*Electronic Journal of Mathematical Analysis and Applications*
Vol. 5(2) July 2017, pp. 40-46.
ISSN: 2090-729X(online)
http://fcag-egypt.com/Journals/EJMAA/

# ON A NEW FIXED POINT THEOREM IN HILBERT ALGEBRAS SPACES AND APPLICATION

F. CHOUIA AND T. MOUSSAOUI

Abstract. In this paper, a new fixed point theorem is proved for some potential operators on Hilbert algebras spaces by using critical point theory. An application is given to some nonlinear integral equations.

## 1. Introduction

Nonlinear integral equations are important to describe some real world problems such as control theory, electrical circuits, mathematical physics and technology and mechanics of fluids. Also many problems of mathematical physics can be stated in the form of nonlinear integral equations, see [3, 12, 13] and the references therein. Integral equations involving the product of operators may be considered only in the framework of Banach algebras, see [2, 5, 6, 7, 8]. The authors in these last papers discuss certain nonlinear integral and nonlinear functional equations by using fixed point theorems in Banach algebras. In this work, we present a new fixed point theorem on Hilbert algebras where the proof is based on critical point theory and we apply it to solve certain nonlinear integral equations. The critical point theory is a modern evolution of an old part of mathematical analysis which is called calculus of variations.
We introduce some preliminary concepts and results of critical point theory in the first section. Our main result contain existence of solutions for the equation $Au.Bu = u$ in a Hilbert algebra $H$. In the last section, we consider as application an equation of the form

$$u(t) = F(t, u(t)).G\Big(t, \int_0^1 k(t,s)u(s)ds\Big), t \in [0,1]$$

in the Hilbert algebra space $L^2(0,1)$, which is equipped with the scalar product

$$(u,v)_{L^2} = \int_0^1 u(s)v(s)ds$$

2010 *Mathematics Subject Classification.* 35A15, 35B38, 45G10, 47G40, 47H10.
*Key words and phrases.* Fixed point, critical point, potential operator, coercive, Ekeland minimization theorem, nonlinear integral equation.
Submitted July 14, 2016.

and endowed with the norm $\|u\|_{L^2} = (\int_0^1 u^2(s)ds)^{1/2}$, where $F : [0,1] \times \mathbb{R} \to \mathbb{R}^-$, $k : [0,1] \times [0,1] \to \mathbb{R}^+$ and $G : [0,1] \times \mathbb{R} \to \mathbb{R}^+$ are continuous functions.
Let $H$ be a Hilbert space supplied with the scalar product $(.,.)_H$, in brief $(.,.)$ and $H^*$ be the dual space of $H$, where the duality pairing between $H$ and $H^*$ is denoted by $< .,. >_{H,H^*}$, in brief $< .,. >$. The Hilbert space $H$ is said to be a Hilbert algebra if $H$ is both Banach algebra and a Hilbert space. In the following we introduce some definitions and theorems which we use in the sequel.

**Definition 1.1.** [8] An operator $A : H \to H$ is called compact if $\overline{A(H)}$ is a compact subset of $H$.

**Definition 1.2.** [11] A functional $\varphi : H \to \mathbb{R}$ is
1. convex if $\varphi(\alpha u + (1-\alpha)v) \leq \alpha\varphi(u) + (1-\alpha)\varphi(v)$ for all $u, v$ in $H$ and all $\alpha \in (0,1)$.
2. strictly convex if $\varphi(\alpha u + (1-\alpha)v) < \alpha\varphi(u) + (1-\alpha)\varphi(v)$ for all $u, v$ in $H$ with $u \neq v$ and all $\alpha \in (0,1)$.

**Definition 1.3.** [4] A functional $\varphi : H \to \mathbb{R}$ is coercive if $\lim\limits_{\|u\| \to +\infty} \varphi(u) = +\infty$.

**Definition 1.4.** [4] A functional $\varphi : H \to \mathbb{R}$ is said to be Gâteaux differentiable at $u_0$ if there exists $u^* \in H^*$ such that

$$\lim_{t \to 0} t^{-1}(\varphi(u_0 + th) - \varphi(u_0)) =< u^*, h >= u^*(h)$$

for all $h \in H$. The functional $u^*$ is called the Gâteaux derivative of $\varphi$ at $u_0$ and we denote it by $\varphi'(u_0)$.

**Definition 1.5.** [4] A mapping $A : H \to H^*$ is said to be a potential operator with a potential $a : H \to \mathbb{R}$ if $a$ is Gâteaux differentiable and

$$\lim_{t \to 0} t^{-1}(a(u + tv) - a(u)) =< A(u), v >$$

for all $u$ and $v$ in $H$. For a potential, it is assumed that $a(0) = 0$. For more about potential operators, see [4].

**Definition 1.6.** [11] Let $H$ be a Hilbert space, $\Omega \subset H$ an open subset, and $\varphi : \Omega \to \mathbb{R}$ a Gâteaux differentiable functional. $u \in \Omega$ is called a critical point of $\varphi$ if $\varphi'(u) = 0$, i.e., $\varphi'(u).v = 0$, for every $v \in H$. If further $\varphi(u) = c$, we say that $u$ is a critical point of $\varphi$ at level $c$.

*Remark* 1.7. [11] Clearly, every local minimum point of a Gâteaux differentiable functional $\varphi$ is a critical point.

**Definition 1.8.** [14] Let $A : H \longrightarrow H^*$ be an operator.
$(a)$ $A$ is said to be demicontinuous if

$$u_n \longrightarrow u \text{ as } n \longrightarrow +\infty \quad \textit{implies } Au_n \rightharpoonup Au \text{ as } n \longrightarrow +\infty.$$

$(b)$ $A$ is said to be hemicontinuous if the real function

$$t \mapsto < A(u + tv), w > \quad \text{is continuous on } [0,1] \text{ for all } u, v, w \in H.$$

*Remark* 1.9. [9] For monotone operators $A : H \longrightarrow H^*$ with $Dom(A) = H$, demicontinuity and hemicontinuity are equivalent.

**Definition 1.10.** [4] A mapping $A : H \to H^*$ is said to be monotone if $< Au - Av, u - v > \geq 0$ for all $u, v$ in $H$ and strictly monotone if equality implies $u = v$.

*Remark* 1.11. [4] A Gâteaux differentiable functional $\varphi : H \to \mathbb{R}$ is convex if and only if its potential operator is hemicontinuous and monotone.

**Definition 1.12.** [4] Let $\varphi \in C^1(H, \mathbb{R})$. If any sequence $(u_n) \subset H$ for which $(\varphi(u_n))$ is bounded in $\mathbb{R}$ and $\varphi'(u_n) \to 0$ as $n \to +\infty$ in $H^*$ possesses a convergent subsequence, then we say that $\varphi$ satisfies the Palais-Smale condition ((PS) condition for short).

**Theorem 1.13.** [11] *Let $H$ be a Hilbert space and let $\varphi : H \to \mathbb{R}$ be a continuous, convex and coercive functional. Then $\varphi$ has a global minimum point.*

**Theorem 1.14.** [11] *Let $\varphi : H \to \mathbb{R}$ be strictly convex. Then $\varphi$ has at most one minimum.*

*Remark* 1.15. In the light of Theorem 1.13 and Theorem 1.14, we note that every stictly convex, continuous and coercive functional has one global minimum point.

**Theorem 1.16.** [10] *Let $H$ be a Hilbert space and $\varphi \in C^1(H, \mathbb{R})$. Suppose that the functional $\varphi$ is bounded from below and verifies the Palais-Smale condition at level $c$ with $c = \inf\limits_{u \in H} \varphi(u)$. Then there exists a critical point for $\varphi$ at level $c$.*

**Proposition 1.17.** [1] *let $H$ be a Hilbert space, $\Omega$ an open subset of $H$, and $\varphi : \Omega \to \mathbb{R}$ a mapping of class $C^1$, i.e., it is Gâteaux differentiable with continuous derivative. Given $u, v \in \Omega$ , if $u + sv \in \Omega$ for all $s \in [0, 1]$, then*

$$\varphi(u + v) = \varphi(u) + \int_0^1 < D\varphi(u + sv), v > ds.$$

Indeed, this result connects between the potential operator $\Phi$ and the Gâteaux differentiable functional $\varphi$ for it can be checked that

$$\varphi(u) = \int_0^1 < \Phi(su), u > ds.$$

## 2. Main result

Let $H$ be a Hilbert algebra space and let the operators $A, B : H \to H$, where $A$ is continuous and $B$ is compact and $A.B$ is potential. In this section, we prove the existence of solution for the abstract equation $(A.B)(u) = u$ or equivalently $Au.Bu = u$ in $H$ by using the minimization principle for some differentiable functionals. We have the following result.

**Theorem 2.1.** *Let $A, B$ be as above. Suppose that:*

1. *For all $v \in H$, the operator $-A_v : u \mapsto -A_v(u) = -Au.Bv$ is monotone and hemicontinuous.*
2. *For every $v \in H$,*
$$(A(su).Bv, u) \leq 0; \forall u \in H, \forall s \in [0, 1].$$
3. *For all $u \in H$,*
$$(Au.B(sv), v) \leq 0, \forall v \in H, \forall s \in [0, 1].$$

*Then, the operator $A.B$ has a fixed point in $H$.*

*Proof.* Since the product $A.B$ is potential, there exists a Gâteaux differentiable functional $S : H \to \mathbb{R}$ such that $S' = A.B$. Let $J = K - S$, where $K$ is defined by $Ku = \frac{1}{2}\|u\|^2$ for $u \in H$, then $K' = I$, $J \in C^1(H, \mathbb{R})$ and $J' = I - S' = I - A.B$ i.e., $\forall u \in H : J'(u) = u - (A.B)u$. Since $A.B$ is a potential operator, it can be represented as

$$S(u) = \int_0^1 \Big((A.B)(su), u\Big) ds.$$

Define the functional $J : H \to \mathbb{R}$ by

$$J(u) = \frac{1}{2}\|u\|^2 - \int_0^1 \Big((A.B)(su), u\Big) ds = \frac{1}{2}\|u\|^2 - \int_0^1 \Big(A(su).B(su), u\Big) ds.$$

Our proof is based on Theorem 1.13, Theorem 1.14 and Theorem 1.16 and it will be done in the following steps.

***Step 1***: Define a mapping $A_v : H \to H$ and a functional $J_v : H \to \mathbb{R}$ for any $v \in H$ by

$$A_v(u) = Au.Bv, \forall u \in H,$$

$$J_v(u) = \frac{1}{2}\|u\|^2 - \int_0^1 \Big(A(su).Bv, u\Big) ds, \forall u \in H.$$

It is clear that $J_v'(u) = u - Au.Bv = (I - A_v)(u)$ for all $u \in H$.

- $J_v$ is continuous as a sum of two continuous functionals on $H$.
- $J_v$ is a strictly convex functional. Indeed, since the mapping $-A_v$ is monotone and hemicontinuous, then the functional

  $$u \mapsto -\int_0^1 \Big(A(su).Bv, u\Big) ds$$

  is convex and it is well known that the functional

  $$u \mapsto \frac{1}{2}\|u\|^2$$

  is strictly convex, and so the functional $J_v$ is strictly convex.
- For any $v \in H$, $J_v$ is coercive. Indeed, we have

  $$\begin{aligned} J_v(u) &= \frac{1}{2}\|u\|^2 - \int_0^1 \Big(A(su).Bv, u\Big) ds \\ &\geq \frac{1}{2}\|u\|^2. \end{aligned}$$

  Thus, if $\|u\| \to +\infty$, then $J_v(u) \to +\infty$. An application of Theorem 1.13 and Theorem 1.14 yields that the functional $J_v$ has a unique global minimum $w \in H$ with $J_v'(w) = 0$ or equivalently $0 = w - Aw.Bv$ i.e., $A_v w = w$.

***Step 2***: Define the operator $N : H \to H$, and the functional $\varphi : H \to \mathbb{R}$ by

$$Nv = w \text{ for } v \in H,$$

$$\varphi(v) = \frac{1}{2}\|v\|^2 - \int_0^1 \Big(Aw.B(sv), v\Big) ds,$$

where $w$ is the unique solution of the equation $AwBv = w, v \in H$, and we show that $\varphi$ verifies all conditions of Theorem 1.16.

- $\varphi$ is bounded from below. Indeed, by using assumption 3, we have:

$$\begin{aligned}\varphi(v) &= \frac{1}{2}\|v\|^2 - \int_0^1 \Big(Au.B(sv), v\Big) ds \\ &\geq \frac{1}{2}\|v\|^2 \geq 0.\end{aligned}$$

- $\varphi$ verifies the (PS) condition. Take a sequence $(v_n) \subset H$ such that $\lim\limits_{n\to+\infty} \varphi'(v_n) = 0$ and $(\varphi(v_n))$ is bounded i.e., there is some positive constant $M$ such that $|\varphi(v_n)| \leq M, \forall n \in \mathbb{N}$. By hypothesis 3, we have

$$\begin{aligned}M \geq \varphi(v_n) &= \frac{1}{2}\|v_n\|^2 - \int_0^1 \Big(Au.B(sv_n), v_n\Big) ds \\ &\geq \frac{1}{2}\|v_n\|^2,\end{aligned}$$

which involves that $(v_n)$ is bounded in $H$. We note that $\varphi'(v_n) = v_n - Aw_n.Bv_n$, with $\lim\limits_{n\to+\infty} \varphi'(v_n) = 0$. Since the sequence $(v_n)$ is bounded and the operator $v \mapsto AwBv$ is compact, the sequence $(Aw_n.Bv_n)$ is relatively compact, and so there exists a subsequence $(v_{n_k}) \subset (v_n)$ such that $Aw_{n_k}.Bv_{n_k} \to w^*$, hence $v_{n_k} \to w^*$ in $H$. Indeed,

$$\|v_{n_k} - w^*\| \leq \|v_{n_k} - Aw_{n_k}.Bv_{n_k}\| + \|Aw_{n_k}.Bv_{n_k} - w^*\| \longrightarrow 0$$

as $k \to +\infty$. Thus the (PS) condition is satisfied. An Application of Theorem 1.16 yields that there is a critical point for the functional $\varphi$ which is a fixed point for the operator $A.B$.

□

## 3. Application

Consider the nonlinear integral equation

$$u(t) = F(t, u(t)).G\Big(t, \int_0^1 k(t,s)u(s)ds\Big), \quad t \in [0,1], \tag{3.1}$$

where $F : [0,1] \times \mathbb{R} \to \mathbb{R}^-$, $k : [0,1] \times [0,1] \to \mathbb{R}^+$ and $G : [0,1] \times \mathbb{R} \to \mathbb{R}^+$ are continuous functions. Define two operators $A$ and $B$ on $L^2(0,1)$ by

$$Au(t) = F(t, u(t)),$$

$$Bu(t) = G\Big(t, \int_0^1 k(t,s)u(s)ds\Big).$$

We know that $L^2(0,1)$ is a Hilbert algebra space equipped with the scalar product

$$(u,v)_{L^2} = \int_0^1 u(s)v(s)ds.$$

Assume that:

(H1) $F(t,u)$ is decreasing with respect to its second variable,

(H2) there exists $\alpha < 0$, such that

$$F(t,u) \leq \alpha u, \quad \text{for all } \ t \in [0,1] \ \text{ and all } \ u \in \mathbb{R},$$

(H3) $G$ satisfies the following condition:

$$G\Big(t, \int_0^1 k(t,s)u(s)ds\Big).u(t) \geq 0, \quad \text{for all } \; t \in [0,1] \; \text{ and all } \; u \in L^2(0,1).$$

Then we have the following result.

**Theorem 3.1.** *If the hypotheses (H1)-(H3) hold, then the nonlinear integral equation (3.1) has a solution in* $L^2(0,1)$.

*Proof.* We are going to verify all the hypotheses of Theorem 2.1 in the following claims.

**Claim 1:** The mapping $u \mapsto -A_v(u) = -Au.Bv$ is monotone and hemicontinuous.
Indeed, let $u_1, u_2 \in L^2(0,1)$, then

$$\begin{aligned}(-A_v u_1 + A_v u_2, u_1 - u_2) = -\int_0^1 &(F(t,u_1(t)) - F(t,u_2(t)))(u_1(t) - u_2(t)) \\ &\times G\Big(t, \int_0^1 k(t,s)v(s)ds\Big)dt.\end{aligned}$$

Since the functions $k$ and $G$ are positive and $F(t,.)$ is decreasing, then we have

$$(-A_v u_1 + A_v u_2, u_1 - u_2) \geq 0, \forall u_1, u_2 \in L^2(0,1),$$

i.e., $-A_v$ is monotone.
Since $F$ is continuous then $-A_v$ is demicontinuous. From the monotonicity of $-A_v$ and by Remark 1.9, we have that $-A_v$ is hemicontinuous on $L^2(0,1)$.

**Claim 2:** The operator $B$ is compact.
This follows from the fact that $k, G$ are continuous.

**Claim 3:** Under assumption (H2), the second condition of Theorem 2.1 holds.
Indeed, let $v \in L^2(0,1)$ and $s \in [0,1]$ then,

$$\begin{aligned}(A(su).Bv, u) &= \int_0^1 F(t, su(t)).G\Big(t, \int_0^1 k(t,\tau)v(\tau)d\tau\Big)u(t)dt \\ &\leq \int_0^1 (\alpha su(t)).G\Big(t, \int_0^1 k(t,\tau)v(\tau)d\tau\Big)u(t)dt \\ &= \alpha \int_0^1 su^2(t).G\Big(t, \int_0^1 k(t,\tau)v(\tau)d\tau\Big)dt.\end{aligned}$$

It is clear that

$$\int_0^1 su^2(t).G\Big(t, \int_0^1 k(t,\tau)v(\tau)d\tau\Big)dt \geq 0$$

and so

$$(A(su).Bv,u) \leq 0, \quad \forall s \in [0,1], \ \ \forall u \in L^2(0,1).$$

**Claim 4:** Under assumption (H3), the third condition of Theorem 2.1 holds. Indeed, let $u \in L^2(0,1)$ then,

$$(Au.B(sv),v) \quad = \quad \int_0^1 F(t,u(t)).G\Big(t, \int_0^1 k(t,\tau)sv(\tau))d\tau\Big)v(t)dt, \ s \in [0,1].$$

By assumption (H3) we see that

$$(Au.B(sv),v) \leq 0, \ \ \forall s \in [0,1], \forall v \in L^2(0,1).$$

**Conclusion:** All the assertions of Theorem 2.1 are verified, then the Equation (3.1) has a solution in $L^2\,(0,1)$, and this complete the proof.

□

## References

[1] A. Avez, Calcul Différentiel, Masson, 1983.
[2] J. Banas, L. Lecko, Fixed points of the product of operators in Banach algebra, Panamer. Math. J., **12**(2) (2002), 101-109.
[3] L. Bers, P. Hilton, H. Hochstadt, Integral Equations, A Wiley-Interscience publication, 1973.
[4] J. Chabrowski, Variational Methods for Potential Operator Equations, Mathematisch Centrum, Amsterdam, 1979.
[5] S. Djebali, K. Hammache, Furi-Pera fixed point theorems in Banach algebras with applications, Acta Universitatis palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, **74**(1) (2008), 55-75.
[6] S. Djebali, K. Hammache, Fixed point theorems for nonexpansive maps in Banach spaces, Nonlinear Analysis **73** (2010), 3440-3449.
[7] B.C. Dhage, On some variants of Schauder's fixed point principle and applications to nonlinear integral equations, Jour. Math. Phys. Sci., **25** (1988), 603-611.
[8] B.C. Dhage, On a fixed point theorem in Banach algebras with applications, Appl. Math. Lett., **18** (2005), 273-280.
[9] L. Gasinski, N. S. Papageorgiou, Nonsmooth Critical Point Theory and Nonlinear Boundary Value Problems, Chapman-Hall CRC, 2005.
[10] O. Kavian, Introduction à la Théorie des Points Critiques et Applications aux Problèmes Elliptiques, Springer-Verlag, 1993.
[11] J. Mawhin, M. Willem, Critical Point Theory and Hamiltonian Systems, Springer-verlag, New York, 1989.
[12] S.G. Mikhlin, Integral Equations, and their Applications to Certain Problems in Mechanics, Mathematical Physics and Technology, Pergamon Press, New York, 1957.
[13] D. O'Regan, M. Meehan, Existence Theory for Nonlinear Integral and Integrodifferential Equations, Kluwer Academic Publishers, 1998.
[14] E. Zeidler, Nonlinear Functional Analysis and Its Applications, II/B, Springer Verlag, Berlin, New York, 1986.

F. Chouia
Laboratory of Fixed Point Theory and Applications, Department of Mathematics, E.N.S., Kouba, Algiers, Algeria
*E-mail address*: `chouiafat@yahoo.com`

T. Moussaoui
Laboratory of Fixed Point Theory and Applications, Department of Mathematics, E.N.S., Kouba, Algiers, Algeria
*E-mail address*: `moussaoui@ens-kouba.dz`