[10]. Let (X,d) be a complete b-metric space with $k \geq 1$ $T : X \to X$ be a contraction with $\alpha \in [0,1)$ and $k\alpha < 1$. Then T has a unique fixed point in X.
Definition 2.[11]. Let (X,d) be a metric space. Let $T : X \to X$ be a self mapping. Then T is called kannan mapping if

$$d(Tx,Ty) \leq \alpha[d(x,Tx) + d(y,Ty)] \quad \text{for all} \quad x,y \in X \text{ and } \quad \alpha \in [0,1/2) \qquad (1)$$

The plane of this paper is as follows: In Section 2, we introduce the concepts of right and left dislocated b-metric (rd and ld b-metric) space. We provide Some examples of these newly constructed spaces. Also, we investigate several properties and generalizations of Theorem 1.1 in right as well as in left dislocated b-metric space.

2. Fixed Point Theorems in Right and Left Dislocated b-Metric Spaces

Definition 2.1. Let X be a non empty set. Let $k \geq 1$ be a real number then a mapping $d : X \times X \to [0,\infty)$ is called right dislocated b-metric if $\forall$ $x,y,z \in X$ satisfying
d_1) $d(x,y) = d(y,x) = 0$ implies that $x = y$;
d_2) $d(x,y) \leq k[d(x,z) + d(y,z)]$.
And the pair (X,d) is called right dislocated b-metric (rd b-metric) space.
Examples.

(1) Every b-metric and b-dislocated metric space is a rd b-metric space.
(2) If $X = \{0\} \cup \{\frac{1}{n}, \forall\ n \in N\}$. Define $d(0,0) = 0, d(0,\frac{1}{n}) = 0, d(\frac{1}{n},0) = 1, d(\frac{1}{n},\frac{1}{m}) = 1$ $\forall m,n \in N$. Then

$$d(x,y) \leq \frac{k}{2}[d(x,z) + d(y,z)]$$

for $k > 2$, it is rd b-metric space.
(3) Let (X,d_1) be a rd-metric space and

$$d(x,y) = [d_1(x,y)]^p$$

where $p > 1$ is a real number. Then condition (d_1) is obviously hold for (d_2) if $1 < p < \infty$, then the convexity of the function $f(x) = x^p$ $(x > 0)$ implies that $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ hold. Thus

$$\begin{aligned} d(x,y) &= [d_1(x,y)]^p \leq [d_1(x,z) + d_1(y,z)]^p \\ &\leq 2^{p-1}([d_1(x,z)]^p + [d_1(y,z)]^p) \\ d(x,y) &\leq d(x,z) + d(y,z). \end{aligned}$$

Thus (d_2) is also satisfied for $k = 2^{p-1}$ where $p > 1$. Hence d is rd b-metric.

However, if (X,d_1) is a right dislocated metric space then it is not necessary that (X,d) is a right dislocated metric space, For example for $X = N$, $d_1(x,y) = \frac{1}{n}$ is a

rd-metric and $d(x,y) = \left(\frac{1}{n}\right)^2$ is a rd b-metric but not a rd-metric on X.

Remark. Example 1 convey that the notion of rd b-metric space extends the idea of b-metric spaces and b-dislocated metric spaces. Example 2 is artificially constructed through which we want to say that the it is complete rd b-metric space but it is not a b-metric nor b-dislocated metric space. Example 3 is the method of constructing rd b-metric spaces from the usual rd metric spaces. Through which we have constructed an example which is rd b-metric space but not a usual rd-metric space.

Remark. The rd b-metric space in Example 2 is a complete space while the rd b-metric space defined in Example 3 is not complete because the sequence $\{x_n\} = n$ is a Cauchy sequence in the given space which converges to 0 which does not belongs to N.

Remark. In the distance space (X,d) we will denote the set $\{x \in X : d(x,x) = 0\}$ by X_0.

The set X_0 in Example 2 is $\{0\}$ while in Example 3 it is empty.

Definition 2.2. Let X be a non empty set. Let $k \geq 1$ be a real number then a mapping $d : X \times X \to [0,\infty)$ is called left dislocated b-metric if $\forall\ x,y,z \in X$ satisfying

ld_1) $d(x,y) = d(y,x) = 0$ implies that $x = y$;

ld_2) $d(x,y) \leq k[d(z,x) + d(z,y)]$.

And the pair (X,d) is called left dislocated b-metric (ld b-metric) space.

Examples.

(1) Every b-metric and b-dislocated metric space is a ld b-metric space.

(2) Let (X,d) be a rd b-metric space and d^* be defined as

$$d(y,x) = d^*(x,y)\ \forall\ x,y \in X.$$

Then (X,d^*) is a ld b-metric space.

(3) Let $X = N$. Define $d(n,m) = \left(\frac{1}{m}\right)^2\ \forall\ n,m \in N$. Then (X,d) is a ld b-metric space.

Remark. It is clear from the above examples that the two newly constructed spaces are independent of each other.

Definition 2.3. A sequence $\{x_n\}$ in X is called rd b-convergent in X if there exists $x \in X$ such that $\lim_{n\to\infty} d(x,x_n) = 0$. In this case x is called the rd b-limit of the sequence $\{x_n\}$.

Unlike b-metric space rd b-metric space need not be left and right convergent. But in case of rd b-metric space it is rd b-convergent only. **Definition 2.4.** A sequence $\{x_n\}$ in X is called ld b-convergent in X if there exists $x \in X$ such that $\lim_{n\to\infty} d(x_n,x) = 0$. In this case x is called the ld b-limit of the sequence $\{x_n\}$.

In case of ld b-metric space a convergent sequence need only to be ld b-convergent.

Remark. Since the notion of ld b-metric space is look like a dual notion of rd b-metric space. Therefore, we state the following definitions and some basic properties for right dislocated b-metric spaces only which may be easily carried out for left dislocated b-metric spaces.

Definition 2.5. A sequence $\{x_n\}$ in rd b-metric space is called Cauchy sequence if for $\epsilon > 0$ there exist $n_0 \in N$, such that for $m > n \geq n_0$, we have $d(x_n,x_m) < \epsilon$.

(1) A sequence $\{x_n\}$ in rd b-metric space is called right Cauchy sequence if for $\epsilon > 0$ there exist $n_0 \in N$, such that for $m > n \geq n_0$, we have $d(x_n, x_m) < \epsilon$.
(2) A sequence $\{x_n\}$ in rd b-metric space is called left Cauchy sequence if for $\epsilon > 0$ there exist $n_0 \in N$, such that for $m > n \geq n_0$, we have $d(x_m, x_n) < \epsilon$.

Definition 2.6. A rd $b-$metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point in X.
Definition 2.7. Let (X, d) be a rd b-metric space. A mapping $T : X \to X$ is called contraction if $k \geq 1$ there exists a constant $\alpha \in [0, 1)$ with $k\alpha < 1$ and for all $x, y \in X$ satisfying

$$d(Tx, Ty) \leq \alpha d(x, y).$$

Lemma 1. Let (X, d) be a rd b-metric space and $\{x_n\}$ be a sequence in X. Then

(1) (X_0, d) is a metric space if X_0 is non-empty set.
(2) If $x \in X$ is a rd-limit of $\{x_n\}$ then $x \in X_0$.

Proof. (1) is easy to prove. We prove (2): let for any $\epsilon > 0$ there exists $n_0 \in N$ such that for all $n \in N$ we have

$$d(x, x) \leq k[d(x, x_n) + d(x, x_n)] < \epsilon.$$

Since $\epsilon > 0$ is arbitrary. Therefore $d(x, x) = 0$. So $x \in X_0$.
In similar manner, one can easily prove the ld b-metric space analogue of Lemma 1.
Lemma 2. Every subsequence of rd b-convergent sequence to x_0 is rd b-convergent to x_0.
It is obvious that converse of Lemma 2 may not be true.
Lemma 3. Limit of convergent sequence in rd b-metric space is unique.
Proof. Let $\{x_n\}$ be a convergent sequence in rd b-metric space. Let $\{x_n\}$ have two limits x and y with $x \neq y$ by using (d_2) in the definition of rd b-metric space we have

$$d(x, y) \leq k[d(x, x_n) + d(y, x_n)].$$

Taking limit $n \to \infty$, since x and y are limits of a convergent sequence $\{x_n\}$ therefore $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y, x_n) = 0$ which implies that $d(x, y) = 0$. Similarly we can show that $d(y, x) = 0$ so by (d_1) in the definition of rd b-metric space $x = y$. Hence limit in rd b-metric space is unique.
Lemma 4. Let (X, d) be a rd b-metric space and $\{x_n\}$ be a sequence in rd b-metric space such that

$$d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n) \tag{2}$$

for $n = 1, 2, 3, ...$ and $0 \leq \alpha k < 1$ where $\alpha \in [0, 1)$ and k is defined in rd b-metric space. Then $\{x_n\}$ is a Cauchy sequence in X.
Proof. Let for $n, m \in N$ and $m > n$. Applying (d_2) to triplets we have (x_n, x_{n+1}, x_m), (x_{n+1}, x_{n+2}, x_m), ..., we obtain

$$\begin{aligned} d(x_n, x_m) &\leq k[d(x_n, x_{n+1}) + d(x_m, x_{n+1}).................] \\ &\leq kd(x_n, x_{n+1}) + k^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}).........] \\ &\leq kd(x_n, x_{n+1}) + k^2 d(x_{n+1}, x_{n+2}) + k^3 d(x_{n+2}, x_{n+3})........... \end{aligned}$$

Now using (2) we get the following

$$\begin{aligned} d(x_n, x_m) &\leq k\alpha^n d(x_0, x_1) + k^2\alpha^{n+1} d(x_0, x_1) + k^3\alpha^{n+2} d(x_0, x_1) + \dots\dots\dots\dots \\ &\leq (1 + k\alpha + (k\alpha)^2 + \dots\dots\dots) k\alpha^n d(x_0, x_1). \\ &= k\alpha^n \left(\frac{1}{1 - k\alpha}\right) d(x_0, x_1). \end{aligned}$$

Taking limit $m, n \to \infty$ we have

$$\lim_{m,n\to\infty} d(x_n, x_m) = 0.$$

Hence $\{x_n\}$ is a right Cauchy sequence in rd b-metric space X. Similarly we can show that $\{x_n\}$ is a left Cauchy sequence. Thus $\{x_n\}$ is a Cauchy sequence in X.

Lemma 5. Let (X, d) be a rd b-metric space. If $T : X \to X$ is a contraction. Then T is rd b-continuous.

Proof. Let $x_0 \in X$ be arbitrary element in rd b-metric space X. Let for $\epsilon > 0$ and put $\delta \leq \epsilon$. Then

$$d(x_0, x) < \delta \ \Rightarrow \ d(Tx_0, Tx) \leq \alpha d(x_0, x) < \alpha\delta \leq \alpha\epsilon < \epsilon.$$

Thus

$$d(Tx_0, Tx) < \epsilon.$$

Theorem 1. Let (X, d) be a complete rd b-metric space. If $T : X \to X$ is a contraction. Then T has a unique fixed point.

Proof. Using Lemma 4, we obtain that $\{x_n\}$ is a Cauchy sequence in complete rd b-metric space. So there must exists $x \in X$ such that

$$\lim_{n\to\infty} x_n = x.$$

Now by Lemma 2 and 5, we have

$$Tx = T \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = x.$$

Thus x is a fixed point of T.

For uniqueness we assume that $x \neq y$ be two fixed points of T. So

$$d(x, y) = d(Tx, Ty) \leq \alpha d(x, y)$$

which is a contradiction because $\alpha \in [0, 1)$. Therefore $d(x, y) = 0$. Similarly, we can show that $d(y, x) = 0$. So by (d_1) $x = y$. Thus fixed point of T is unique.

Now, we introduce an application of Theorem 1 to T^n where $n \in N$ as follows.

Theorem 2. Let (X, d) be a complete rd b-metric space. If $T : X \to X$ satisfies that T^n is contraction for some $n \in N$. Then T has a unique fixed point.

Proof. For any $n \in N$, since T^n is a contraction so by Theorem 1 T^n has a unique fixed point say $x \in X$. Then

$$Tx = T(T^n x) = T^n(Tx).$$

So Tx is the fixed point of T^n. Since fixed point is unique thus $Tx = x$. Therefor, T has a fixed point in X.

For uniqueness we assume that $x \neq y$ be two fixed points of T. So

$$d(x, y) = d(T^n x, T^n y) \leq \alpha d(x, y)$$

which is a contradiction because $\alpha \in [0, 1)$. Therefore $d(x, y) = 0$. Similarly, we can show that $d(y, x) = 0$. So by (d_1) $x = y$. Thus fixed point of T is unique.

In Theorem 1 if the mapping T is replaced by T_α, $\alpha \in A$ where A is an indexed

family set. Then we state another application of Theorem 1 without any proof.
Theorem 3. Let (X, d) be a complete rd b-metric space. If $T_\alpha : X \to X$ satisfies that T_α^n is contraction for some $n \in N$ and $\alpha \in A$. Then T_α has a unique fixed point for each $\alpha \in A$.
Theorem 4. Let (X, d) be a complete rd b-metric space with $k \geq 1$. Let $T : X \to X$ be a continuous self mapping for $\delta \in [0, 1)$ with $\delta(1 + k) < 1$ and satisfying the condition

$$d(Tx, Ty) \leq \delta[d(x, Tx) + d(y, Ty)] \tag{3}$$

$\forall\ x, y \in X$. Then T has a unique fixed point.
Proof. Let x_0 be arbitrary in X we define a sequence $\{x_n\}$ in X by the rule

$$x_0, x_1 = Tx_0, x_2 = Tx_1, ..., x_{n+1} = Tx_n.$$

Consider

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n).$$

By using condition (3) we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \delta[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)]$$
$$d(x_n, x_{n+1}) \leq \delta[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$$
$$d(x_n, x_{n+1}) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n).$$

Since $\delta(1 + k) < 1$ therefore, take $h = \frac{\delta}{1-\delta} < \frac{1}{k}$. Hence

$$d(x_n, x_{n+1}) \leq h d(x_{n-1}, x_n).$$

So by Lemma 4 we can say that $\{x_n\}$ is a Cauchy sequence in complete rd b-metric space X. So there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$ also since T is continuous so

$$\lim_{n\to\infty} Tx_n = Tz \ \Rightarrow \ Tz = z.$$

Thus z is the fixed point of T in X.

Now it is an easy job to show that the fixed point z of T is unique.

Conclusion.
In this work two new concepts of distance spaces are introduced and some properties of these new spaces are investigated. Moreover, fixed point theorems are established which generalize the results of [1], [3],[4] and [10] in the setting of rd b-metric as well as in ld b-metric space.

Competing Interest.
The authors declare that they have no competing interests.

Authors Contributions.
Both the authors contributed equally in this article and approved the final version.

Acknowledgement.
The authors are grateful to the editor in chief and anonymous reviewers for their valuable comments and remarks to improve this manuscript.

References

[1] P. Hitzler and A.K. Seda, Dislocated Topologies. *J. Electr. Engin.*, **51**(2000), 3-7.
[2] S.G. Mathews, Metrics Domain for Completeness, Ph.D Thesis, Department of Computer Science, University of Warwick, U.K, (1986), 1-127.
[3] F.M. Zeyada, M.A. Ahmed and G.F. Hassan, A Generalization of Fixed Point Theorem Due to Hitzler and Seda in Dislocated Quasi Metric Space, *Arabian J. Sci. Engg.*, **31**(2005), 111-114.

[4] M.A. Ahmed, F.M. Zeyada and G.F. Hassan, Fixed Point Theorems in Generalized Types of Dislocated Metric Spaces and Its Applications, *Thai Journal of Mathematics*, **11**(2013), 67-73.
[5] M.U. Rahman and M. Sarwar, Some Fixed Point Theorems in Generalized Types of Metric Spaces, *Electronic Journal of Mathematical Analysis and Applications*, **3**(2015), 289-296.
[6] S. Czerwik, Contraction Mappings in b-Metric Spaces, *Acta Math. Inform. Univ. Ostraviensis*, **1**(1993), 5-11.
[7] S. Shukla, Partial b-Metric Spaces and Fixed Point Theorems, *Mediterr. J. Math.*, June(2013), 1-9.
[8] N. Hussain, J.R. Roshan, V. Parvaneh, M. Abbas, Common Fixed Point Results for Weak Contractive Mappings in Ordered b-Dislocated Metric Spaces With Applications, *Journal of Inequalities and Applications*, (2013), **486**:2013.
[9] M.U. Rahman and M. Sarwar, Dislocated Quasi b-Metric Space and Fixed Point Theorems, *Electronic Journal of Mathematical Analysis and Applications*, **4**(2016).
[10] M. Kir and H. Kizitune, On Some Well-Known Fixed Point Theorems in b-Metric Space, *Turkish Journal of Analysis and Number Theory*, **1**(2013), 13-16.
[11] R. Kannan, Some Results on Fixed Points, *Bull. Cal. Math. Soc. India*, **60**(1968), 71-76.
[12] M. Sarwar and M.U. Rahman, Fixed Point Theorems for Ciric's and Generalized Contractions in b-Metric Spaces, *International Journal of Analysis and Applications*, **7**(2015), 70-78.

Mujeeb Ur Rahman
Department of Mathematics, Government PG Jahanzeb College Saidu Sharief Swat, Khyber PakhtunKhwa, Pakistan.
E-mail address: mujeeb846@yahoo.com

Muhammad Sarwar
Department of Mathematics, University of Malakand, Dir(L), Khyber PakhtunKhwa, Pakistan.
E-mail address: sarwarswati@gmail.com