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MULTIPLE POSITIVE SOLUTIONS FOR FUNCTIONAL DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. This paper is concerned with the existence of multiple positive solutions for a functional dynamic equations with multi-point boundary conditions on time scales by using fixed point theorems in a cone. As an application, we also give an example to demonstrate our results.

1. Introduction

The theory of dynamic equations on time scales has become important mathematical branch [2, 3, 12] since it was initiated by Hilger [14]. The study of time scales theory has led to many important applications, for example, in the study of insect population models, neural networks, heat transfer, quantum mechanics, epidemic, crop harvest and stock market [5, 6, 15, 16, 24]. Boundary-value problems for scalar dynamic equations on time scales have received considerable attention [4, 19, 20]. Recently, existence and multiplicity of solutions for boundary value problems of dynamic equations have been of great interest in mathematics and its applications to engineering sciences [1, 7, 10, 18, 26]. But very little work has been done to the existence of positive solutions for functional dynamic equations on time scales [17, 22, 23, 25]. In particular, we would like to mention some results of Kaufmann and Raffoul [17] and Tang, Sun and Chen [23] which motivate us to consider our problem.

In [17], authors studied the existence of at least one positive solution to the nonlocal eigenvalue problem for a class of nonlinear functional dynamic equations on time scales

$$u^{\triangle\nabla}(t) + \lambda a(t) f(u(t), u(\theta(t))) = 0, \quad t \in (0, T),$$
$$u(s) = \psi(s) \quad s \in [-r, 0], \quad u(0) = 0, \quad \alpha u(\eta) = u(T).$$

In [23], authors discussed the existence of single and multiple positive solutions of the boundary value problems for a p-Laplacian functional dynamic equations on

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time scales

$$(\varphi_p(u^{\triangle}(t)))^{\nabla} + h(t)f(u(t), u(\theta(t))) = 0, \quad t \in (0, T),$$
$$u(t) = \psi(t) \quad s \in [-r, 0], \quad u(0) - \beta u^{\triangle}(0) = \gamma u^{\triangle}(\eta) \quad u^{\triangle}(T) = 0.$$

In [21], authors considered the existence of positive solutions of the boundary value problems for the following second order multipoint boundary value problem on time scales

$$u^{\triangle\nabla}(t) + f(t, u(t)) = 0, \quad t \in [0, 1],$$
$$\beta u(0) - \gamma u^{\triangle}(0) = 0 \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \quad m \geq 3.$$

Motivated by those works and the references therein, in this paper we shall consider the following functional multi point problem on time scales:

$$u^{\triangle\triangle}(t) + f(u(t), u(\theta_1(t)), u(\theta_2(t))) = 0, \ t \in [0, T] \tag{1}$$
$$u(s) = \varphi_1(s), \ s \in [-r, 0], \ u(s) = \varphi_2(s), \ s \in [T, p],$$
$$\alpha u(0) - \beta u^{\triangle}(0) = 0, \ \delta u(T) + \gamma u^{\triangle}(T) = \sum_{i=1}^{m-2} a_i u(\xi_i), \tag{2}$$

where $-r, 0, T, p \in \mathbb{T}$ and an closed interval $[0, T]$ is defined by $[0, T] = \{t \in \mathbb{T} : 0 \leq t \leq T\}$. Other types of intervals are defined similarly. Some preliminary definitions and theorems on time scales can be found in the books [8, 9].

In this paper, we study more general problem and some new results are obtained for the existence of at least one, three and four positive solutions for the above problem by using cone theory techniques [7, 13]. The results are even new for the special cases of differential equations and difference equations, as well as in the general time scale setting.

The plan of this paper is as follows. In Section 2, we provide some necessary backgrounds. In particular, we construct the Green's function of the linear boundary value problem and develop upper and lower bounds on the Green's function. In Section 3, we establish the main results of the paper. Finally, one example is also included to illustrate the main results.

2. The Preliminary Lemmas

Throughout the paper we assume that the following conditions are satisfied:

(H_1) $\alpha, \beta, \gamma \geq 0$, $\delta > 0$, $0 < \beta + \alpha \leq 1$, $0 < \xi_1 < \xi_2 < ... < \xi_{m-2} < T$,
(H_2) $D = \alpha(\delta T + \gamma - \sum_{i=1}^{m-2} a_i \xi_i) + \beta(\delta - \sum_{i=1}^{m-2} a_i) > 0$, $0 < \sum_{i=1}^{m-2} a_i \xi_i < T$, $\sum_{i=1}^{m-2} a_i < \delta$ with $a_i \in (0, \infty)$,
(H_3) $f : [0, \infty) \times [0, \infty) \times [0, \infty) \to [0, \infty)$ is continuous,
(H_4) $\varphi_1 : [-r, 0] \to [0, \infty)$, $\varphi_2 : [T, p] \to [0, \infty)$ are continuous where $r > 0$ and $p > T$,
(H_5) $\theta_1 : [0, T] \to [-r, T]$, $\theta_2 : [0, T] \to [0, p]$ are continuous and nondecreasing with $\theta_1(0) < 0$, $\theta_1(T) > 0$ and $\theta_2(T) > T$,
(H_6) $\upsilon = \sup\{t \in [0, T] : \theta_1(t) \leq 0\}$, $\mu = \sup\{t \in [0, T] : \theta_2(t) \leq T\}$ and $\upsilon < \mu$.

Remark Let $\mathbb{T} = \mathbb{R}$. If the assumption (H_5) satisfies, then (H_6) satisfies.

To prove the main results in this paper, we will employ several lemmas. These lemmas are based on the linear boundary value problem

$$u^{\triangle\triangle}(t)+y(t)=0,\ t\in[0,T] \tag{3}$$

$$\alpha u(0)-\beta u^{\triangle}(0)=0,\ \delta u(T)+\gamma u^{\triangle}(T)=\sum_{i=1}^{m-2}a_iu(\xi_i) \tag{4}$$

Lemma 1 Let $D=\alpha(\delta T+\gamma-\sum_{i=1}^{m-2}a_i\xi_i)+\beta(\delta-\sum_{i=1}^{m-2}a_i)\neq 0$ and $0<\xi_1<\xi_2<...<\xi_{m-2}<T$, then for $y\in C([0,T])$, the boundary value problem $(3)-(4)$ has the unique solution

$$u(t)=\frac{\beta+\alpha t}{D}\int_0^T(\delta T-\delta s+\gamma)y(s)\Delta s-\frac{\beta+\alpha t}{D}\sum_{i=1}^{m-2}a_i\int_0^{\xi_i}(\xi_i-s)y(s)\Delta s$$
$$-\int_0^t(t-s)y(s)\Delta s.$$

Lemma 2 Suppose $D=\alpha(\delta T+\gamma-\sum_{i=1}^{m-2}a_i\xi_i)+\beta(\delta-\sum_{i=1}^{m-2}a_i)\neq 0$ and $0<\xi_1<\xi_2<...<\xi_{m-2}<T$, then the Green's function for the boundary value problem $(3)-(4)$ is given by

$$G(t,s)=\begin{cases} G_1(t,s), & \xi_0\le s\le\xi_1, (\xi_0=0),\\ G_2(t,s), & \xi_1\le s\le \xi_2,\\ . & \\ . & \\ . & \\ G_{m-2}(t,s), & \xi_{m-3}\le s\le\xi_{m-2},\\ H(t,s) & \xi_{m-2}\le s\le T,\end{cases}$$

where for all $i=1,2,...,m-2$,

$$G_i(t,s)=\begin{cases}\frac{(\beta+\alpha t)(\delta(T-s)+\gamma-\sum_{j=i}^{m-2}a_j(\xi_j-s))}{D}-(t-s), & s\le t,\\ \frac{(\beta+\alpha t)(\delta(T-s)+\gamma-\sum_{j=i}^{m-2}a_j(\xi_j-s))}{D}, & t\le s,\end{cases}$$

and

$$H(t,s)=\begin{cases}\frac{(\beta+\alpha t)(\delta(T-s)+\gamma)}{D}-(t-s), & s\le t,\\ \frac{(\beta+\alpha t)(\delta(T-s)+\gamma)}{D}, & t\le s.\end{cases}$$

Using the above Green's function, the solution of the problem $(3)-(4)$ is expressed as

$$u(t)=\int_0^T G(t,s)y(s)\Delta s.$$

Lemma 3 Assume that the conditions $(H_1)-(H_2)$ are satisfied. Then
i) $G(t,s)\ge 0$ for all $t,s\in[0,T]$,

ii) There exist a number $\eta\in(0,1)$ and a continuous function $\phi:[0,T]\to(0,\infty)$ such that $G(t,s)\le\phi(s)$ and $G(t,s)\ge\eta\phi(s)$ for all $t,s\in[0,T]$, where $\phi(s)=\frac{(\beta+\alpha s)(\delta T+\gamma)}{D}$ and $\eta=\frac{\beta\gamma}{\delta T+\gamma}$.

Proof. *i)* Since $D>0$, we can easily get that $G(t,s)\ge 0$ for all $t,s\in[0,T]$.

ii) Now, we will find an upper and a lower bound for the function $G(t,s)$ for all $t,s\in[0,T]$.

Upper bound:

Case 1. Consider $\xi_{i-1} \le s \le \xi_i$ $(i = 1, 2, 3, ..., m-2)$, $s \le t$. Then

$$\begin{aligned}
G(t,s) &= \tfrac{(\beta+\alpha t)(\delta(T-s)+\gamma-\sum_{j=i}^{m-2} a_j(\xi_j - s))}{D} - (t-s) \\
&= \tfrac{(\beta+\alpha s)(\delta(T-t)+\gamma)+\sum_{j=1}^{i-1} a_j(\beta+\alpha\xi_j)(t-s)+\sum_{j=i}^{m-2} a_j(t-\xi_j)(\beta+\alpha s)}{D} \\
&\le \tfrac{(\beta+\alpha s)(\delta(T-t)+\gamma)+\sum_{j=1}^{i-1} a_j(\beta+\alpha s)(t-s)+\sum_{j=i}^{m-2} a_j(t-\xi_j)(\beta+\alpha s)}{D} \\
&= \tfrac{(\beta+\alpha s)(\delta T-\delta t+\gamma+\sum_{j=1}^{i-1} a_j t+\sum_{j=i}^{m-2} a_j t - s\sum_{j=1}^{i-1} a_j - \sum_{j=i}^{m-2} a_j\xi_j)}{D} \\
&\le \tfrac{(\beta+\alpha s)(\delta T+\gamma+t(\sum_{j=1}^{m-2} a_j-\delta))}{D} \\
&\le \tfrac{(\beta+\alpha s)(\delta T+\gamma)}{D} = \phi(s).
\end{aligned}$$

Case 2. For $\xi_{i-1} \le s \le \xi_i$ $(i = 1, 2, 3, ..., m-2)$, $s \ge t$, we have

$$\begin{aligned}
G(t,s) &= \tfrac{(\beta+\alpha t)(\delta(T-s)+\gamma-\sum_{j=i}^{m-2} a_j(\xi_j - s))}{D} \\
&\le \tfrac{(\beta+\alpha t)(\delta(T-s)+\gamma)}{D} \\
&\le \tfrac{(\beta+\alpha s)(\delta T+\gamma)}{D} = \phi(s).
\end{aligned}$$

Case 3. For $\xi_{m-2} \le s \le T$, $s \le t$, we obtain

$$\begin{aligned}
G(t,s) &= \tfrac{(\beta+\alpha t)(\delta(T-s)+\gamma)}{D} - (t-s) \\
&= \tfrac{(\beta+\alpha s)(\delta T+\gamma-s\sum_{j=1}^{m-2} a_j+t(\sum_{j=1}^{m-2} a_j-\delta))}{D} \\
&\le \tfrac{(\beta+\alpha s)(\delta T+\gamma)}{D} = \phi(s).
\end{aligned}$$

Case 4. For $\xi_{m-2} \le s \le T$, $s \ge t$, we clearly have

$$G(t,s) \le \tfrac{(\beta+\alpha s)(\delta T+\gamma)}{D} = \phi(s).$$

Lower bound:

Case 1. For $\xi_{i-1} \le s \le \xi_i$ $(i = 1, 2, 3, ..., m-2)$, $s \le t$, we get

$$\begin{aligned}
G(t,s) &= \tfrac{(\beta+\alpha t)(\delta(T-s)+\gamma-\sum_{j=i}^{m-2} a_j(\xi_j - s))}{D} - (t-s) \\
&= \tfrac{(\beta+\alpha s)(\delta(T-t)+\gamma)+\sum_{j=1}^{i-1} a_j(\beta+\alpha\xi_j)(t-s)+\sum_{j=i}^{m-2} a_j(t-\xi_j)(\beta+\alpha s)}{D} \\
&\ge \tfrac{(\beta+\alpha s)(\delta(T-t)+\gamma)+\sum_{j=i}^{m-2} a_j(t-\xi_j)(\beta+\alpha s)}{D} \\
&= \tfrac{(\beta+\alpha s)(\delta(T-t)+\gamma+\sum_{j=i}^{m-2} a_j(t-\xi_j))}{D} \\
&= \tfrac{(\beta+\alpha s)(\delta T+\gamma+(\sum_{j=i}^{m-2} a_j-\delta)t-\sum_{j=i}^{m-2} a_j\xi_j)}{D} \\
&\ge \tfrac{(\beta+\alpha s)(\delta T+\gamma+(\sum_{j=i}^{m-2} a_j-\delta)\xi_{m-2}-\sum_{j=i}^{m-2} a_j\xi_{m-2})}{D} \\
&= \tfrac{(\beta+\alpha s)(\delta T+\gamma)(\delta(T-\xi_{m-2})+\gamma)}{D(\delta T+\gamma)} \\
&\ge \eta\phi(s).
\end{aligned}$$

Case 2. For $\xi_{i-1} \le s \le \xi_i$ $(i = 1, 2, 3, ..., m-2)$, $s \ge t$, we get

$$G(t,s) = \tfrac{(\beta+\alpha t)(\delta(T-s)+\gamma-\sum_{j=i}^{m-2} a_j(\xi_j - s))}{D}$$

$$= \tfrac{(\beta+\alpha t)(\delta T+\gamma+(\sum_{j=i}^{m-2} a_j-\delta)s-\sum_{j=i}^{m-2} a_j\xi_j)}{D}$$
$$\geq \tfrac{(\beta+\alpha t)(\delta T+\gamma+(\sum_{j=i}^{m-2} a_j-\delta)\xi_{m-2}-\sum_{j=i}^{m-2} a_j\xi_{m-2})}{D}$$
$$\geq \tfrac{(\beta+\alpha s)(\delta T+\gamma)\beta(\delta(T-\xi_{m-2})+\gamma)}{D(\delta T+\gamma)}$$
$$\geq \eta\phi(s).$$

Case 3. For $\xi_{m-2} \leq s \leq T$, $s \leq t$, we obtain

$$G(t,s) = \tfrac{(\beta+\alpha t)(\delta(T-s)+\gamma)}{D} - (t-s)$$
$$= \tfrac{(\beta+\alpha s)(\delta(T-t)+\gamma)+\sum_{j=1}^{m-2} a_j(\beta+\alpha\xi_j)(t-s)}{D}$$
$$\geq \tfrac{(\beta+\alpha s)(\delta T+\gamma)\gamma}{D(\delta T+\gamma)}$$
$$\geq \eta\phi(s).$$

Case 4. For $\xi_{m-2} \leq s \leq T$, $s \geq t$, we clearly have

$$G(t,s) = \tfrac{(\beta+\alpha t)(\delta(T-s)+\gamma)}{D}$$
$$\geq \tfrac{(\beta+\alpha s)(\delta T+\gamma)\beta\gamma}{D(\delta T+\gamma)}$$
$$\geq \eta\phi(s).$$

Lemma 4 Let the conditions $(H_1)-(H_2)$ be satisfied; then for $y \in C([0,T],[0,\infty))$, the solution of the boundary value problem $(3)-(4)$ satisfies $u(t) \geq \eta\|u\|$, $t \in [0,T]$.
Proof. By using Lemma 3, we get

$$u(t) = \int_0^T G(t,s)y(s)\Delta s \leq \int_0^T \phi(s)y(s)\Delta s, \quad t \in [0,T],$$

and so

$$\|u\| \leq \int_0^T \phi(s)y(s)\Delta s,$$

Now, by using Lemma 3 again , we obtain for $t \in [0,T]$,

$$u(t) = \int_0^T G(t,s)y(s)\Delta s \geq \eta \int_0^T \phi(s)y(s)\Delta s \geq \eta\|u\|.$$

This completes the proof.

The following two theorems are crucial in our arguments.

Theorem 1 Let $E = (E, \|.\|)$ be a Banach space, and let $P \subset E$ be a cone in E. Assume Ω_1, Ω_2 are bounded open subsets of E with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$, and let

$$A : P \cap (\overline{\Omega_2}\backslash\Omega_1) \to P$$

be a completely continuous operator such that, either
(a) $\|Au\| \leq \|u\|, u \in P \cap \partial\Omega_1$ and $\|Au\| \geq \|u\|, u \in P \cap \partial\Omega_2$ or
(b) $\|Au\| \geq \|u\|, u \in P \cap \partial\Omega_1$ and $\|Au\| \leq \|u\|, u \in P \cap \partial\Omega_2$.
Then A has a fixed point in $P \cap (\overline{\Omega_2}\backslash\Omega_1)$.

Theorem 2 Let P be a cone in the real Banach space E, $A : \overline{P_c} \to \overline{P_c}$ be completely continuous and ψ be a nonnegative continuous concave functional on P with $\psi(u) \leq \|u\|$ for all $u \in \overline{P_c}$. Suppose there exist $0 < d < a < b \leq c$ such that the following conditions hold:

(i) $\{u \in P(\psi, a, b) : \psi(u) > a\} \neq \emptyset$ and $\psi(Au) > a$ for all $u \in P(\psi, a, b)$;
(ii) $\|Au\| < d$ for $u \in \overline{P}_d$;
(iii) $\psi(Au) > a$ for $u \in P(\psi, a, c)$ with $\|Au\| > b$.

Then A has at least three fixed points u_1, u_2 and u_3 satisfying

$$\|u_1\| < d, \quad \psi(u_2) > a, \quad d < \|u_3\| \text{ with } \psi(u_3) < a,$$

where $P_c = \{u \in P : \|u\| < c\}$ and $P(\psi, a, b) = \{u \in P : a \leq \psi(u),\ \|u\| \leq b\}$.

3. Main Results

In this section, we present sufficient conditions for the existence of the positive solutions of our problem. Firstly, we prove the existence of at least one positive solution by applying Theorem 1. Secondly, we use Theorem 2 to prove the existence of at least three positive solutions. Finally, we obtain that there exist at least four positive solutions of our problem.

We note that $u(t)$ is a solution of $(1)-(2)$ if and only if

$$u(t) = \begin{cases} \varphi_1(t), & t \in [-r, 0], \\ \int_0^T G(t,s) f(u(s), u(\theta_1(s)), u(\theta_2(s)))\Delta s, & t \in [0, T], \\ \varphi_2(t), & t \in [T, p]. \end{cases}$$

Let E denote the Banach space $C([0,T])$ with the norm $\|u\| = \max_{t\in[0,T]} |u(t)|$. Define the cone $P \subset E$ by $P = \{u \in E : u(t) \geq \eta\|u\|,\ \forall t \in [0,T]\}$.

For each $u \in E$, extend $u(t)$ to $[-r, T]$ with $u(t) = \varphi_1(t)$ for $t \in [-r, 0]$ and extend $u(t)$ to $[0, p]$ with $u(t) = \varphi_2(t)$ for $t \in [T, p]$.

Define an operator $A : P \to E$ by

$$Au(t) = \int_0^T G(t,s) f(u(s), u(\theta_1(s)), u(\theta_2(s)))\Delta s, \quad \text{for } t \in [0, T].$$

Let u_1 be a fixed point of A in the cone P. Define

$$u(t) = \begin{cases} \varphi_1(t), & t \in [-r, 0], \\ u_1(t), & t \in [0, T], \\ \varphi_2(t), & t \in [T, p]. \end{cases}$$

Then, u is a positive solution of the problem $(1)-(2)$.

Fix $\upsilon \in \mathbb{T}$ which is defined in (H_6) and define the following sets

$$Y_1 = \{t \in [0,T] : \theta_1(t) \leq 0,\ \theta_2(t) \leq T\},$$
$$Y_2 = \{t \in [0,T] : \theta_1(t) > 0,\ \theta_2(t) \geq T\},$$
$$Y_3 = \{t \in [0,T] : \theta_1(t) > 0,\ \theta_2(t) < T\}.$$

It is obvious that the sets are pairwise disjoint and $Y_1 \bigcup Y_2 \bigcup Y_3 = [0,T]$.

For notational convenience, we denote m, k and M by

$$m = \eta^2 \int_{Y_1} \phi(s)\Delta s, \quad k = \eta \int_{Y_1} \phi(s)\Delta s \quad \text{and} \quad M = \int_0^T \phi(s)\Delta s.$$

Theorem 3 Suppose that the assumptions $(H_1)-(H_6)$ hold and f satisfies the following conditions:

(A_1) $$\lim_{u_1,u_3\to 0^+} \frac{f(u_1, \varphi_1(s), u_3)}{u_1} \leq 1/M, \text{ uniformly in } s \in [-r, 0],$$

$$\lim_{u_1,u_2\to 0^+} \frac{f(u_1, u_2, \varphi_2(s))}{u_1} \leq 1/M, \text{ uniformly in } s \in [T, p]$$

and
$$\lim_{u_1\to 0^+} \frac{f(u_1,u_1(s),u_1(s))}{u_1} \le 1/M, \text{ uniformly in } s\in[0,T].$$

(A_2) $$\lim_{u_1,u_3\to+\infty} \frac{f(u_1,\varphi_1(s),u_3)}{u_1} \ge 1/m, \text{ uniformly in } s\in[-r,0].$$

Then the problem $(1)-(2)$ has at least one positive solution.

Proof. We use Theorem 1 to prove that A has a fixed point in our cone P. First, it is obvious that A is completely continuous and $A(P)\subseteq P$.

By the condition (A_1), there exists an $N_1>0$ such that if $0<u_1<N_1$ and $0<u_3<N_1$, then

$$f(u_1,\varphi_1(s),u_3) \le \frac{1}{M}u_1,\ \ for\ s\in[-r,0], \tag{5}$$

and there exists an $N_2>0$ such that if $0<u_1<N_2$ and $0<u_2<N_2$, then

$$f(u_1,u_2,\varphi_2(s)) \le \frac{1}{M}u_1,\ \ for\ s\in[T,p]. \tag{6}$$

and similarly, there exists an $N_3>0$ such that if $0<u_1<N_3$, then

$$f(u_1,u_1(s),u_1(s)) \le \frac{1}{M}u_1,\ \ for\ s\in[0,T]. \tag{7}$$

Let $r=\min\{N_1,N_2,N_3\}$ and $\Omega_1=\{u\in E:\|u\|<r\}$. We shall prove that $\|Au\|\le\|u\|$ for $u\in P\cap\partial\Omega_1$. Let $u\in P\cap\partial\Omega_1$. Then, for all $t\in[0,T]$, we have $0\le u(t)\le r$. Thus, by $(5),(6),(7)$ and Lemma 3, for $t\in[0,T]$, we find

$$\begin{aligned}
Au(t) &= \textstyle\int_0^T G(t,s)f(u(s),u(\theta_1(s)),u(\theta_2(s)))\Delta s\\
&= \textstyle\int_{Y_1} G(t,s)f(u(s),\varphi_1(\theta_1(s)),u(\theta_2(s)))\Delta s\\
&\quad + \textstyle\int_{Y_2} G(t,s)f(u(s),u(\theta_1(s)),\varphi_2(\theta_2(s)))\Delta s\\
&\quad + \textstyle\int_{Y_3} G(t,s)f(u(s),u(\theta_1(s)),u(\theta_2(s)))\Delta s\\
&\le \textstyle\int_{Y_1}\phi(s)\frac{1}{M}u(s)\Delta s + \int_{Y_2}\phi(s)\frac{1}{M}u(s)\Delta s + \int_{Y_3}\phi(s)\frac{1}{M}u(s)\Delta s\\
&= \textstyle\frac{1}{M}\int_0^T \phi(s)u(s)\Delta s\\
&\le \textstyle\frac{1}{M}\int_0^T \phi(s)\|u\|\Delta s = \|u\|.
\end{aligned}$$

Therefore, we get $\|Au\|\le\|u\|$ for $u\in P\cap\partial\Omega_1$.

In view of (A_2), for all $u_1,u_3\ge N$, there exists $N>0$ such that

$$f(u_1,\varphi_1(s),u_3) \ge \frac{1}{m}u_1,\ \ for\ s\in[-r,0] \tag{8}$$

Now, set

$$R = r+\frac{1}{\eta}N. \tag{9}$$

Let $\Omega_2=\{u\in E:\|u\|<R\}$. We shall prove that $\|Au\|\ge\|u\|$ for $u\in P\cap\partial\Omega_2$. Let $u\in P\cap\partial\Omega_2$, then $\|u\|=R$. So from (9) and the fact that $u\in P$, we get

$$u(t) \geq \eta\|u\| \geq N, \;\; for \; t \in [0,T]. \tag{10}$$

Considering (8) and (10), we obtain

$$\begin{aligned} Au(t) &= \int_0^T G(t,s) f(u(s), u(\theta_1(s)), u(\theta_2(s)))\Delta s \\ &\geq \int_0^T \eta\phi(s) f(u(s), u(\theta_1(s)), u(\theta_2(s)))\Delta s \\ &\geq \int_{Y_1} \eta\phi(s) f(u(s), \varphi_1(\theta_1(s)), u(\theta_2(s)))\Delta s \\ &\geq \int_{Y_1} \eta\phi(s) \tfrac{1}{m} u(s)\Delta s \\ &\geq \tfrac{\eta}{m} \int_{Y_1} \phi(s)\eta\|u\|\Delta s \end{aligned}$$

and so we obtain

$$\|Au\| \geq \|u\| \tfrac{\eta^2}{m} \int_{Y_1} \phi(s)\Delta s = \|u\|.$$

Therefore, we get $\|Au\| \geq \|u\|$ for $u \in P \cap \partial\Omega_2$.

Then, it follows from Theorem 1 that A has a fixed point u_1 such that $r \leq \|u_1\| \leq R$. It is clear that u is a positive solution of $(1)-(2)$ with the form

$$u(t) = \begin{cases} \varphi_1(t), & t \in [-r,0], \\ u_1(t), & t \in [0,T], \\ \varphi_2(t), & t \in [T,p]. \end{cases}$$

The proof is complete.

Theorem 4 Suppose that the assumptions $(H_1)-(H_6)$, (A_1) hold and f satisfies the following condition:

(A_2') $\displaystyle\lim_{u_1,u_3\to+\infty} \frac{f(u_1,\varphi_1(s),u_3)}{u_1} \geq 1/m_1$, uniformly in $s \in [-r,0]$,
or
$\displaystyle\lim_{u_1,u_2\to+\infty} \frac{f(u_1,u_2,\varphi_2(s))}{u_1} \geq 1/m_2$, uniformly in $s \in [T,p]$,

where $m_i = \eta^2 \int_{Y_i} \phi(s)\Delta s, \quad i = 1,2.$

Then the problem $(1)-(2)$ has at least one positive solution.
Proof. The proof is similar to the proof of the Theorem 3.

In order to establish existence criteria of at least three positive solutions of the problem $(1)-(2)$, we define a nonnegative continuous concave functional on P by

$$\psi(u) = \min_{t\in[0,T]} u(t).$$

Theorem 5 Assume that the assumptions $(H_1)-(H_6)$ are satisfied. Let

$$0 < d < a < \frac{a}{\eta} \leq c$$

and suppose that f satisfies the following conditions:

(C_1) $\quad f(u_1,\varphi_1(s),u_3) > \dfrac{a}{k}$, for $a \leq u_1, u_3 \leq \dfrac{a}{\eta}$, uniformly in $s \in [-r,0]$.

(C_2) $\quad f(u_1,\varphi_1(s),u_3) < \dfrac{d}{M}$, for $0 \leq u_1, u_3 \leq d$, uniformly in $s \in [-r,0]$,

$f(u_1,u_2,\varphi_2(s)) < \dfrac{d}{M}$, for $0 \leq u_i \leq d$, $i = 1,2$, uniformly in $s \in [T,p]$,

$$f(u_1,u_2,u_3) < \frac{d}{M}, \text{ for } 0 \le u_i \le d,\ i=1,2,3.$$

(C_3) $f(u_1,\varphi_1(s),u_3) \le \dfrac{c}{M}$, for $0 \le u_1, u_3 \le c$, uniformly in $s \in [-r,0]$,

$$f(u_1,u_2,\varphi_2(s)) \le \frac{c}{M}, \text{ for } 0 \le u_i \le c,\ i=1,2, \text{ uniformly in } s \in [T,p],$$

$$f(u_1,u_2,u_3) \le \frac{c}{M}, \text{ for } 0 \le u_i \le c,\ i=1,2,3.$$

Then the problem $(1)-(2)$ has at least three positive solutions of the form

$$u(t) = \begin{cases} \varphi_1(t), & t \in [-r,0], \\ u_i(t), & t \in [0,T], \quad i=1,2,3, \\ \varphi_2(t), & t \in [T,p], \end{cases}$$

where $\|u_1\| < d, \quad \psi(u_2) > a, \quad d < \|u_3\|$ with $\psi(u_3) < a$.

Proof. First, we prove that $A : \overline{P_c} \to \overline{P_c}$. Let $u \in \overline{P_c}$. Then, we have $0 \le u(t) \le c$, $t \in [0,T]$. By condition (C_3), for $t \in [0,T]$, we obtain

$$\begin{aligned} Au(t) &= \textstyle\int_0^T G(t,s) f(u(s),u(\theta_1(s)),u(\theta_2(s)))\Delta s \\ &= \textstyle\int_{Y_1} G(t,s) f(u(s),\varphi_1(\theta_1(s)),u(\theta_2(s)))\Delta s \\ &\quad + \textstyle\int_{Y_2} G(t,s) f(u(s),u(\theta_1(s)),\varphi_2(\theta_2(s)))\Delta s \\ &\quad + \textstyle\int_{Y_3} G(t,s) f(u(s),u(\theta_1(s)),u(\theta_2(s)))\Delta s \\ &\le \textstyle\int_{Y_1} \phi(s) \frac{c}{M}\Delta s + \int_{Y_2} \phi(s)\frac{c}{M}\Delta s + \int_{Y_3}\phi(s)\frac{c}{M}\Delta s \\ &= \textstyle\frac{c}{M}\int_0^T \phi(s)\Delta s = c. \end{aligned}$$

Therefore, we get $\|Au\| \le c$. This implies $Au \in \overline{P_c}$ for $u \in \overline{P_c}$.

We now show that all the conditions of Theorem 2 are satisfied. By (C_2) and the argument above, we can get that $A : \overline{P_d} \to P_d$. Hence condition (ii) of Theorem 2 holds.

We now verify that (i) of Theorem 2 is fulfilled. We note that $u(t) = \frac{a}{\eta}$, $t \in [0,T]$ is a member of $P(\psi, a, \frac{a}{\eta})$ since $\psi(u) = \frac{a}{\eta} > a$. Therefore $P(\psi, a, \frac{a}{\eta}) \neq \emptyset$. Now let $u \in P(\psi, a, \frac{a}{\eta})$. Then, we have $a \le u(t) \le \frac{a}{\eta}$, $t \in [0,T]$. Combining this with (C_1), we get

$$f(u_1,\varphi_1(s),u_3) > \frac{a}{k}, \text{ for } a \le u_1,u_3 \le \frac{a}{\eta}, \text{ uniformly in } s \in [-r,0],$$

Thus,

$$\begin{aligned} \psi(Au) = \min_{t\in[0,T]} Au(t) &= \min_{t\in[0,T]} \textstyle\int_0^T G(t,s) f(u(s),u(\theta_1(s)),u(\theta_2(s)))\Delta s \\ &\ge \textstyle\int_0^T \eta\phi(s) f(u(s),u(\theta_1(s)),u(\theta_2(s)))\Delta s \\ &\ge \eta \textstyle\int_{Y_1} \phi(s) f(u(s),\varphi_1(\theta_1(s)),u(\theta_2(s)))\Delta s \\ &> \eta \textstyle\int_{Y_1} \phi(s)\frac{a}{k}\Delta s = a \end{aligned}$$

Then condition (i) of Theorem 2 is satisfied.

Finally, we show that (iii) of Theorem 2 is also satisfied. In fact, let $u \in P(\psi,a,c)$ with $\|Au\| > \frac{a}{\eta}$, we get

$$\psi(Au) = \min_{t\in[0,T]} Au(t) \ge \eta\|Au\| > \eta\tfrac{a}{\eta} = a,$$

that is to say condition (iii) of Theorem 2 holds.

Since all conditions of Theorem 2 are verified, the operator A has at least three fixed points satisfying

$$\|u_1\| < d, \quad \min_{t\in[0,T]} u_2(t) > a, \quad d < \|u_3\| \text{ with } \min_{t\in[0,T]} u_3(t) < a.$$

Now, let

$$u(t) = \begin{cases} \varphi_1(t), & t \in [-r, 0], \\ u_i(t), & t \in [0, T], \quad i = 1, 2, 3, \\ \varphi_2(t), & t \in [T, p], \end{cases}$$

which are three positive solutions of the problem $(1) - (2)$.

Theorem 6 Suppose that the assumptions $(H_1) - (H_6)$ hold. If there exist

$$0 < d < a < \frac{a}{\eta} \le c$$

such that the assumptions $(C_1) - (C_3)$ and (A_2) are satisfied.

Then the problem $(1) - (2)$ has at least four positive solutions of the form

$$u(t) = \begin{cases} \varphi_1(t), & t \in [-r, 0], \\ u_i(t), & t \in [0, T], \quad i = 1, 2, 3, 4, \\ \varphi_2(t), & t \in [T, p], \end{cases}$$

where $\|u_1\| < d, \quad \psi(u_2) > a, \quad d < \|u_3\|$ with $\psi(u_3) < a, \quad c < \|u_4\|$.

Proof. First, it follows from Theorem 5 that the problem $(1)-(2)$ has at least three positive solutions.

We now show that the condition of Theorem 1 is satisfied. Let $\Omega_1 = \{u \in E : \|u\| < c\}$. Then, from the proof of Theorem 5, we have $\|Au\| \le c = \|u\|$ for $u \in P \cap \partial\Omega_1$.

Now, set

$$R = c + \tfrac{1}{\eta} N,$$

where N is given in the proof of Theorem 3.

Let $\Omega_2 = \{u \in E : \|u\| < R\}$. Then by the proof of Theorem 3, we know that $\|Au\| \ge \|u\|$ for $u \in P \cap \partial\Omega_2$.

Therefore, by Theorem 1, A has a fixed point u_4 satisfying $c < \|u_4\| \le R$. Thus, clearly, u is a positive solution of $(1) - (2)$ with the form

$$u(t) = \begin{cases} \varphi_1(t), & t \in [-r, 0], \\ u_i(t), & t \in [0, T], \quad i = 1, 2, 3, 4, \\ \varphi_2(t), & t \in [T, p], \end{cases}$$

which are four positive solutions of the problem $(1) - (2)$ such that

$$\|u_1\| < d, \quad \min_{t\in[0,T]} u_2(t) > a, \quad d < \|u_3\| \text{ with } \min_{t\in[0,T]} u_3(t) < a, \quad c < \|u_4\|.$$

Example Let $\mathbb{T} = [-3, 1] \cup \{1 + \frac{1}{3^n} : n \in \mathbb{N}_0\} \cup \{\frac{3}{2}, \frac{5}{2}, 3\}$ be a time-scale. We consider the following dynamic equation on time scale $\mathbb{T}$:

$$u^{\triangle\triangle}(t) + \frac{1000[u^3(t) + u(t)u(t+\frac{1}{2})]}{u^2(t) + u^2(t-1) + u(t+\frac{1}{2}) + 1} = 0, \ for\ t \in [0, 2], \tag{11}$$

$$u(s) = \varphi_1(s) \equiv s^2, \ s \in [-3, 0], \ u(s) = \varphi_2(s) \equiv 0, \ s \in [2, 3],$$

$$\frac{1}{4}u(0) - \frac{1}{2}u^{\triangle}(0) = 0, \ 2u(2) + \frac{1}{2}u^{\triangle}(2) = \frac{1}{2}u(\frac{1}{2}) + \frac{1}{3}u(\frac{1}{3}) + \frac{1}{4}u(\frac{1}{4}), \tag{12}$$

where $\alpha = \frac{1}{4}$, $\beta = \gamma = \frac{1}{2}$, $\delta = 2$, $\xi_1 = \frac{1}{4}$, $\xi_2 = \frac{1}{3}$, $\xi_3 = \frac{1}{2}$, $a_1 = \frac{1}{4}$, $a_2 = \frac{1}{3}$, $a_3 = \frac{1}{2}$, $p = r = 3$, $\theta_1 : [0,2] \to [-3,2]$, $\theta_2 : [0,2] \to [0,3]$, $\theta_1(t) = t-1$, $\theta_2(t) = t + \frac{1}{2}$, $\upsilon = 1$, $\mu = \frac{3}{2}$ and

$$f(u_1,u_2,u_3) = \frac{1000(u_1^3 + u_1u_3)}{u_1^2 + u_2^2 + u_3 + 1},$$

$$f(u_1,\varphi_1(s),u_3) = \frac{1000(u_1^3 + u_1u_3)}{u_1^2 + s^4 + u_3 + 1},$$

$$f(u_1,u_2,\varphi_2(s)) = \frac{1000u_1^3}{u_1^2 + u_2^2 + 1}.$$

Then we get $\eta = \frac{1}{18}$, $Y_1 = [0,1]$, $Y_2 = \left[\frac{3}{2},2\right]$, $Y_3 = \left[1,\frac{3}{2}\right)$ and after some simple calculation, we find

$$D = \alpha(\delta T + \gamma - \textstyle\sum_{i=1}^{m-2} a_i\xi_i) + \beta(\delta - \sum_{i=1}^{m-2} a_i) \cong 1,07,$$

$$m = \eta^2 \int_{Y_1} \phi(s)\Delta s = \frac{1}{324}\int_{Y_1} \frac{(\beta+\alpha s)(\delta T+\gamma)}{D}\Delta s \cong 0,0011,$$

$$M = \int_0^T \phi(s)\Delta s = \int_0^T \frac{(\beta+\alpha s)(\delta T+\gamma)}{D}\Delta s = 8.$$

Clearly, the conditions $(H_1) - (H_6)$ hold. Now, we check that the conditions in Theorem 3 are satisfied. Observe that

$$\lim_{u_1,u_3\to 0^+} \frac{f(u_1,\varphi_1(s),u_3)}{u_1} = \lim_{u_1,u_3\to 0^+} \frac{1000(u_1^3 + u_1u_3)}{u_1(u_1^2 + s^4 + u_3 + 1)} = 0 \le 1/8,$$

$$\lim_{u_1,u_2\to 0^+} \frac{f(u_1,u_2,\varphi_2(s))}{u_1} = \lim_{u_1,u_2\to 0^+} \frac{1000u_1^3}{u_1(u_1^2 + u_2^2 + 1)} = 0 \le 1/8,$$

$$\lim_{u_1\to 0^+} \frac{f(u_1,u_1(s),u_1(s))}{u_1} = \lim_{u_1\to 0^+} \frac{1000(u_1^3 + u_1u_1)}{u_1(u_1^2 + u_1^2 + u_1 + 1)} = 0 \le 1/8,$$

and

$$\lim_{u_1,u_3\to +\infty} \frac{f(u_1,\varphi_1(s),u_3)}{u_1} = \lim_{u_1,u_3\to +\infty} \frac{1000(u_1^3 + u_1u_3)}{u_1(u_1^2 + s^4 + u_3 + 1)} = 1000 \ge 1/0.0011.$$

Thus, by Theorem 3, the problem $(11) - (12)$ has at least one positive solution.

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