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A COMPARISON BETWEEN THE IMPROVED $(G'/G)-$ EXPANSION METHOD AND THE TANH METHOD

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Abstract. In this paper, we show that the improved $(G'/G)-$ expansion method is more general than the tanh method and it may give some new exact solutions of nonlinear partial differential equations.

1. Introduction

Recently, many methods have been proposed for obtaining exact traveling wave solutions of partial differential equations. Examples of these methods are tanh method [1], sine-cosine method [2], extended mapping transformation method [3], simplest equation method [4], direct integration to the simplest equation method [5] and the $(G'/G)-$ expansion method [6]. Many improved and extended versions of the $(G'/G)-$ expansion method have been proposed to get more exact solutions of partial differential equations (see for example [7, 8, 9]).

A good effort was done in proving that the $(G'/G)-$ expansion method is equivalent to the tanh method. For example, the equivalence between the $(G'/G)-$ expansion method and the tanh method is proved in [1, 10, 11]. Moreover, in [12], it is shown that the $(G'/G)-$ expansion method is a specific form of the simplest equation method [4].

The improved $(G'/G)-$ expansion method [8] is used to obtain new exact solutions of some models [13, 14]. In this paper, we show that this improved (G'/G)-expansion method is more general than the tanh method and it may give some new exact solutions of nonlinear partial differential equations.

2. The tanh method [1]

In this section, we give the detailed description of the tanh method. Suppose that a nonlinear evolution equation (NLEE) with independent variable u and two independent variables x and t is given by

$$H(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, ...) = 0, \tag{1}$$

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where, H is a polynomial in $u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. To determine u explicitly, one can follow the following steps:

Step 1: Use the traveling wave transformation:

$$u = u(\xi), \quad \xi = x - \nu t, \tag{2}$$

where, ν is a constant to be determined latter. Then, the NLEE (1) is reduced to a nonlinear ordinary differential equation (NLODE) for $u = u(\xi)$:

$$H(u, u', u'', u''', ...) = 0. \tag{3}$$

step 2: Suppose that the NLODE (3) has the following solution:

$$u = \sum_{i=0}^{n} b_i (y)^i, \tag{4}$$

where, y is the solution of the following Riccati equation

$$y' = y^2 + a. \tag{5}$$

The Riccati equation (5) has the following two solutions [3]

Solution 1: when $a < 0$

$$y = -\sqrt{-a}\tanh(\sqrt{-a}(\xi - \xi_0)). \tag{6}$$

In this case

$$u = \sum_{i=0}^{n} b_i(-\sqrt{-a}\tanh(\sqrt{-a}(\xi - \xi_0)))^i. \tag{7}$$

Solution 2: when $a = 0$

$$y = -\frac{1}{\xi - \xi_0}. \tag{8}$$

In this case

$$u = \sum_{i=0}^{n} b_i \left(-\frac{1}{\xi - \xi_0}\right)^i. \tag{9}$$

where, $a, b_i (i = 0, ..., n)$ are constants to be determined later, ξ_0 is an arbitrary constant and n is a positive integer to be determined in step 3.

Step 3: Determine the positive integer n by balancing the highest order derivatives and nonlinear terms in Eq. (3).

Step 4: Substituting Eq. (7) or Eq. (9) into Eq. (3) and equating expressions of different powers of $(\tanh(\sqrt{-a}(\xi - \xi_0))^i$ to zero (or, equating expressions of different powers of $\left(\frac{1}{\xi-\xi_0}\right)$ to zero), we obtain coefficients b_i and the parameter a.

Step 5: Substituting b_i and a into Eq. (7) or Eq. (9) , we can obtain the explicit solutions of Eq. (1) immediately.

3. The improved (G'/G)- expansion method [8]

In this section, we give the detailed description of the improved (G'/G) -expansion method. To determine u in Eq. (1) explicitly using the improved (G'/G)- expansion method, one can follow the following five steps:

Step 1: Use the traveling wave transformation (2) to reduce the NLEE (1) to the NLODE (3)

Step 2: Suppose that the NLODE (3) has the following solution:

$$u=\sum_{i=-n}^{n}\frac{a_i\left(G'/G\right)^i}{\left(1+\sigma\left(G'/G\right)\right)^i}=\sum_{i=-n}^{n}a_i\left(\frac{\left(G'/G\right)}{1+\sigma\left(G'/G\right)}\right)^i, \tag{10}$$

where, σ and $a_i(i=-n,...,n)$ are constants to be determined later, n is a positive integer, and $G=G(\xi)$ satisfies the following second order linear ordinary differential equation(LODE):

$$G''+\mu G=0, \tag{11}$$

where, μ is a real constant. The general solutions of Eq. (11) can be listed as follows. When $\mu<0$, we obtain the hyperbolic function solution of Eq. (11)

$$G=A_1\cosh(\sqrt{-\mu}\xi)+A_2\sinh(\sqrt{-\mu}\xi), \tag{12}$$

where, A_1 and A_2 are arbitrary constants. When $\mu>0$, we obtain the trigonometric function solution of Eq. (11)

$$G=A_1\cos(\sqrt{\mu}\xi)+A_2\sin(\sqrt{\mu}\xi), \tag{13}$$

where, A_1 and A_2 are arbitrary constants. When $\mu=0$, we obtain the linear solution of Eq. (11)

$$G=A_1+A_2\xi, \tag{14}$$

where, A_1 and A_2 are arbitrary constants

Step 3: Determine the positive integer n by balancing the highest order derivatives and nonlinear terms in Eq. (3).

Step 4: Substituting (10) along with (11) into Eq. (11) and then setting all the coefficients of $\left(G'/G\right)^k, (k=1,2,3,...)$of the resulting systems numerator to zero, yields a set of over-determined nonlinear algebraic equations for ν,σ and $a_i(i=-n,...,n)$.

Step 5: Assuming that the constants ν,σ and $a_i(i=-n,...,n)$ can be obtained by solving the algebraic equations in Step 4, then substituting these constants and the known general solutions of Eq. (11) into (10), we can obtain the explicit solutions of Eq. (1) immediately.

4. The first way to compare between the two methods

In the second step of the improved (G'/G)- expansion method let $y=\frac{\left(G'/G\right)}{1+\sigma(G'/G)}$, Eqs. (10), (11) are transformed into

$$u=\sum_{i=-n}^{n}a_iy^i, \tag{15}$$

$$y'+(1+\mu\sigma^2)y^2-2\sigma\mu y+\mu=0. \tag{16}$$

The general solution of Eq. (16) (when $\mu<0$) is given by [1]

$$y=\alpha+\beta\tanh\left(\sqrt{-\mu}(\xi-\xi_0)\right), \tag{17}$$

where, $\alpha=\frac{\sigma\mu}{1+\mu\sigma^2}$ and $\beta=\frac{\sqrt{-\mu}}{1+\mu\sigma^2}$.

Substituting solution (17) into expansion (15) we have

$$u=\sum_{i=-n}^{n}a_i\left(\alpha+\beta\tanh\left(\sqrt{-\mu}(\xi-\xi_0)\right)\right)^i=u_1+u_2, \tag{18}$$

where,

$$u_1 = \sum_{i=0}^{n} a_i \left(\alpha + \beta \tanh\left(\sqrt{-\mu}(\xi - \xi_0)\right)\right)^i = \sum_{i=0}^{n} b_i \left(\tanh\left(\sqrt{-\mu}(\xi - \xi_0)\right)\right)^i, \quad (19)$$

$$u_2 = \sum_{i=-n}^{-1} a_i \left(\alpha + \beta \tanh\left(\sqrt{-\mu}(\xi - \xi_0)\right)\right)^i =$$

$$\sum_{i=1}^{n} a_{-i} \left(\frac{1}{\alpha + \beta \tanh\left(\sqrt{-\mu}(\xi - \xi_0)\right)}\right)^i =$$

$$\sum_{i=1}^{n} a_{-i} \left(\frac{\alpha}{\alpha^2 - \beta^2} - \frac{\alpha}{\alpha^2 - \beta^2} + \frac{1}{\alpha + \beta \tanh\left(\sqrt{-\mu}(\xi - \xi_0)\right)}\right)^i =$$

$$\sum_{i=1}^{n} a_{-i} \left(\frac{\alpha}{\alpha^2 - \beta^2} + \frac{\beta}{\beta^2 - \alpha^2} \frac{\frac{\beta}{\alpha} + \tanh\left(\sqrt{-\mu}(\xi - \xi_0)\right)}{1 + \frac{\beta}{\alpha} \tanh\left(\sqrt{-\mu}(\xi - \xi_0)\right)}\right)^i =$$

$$\sum_{i=1}^{n} a_{-i} \left(\frac{\alpha}{\alpha^2 - \beta^2} + \frac{\beta}{\beta^2 - \alpha^2} \tanh\left(\sqrt{-\mu}(\xi - \xi_0) + z\right)\right)^i, \quad z = \tanh^{-1}\frac{\beta}{\alpha}, \quad (20)$$

therefore, u_2 may be rewritten as

$$u_2 = \sum_{i=0}^{n} c_i \left(\tanh\left(\sqrt{-\mu}(\xi - \xi_0) + z\right)\right)^i, \quad (21)$$

hence,

$$u = u_1 + u_2 = \sum_{i=0}^{n} b_i \left(\tanh\left(\sqrt{-\mu}(\xi - \xi_0)\right)\right)^i + \sum_{i=0}^{n} c_i \left(\tanh\left(\sqrt{-\mu}(\xi - \xi_0) + z\right)\right)^i, \quad (22)$$

When $\mu = 0$, the solution of Eq. (16) is given by

$$y = \frac{1}{\xi + c}, \quad (23)$$

where, c is an arbitrary constant. In this case, we simply obtain the following rational algebraic solution of Eq. (3)

$$u = \sum_{i=-n}^{n} a_i \left(\frac{1}{\xi + c}\right)^i. \quad (24)$$

5. The second way to compare between the two methods

In this section we will proof that The solution formula (10) will give solutions in the form of various forms of the tanh function and the rational function.

Case 1: When $\mu < 0$, we have

$$\frac{G'}{G} = \sqrt{-\mu}\frac{A_2 \cosh(\sqrt{-\mu}\xi) + A_1 \sinh(\sqrt{-\mu}\xi)}{A_1 \cosh(\sqrt{-\mu}\xi) + A_2 \sinh(\sqrt{-\mu}\xi)} = \sqrt{-\mu}\frac{1 + \frac{A_1}{A_2}\tanh(\sqrt{-\mu}\xi)}{\frac{A_1}{A_2} + \tanh(\sqrt{-\mu}\xi)}$$

$$= \sqrt{-\mu}\coth(\sqrt{-\mu}\xi + d), \quad d = \tanh^{-1}\frac{A_1}{A_2}, \quad (25)$$

$$\frac{1+\sigma\,(G'/G)}{(G'/G)} = \frac{G+\sigma G'}{G'} = \frac{G}{G'} + \sigma = \frac{1}{\sqrt{-\mu}}\tanh(\sqrt{-\mu}\xi + d) + \sigma, \quad (26)$$

$$\begin{aligned}\sum_{j=-n}^{-1} a_j \left(\frac{(G'/G)}{1+\sigma\,(G'/G)}\right)^j &= \sum_{j=1}^{n} a_{-j}\left(\frac{1+\sigma\,(G'/G)}{(G'/G)}\right)^j \\ &= \sum_{j=1}^{n} a_{-j}\left(\frac{1}{\sqrt{-\mu}}\tanh(\sqrt{-\mu}\xi+d)+\sigma\right)^j = \sum_{j=0}^{n} b_j(\tanh(\sqrt{-\mu}\xi+d))^j, \quad (27)\end{aligned}$$

$$\begin{aligned}\frac{(G'/G)}{1+\sigma\,(G'/G)} = \frac{G'}{G+\sigma G'} &= \frac{\sqrt{-\mu}}{\tanh(\sqrt{-\mu}\xi+d)+\sigma\sqrt{-\mu}} \\ &= \left(\frac{\mu\sigma}{1+\mu\sigma^2} - \frac{\mu\sigma}{1+\mu\sigma^2} + \frac{\sqrt{-\mu}}{\tanh(\sqrt{-\mu}\xi+d)+\sigma\sqrt{-\mu}}\right) \\ &= \left(\frac{\mu\sigma}{1+\mu\sigma^2} + \frac{\sqrt{-\mu}}{1+\mu\sigma^2}\frac{\frac{1}{\sigma\sqrt{-\mu}}+\tanh(\sqrt{-\mu}\xi+d)}{1+\frac{1}{\sigma\sqrt{-\mu}}\tanh(\sqrt{-\mu}\xi+d)}\right) \\ &= \frac{\mu\sigma}{1+\mu\sigma^2} + \frac{\sqrt{-\mu}}{1+\mu\sigma^2}\tanh(\sqrt{-\mu}\xi+d+k), \quad k=\tanh^{-1}\frac{1}{\sigma\sqrt{-\mu}}, \quad (28)\end{aligned}$$

$$\begin{aligned}\sum_{j=1}^{n} a_j \left(\frac{(G'/G)}{1+\sigma\,(G'/G)}\right)^j &= \sum_{j=1}^{n} a_j\left(\frac{\mu\sigma}{1+\mu\sigma^2} + \frac{\sqrt{-\mu}}{1+\mu\sigma^2}\tanh(\sqrt{-\mu}\xi+d+k)\right)^j \\ &= \sum_{j=0}^{n} c_j(\tanh(\sqrt{-\mu}\xi+d+k))^j, \quad (29)\end{aligned}$$

So, in this case (when $\mu < 0$) Eq. (10) can be rewritten as

$$\begin{aligned}u = \sum_{j=-n}^{n} a_j\left(\frac{(G'/G)}{1+\sigma\,(G'/G)}\right)^j = \sum_{j=0}^{n} b_j(\tanh(\sqrt{-\mu}\xi+d))^j + \\ \sum_{j=0}^{n} c_j(\tanh(\sqrt{-\mu}\xi+d+k))^j. \quad (30)\end{aligned}$$

Case 2: When $\mu > 0$, we have

$$\begin{aligned}\frac{G'}{G} = \sqrt{\mu}\frac{A_2\cos(\sqrt{\mu}\xi)-A_1\sin(\sqrt{\mu}\xi)}{A_1\cos(\sqrt{\mu}\xi)+A_2\sin(\sqrt{\mu}\xi)} &= \sqrt{\mu}\frac{1-\frac{A_1}{A_2}\tan(\sqrt{\mu}\xi)}{\frac{A_1}{A_2}+\tan(\sqrt{\mu}\xi)} \\ &= \sqrt{\mu}\cot(\sqrt{\mu}\xi+d_1), \quad d_1=\tan^{-1}\frac{A_1}{A_2}, \quad (31)\end{aligned}$$

$$\frac{1+\sigma\,(G'/G)}{(G'/G)} = \frac{G+\sigma G'}{G'} = \frac{G}{G'} + \sigma = \frac{1}{\sqrt{\mu}}\tan(\sqrt{\mu}\xi + d_1) + \sigma, \quad (32)$$

$$\sum_{j=-n}^{-1} a_j \left(\frac{(G'/G)}{1+\sigma\,(G'/G)}\right)^j = \sum_{j=1}^{n} a_{-j} \left(\frac{1+\sigma\,(G'/G)}{(G'/G)}\right)^j$$
$$= \sum_{j=1}^{n} a_{-j} \left(\frac{1}{\sqrt{\mu}} \tan(\sqrt{\mu}\xi + d_1) + \sigma\right)^j = \sum_{j=0}^{n} b_j (\tan(\sqrt{\mu}\xi + d_1))^j, \quad (33)$$

$$\frac{(G'/G)}{1+\sigma\,(G'/G)} = \frac{G'}{G+\sigma G'} = \frac{\sqrt{\mu}}{\tan(\sqrt{\mu}\xi + d_1) + \sigma\sqrt{\mu}}$$
$$= \left(-\frac{\mu\sigma}{1-\mu\sigma^2} + \frac{\mu\sigma}{1-\mu\sigma^2} + \frac{\sqrt{\mu}}{\tan(\sqrt{\mu}\xi + d_1) + \sigma\sqrt{\mu}}\right)$$
$$= \left(-\frac{\mu\sigma}{1-\mu\sigma^2} + \frac{\sqrt{\mu}}{1-\mu\sigma^2} \frac{\frac{1}{\sigma\sqrt{\mu}} + \tan(\sqrt{\mu}\xi + d_1)}{1 + \frac{1}{\sigma\sqrt{\mu}} \tan(\sqrt{\mu}\xi + d_1)}\right)$$
$$= -\frac{\mu\sigma}{1+\mu\sigma^2} + \frac{\sqrt{\mu}}{1-\mu\sigma^2} \tan(\sqrt{\mu}\xi + d_1 - k_1), \quad k_1 = \tan^{-1} \frac{1}{\sigma\sqrt{\mu}}, \quad (34)$$

$$\sum_{j=1}^{n} a_j \left(\frac{(G'/G)}{1+\sigma\,(G'/G)}\right)^j = \sum_{j=1}^{n} a_j \left(-\frac{\mu\sigma}{1-\mu\sigma^2} + \frac{\sqrt{\mu}}{1-\mu\sigma^2} \tan(\sqrt{\mu}\xi + d_1 - k_1)\right)^j$$
$$= \sum_{j=0}^{n} c_j (\tan(\sqrt{\mu}\xi + d_1 - k_1))^j, \quad (35)$$

So, in this case (when $\mu > 0$) Eq. (10) can be rewritten as

$$u = \sum_{j=-n}^{n} a_j \left(\frac{(G'/G)}{1+\sigma\,(G'/G)}\right)^j = \sum_{j=0}^{n} b_j (\tan(\sqrt{\mu}\xi + d_1))^j$$
$$+ \sum_{j=0}^{n} c_j (\tan(\sqrt{\mu}\xi + d_1 - k_1))^j. \quad (36)$$

By considering the formula [15]

$$\tan(i\alpha) = i \tanh(\alpha), \quad i = \sqrt{-1}, \quad (37)$$

Eq. (36) can be reformulated as

$$u = \sum_{j=0}^{n} b_j (\tan(\sqrt{\mu}\xi + d_1))^j + \sum_{j=0}^{n} c_j (\tan(\sqrt{\mu}\xi + d_1 - k_1))^j =$$
$$\sum_{j=0}^{n} b_j (-\tan(i\sqrt{-\mu}\xi - d_1))^j + \sum_{j=0}^{n} c_j (-\tan(i\sqrt{-\mu}\xi - d_1 + k_1))^j =$$
$$\sum_{j=0}^{n} b_j (-i \tanh(\sqrt{-\mu}\xi + id_1))^j + \sum_{j=0}^{n} c_j (-i \tanh(\sqrt{-\mu}\xi + id_1 - ik_1))^j =$$
$$\sum_{j=0}^{n} e_j (\tanh(\sqrt{-\mu}\xi + d_2))^j + \sum_{j=0}^{n} f_j (\tanh(\sqrt{-\mu}\xi + d_2 + k_2))^j \quad (38)$$

which is equivalent to the solution (30) in case 1.

Case 3. When $\mu = 0$, in this case we will simply obtain the following rational solution

$$u = \sum_{j=-n}^{n} a_j \left(\frac{(G'/G)}{1 + \sigma (G'/G)} \right)^j = \sum_{j=-n}^{n} a_j \left(\frac{A_2}{A_1 + \sigma A_2 + A_2 \xi} \right). \quad (39)$$

6. Conclusion

It is shown that the improved $(G'/G)-$ expansion method is more general than the tanh method. When using the improved $(G'/G)-$ expansion method, it is inevitable that one solution set for the coefficients a_i will have $a_i = 0$ for $i = -n, ..., -1$ and so the solution u_1 is equivalent to the one obtained from the tanh-function method. But there will also be a solution set which gives a more general result, namely $u_1 + u_2$, that cannot be obtained by the tanh-function method.

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