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ON SOME IDENTITIES AND GENERATING FUNCTIONS FOR HADAMARD PRODUCT

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Abstract. In this paper, we introduce a new operator in order to derive some properties of homogeneous symmetric functions. By making use of the proposed operator, we give some new generating functions for Fibonacci numbers, Tchebychev polynomials of second kind and Hadamard product.

1. Introduction and Notations

Symmetric functions are located in the area of algebraic combinatorics whose main motivation is the calculation of certain identities from discrete mathematics or physics, and, conversely, applies combinatorial techniques to problems in algebra. In particular, they are studied from a combinatorial perspective by examining the fundamental bases and the change of basis coefficients [15]. Nowadays, the study of symmetric functions lies in the intersection of physics and algebra.

Our main motivation in this paper is to introduce a novel operator in order to derive some properties of homogeneous symmetric functions. Thus, some new generating functions are developed for Fibonacci numbers, Tchebychev polynomials of second kinds and Hadamard product.

Basically, Fibonacci numbers are defined by $F_0 = 1$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. It is convenient to write fn for F_{n+1} so that fn is the number of ways to tile a $1 \times n$ strip with 1×1 square bricks and 1×2 rectangular bricks [9]. The number of tilings of a $1 \times n$ strip with k bricks corresponds to the coefficient of t^n in $(1 + t^2)^k$. So, we know that the generating function for Fibonacci numbers is [8]

$$\sum_{n=0}^{\infty} F_n t^n = \frac{1}{1 - t - t^2}.$$

We now define the polynomial $F_n(s)$ by

$$\sum_{n=0}^{\infty} F_n(s) t^n = \frac{1}{1 - st - t^2}.$$

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Then we have $F_n(1) = F_n$, and $F_n(s)$ can be interpreted as the sum of the weights of tilings of a $1 \times n$ strip with 1×1 square bricks weighted by a 1×2 rectangular bricks weighted by 1.

By applying the geometric series and binomial series to $(1-st-t^2)^{-1}$, we derive that

$$F_n(s) = \sum_{j=0}^{\left[\frac{n}{2}\right]} \binom{n-j}{j} s^{n-2j},$$

Accordingly, we have $F_0(s) = 1,\ F_1(s) = s,\ F_2(s) = 1 + s^2, ...$

On the other hand, Shapiro [14] deduced a combinatorial proof of a bilinear generating function for Tchebychev polynomials such that

$$\sum_{n=0}^{\infty} F_n(s)F_n(j)t^{2n} = \frac{1-t^4}{1-sjz^2-(2+s^2+j^2)t^4-sjt^6+t^8}. \tag{1}$$

The Hadamard product $G*H$ of the power series $G(z) = \sum_{k=0}^{\infty} g(k)t^k$ and $H(z) = \sum_{k=0}^{\infty} h(k)t^k$ is defined by

$$G*H = \sum_{k=0}^{\infty} g(k)h(k)t^k.$$

For his part, the Hadamard product $G*H$ of the power series $G(z) = \sum_{k=0}^{\infty} g(k)t^k$ and $H(z) = \sum_{k=0}^{\infty} h(k)t^k$ is defined by

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Using the notation of Hadamard product, we can rewrite (1) as

$$\frac{1}{1-st-t^2} * \frac{1}{1-jt-t^2} = \frac{1-t^2}{1-sjt-(2+s^2+j^2)t^2-sjt^3+t^4}.$$

Definition 1 [1] Let B and P be any two alphabets, then we give $S_n(B-P)$ by the following form:

$$\frac{\Pi_{p\epsilon P}(1-pt)}{\Pi_{b\epsilon B}(1-bt)} = \sum_{n=0}^{\infty} S_n(B-P)t^n, \tag{2}$$

with the condition $S_n(B-P) = 0$ for $n < 0$.

Equation (2) can be rewritten in the following form

$$\sum_{n=0}^{\infty} S_n(B-P)t^n = \left(\sum_{n=0}^{\infty} S_n(B)t^n\right) \times \left(\sum_{n=0}^{\infty} S_n(-P)t^n\right)$$

with

$$S_n(B-P) = \sum_{j=0}^{n} S_{n-j}(-P)S_j(B) \tag{3}$$

We know that the polynomial whose roots are P is written as

$$S_n(x-P) = \sum_{j=0}^{n} S_{n-j}(-P)x^n, \text{ with } card(P) = n.$$

On the other hand, if B has cardinality equal to 1, i.e., $B = \{x\}$, then (2) can be rewritten as follows [1]

$$\sum_{n=0}^{\infty} S_n(x-P)t^n = \frac{\prod_{p\in P}(1-pt)}{(1-xt)} = 1 + \cdots + S_{n-1}(x-P)t^{n-1} + \frac{S_n(x-P)}{(1-xt)}t^n,$$

where $S_{n+k}(x-P) = x^k S_n(x-P)$ for all $k \geq 0$.

The summation is actually limited to a finite number of terms since $S_{-k}(\cdot) = 0$ for all $k > 0$. In particular, we have

$$\prod_{p\in P}(x-p) = S_n(x-P) = S_0(-P)x^n + S_1(-P)x^{n\ -1} + S_2(-P)x^{n\ -2} + \cdots,$$

where $S_k(-B)$ are the coefficients of the polynomials $S_n(x-P)$ for $0 \leq k \leq n$. These coefficients are zero for $k > n$.

For example, if all $p \in P$ are equal, i.e., $P = np$, then we have $S_n(x - np) = (x-p)^n$.

By choosing $p = 1$, i.e., $P = \left\{ \underbrace{1, 1, ...1}_{n} \right\}$, we obtain

$$S_k(-n) = (-1)^k \binom{n}{k} \text{ and } S_k(n) = \binom{n+k-1}{k}. \tag{4}$$

By combining (3) and (4), we obtain the following expression

$$S_n(B - nx) = S_n(B) - \binom{n}{1} S_{n-1}(B)x + \binom{n}{2} S_{n-2}(B)x^2 - \cdots + (-1)^n \binom{n}{n} x^n.$$

Definition 2 Given a function f on $\mathbb{R}^n$, the divided difference operator is defined as follows

$$\partial_{p_i p_{i+1}}(f) = \frac{f(p_1, \cdots, p_i, p_{i+1}, \cdots p_n) - f(p_1, \cdots p_{i-1}, p_{i+1}, p_i, p_{i+2} \cdots p_n)}{p_i - p_{i+1}}.$$

Definition 3 [6] The symmetrizing operator $\delta^k_{e_1 e_2}$ is defined by

$$\delta^k_{p_1 p_2}(g) = \frac{p_1^k g(p_1) - p_2^k g(p_2)}{p_1 - p_2} \text{ for all } k \in \mathbb{N}. \tag{5}$$

Proposition 1 [4] Let $P = \{p_1, p_2\}$ an alphabet, we define the operator $\delta^k_{p_1 p_2}$ as follows:

$$\delta^k_{p_1 p_2} g(p_1) = S_{k-1}(p_1 + p_2)g(p_1) + p_2^k \partial_{p_1 p_2} g\ (p_1), \ \textit{for all } k \in \mathbb{N}.$$

2. **Principal Formulas**

In our main result, we will combine all these results in a unified way such that they can be considered as a special case of the following Theorem.

Theorem 1 Given two alphabets $P=\{p_1,p_2\}$ and $B=\{b_1,b_2,...,b_n\}$ we have

$$\sum_{n=0}^{\infty} S_n(B)\,\delta_{p_1p_2}^{k+n-1}(p_1)t^n=\frac{\sum\limits_{n=0}^{k-1} S_n(-B)(p_1p_2)^n\,\delta_{p_1p_2}^{k-n-1}(p_1)t^n-(p_1p_2t)^k\sum\limits_{n=0}^{\infty} S_{n+k+1}(-B)\,\delta_{p_1p_2}^{n}(p_1)t^{n+1}}{\left(\sum\limits_{n=0}^{\infty} S_n(-B)(p_1t)^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-B)(p_2t)^n\right)}. \tag{6}$$

Proof. formula (2) for $B=P$ allows us to get $\sum\limits_{n=0}^{\infty} S_n(B)t^n=\frac{1}{\sum\limits_{n=0}^{\infty} S_n(-B)t^n}$. On one hand, since $g(p_1)=\sum\limits_{n=0}^{\infty} S_n(B)p_1^nt^n$, we have

$$\begin{aligned}
\delta_{p_1p_2}^{k}g(p_1) &= \delta_{p_1p_2}^{k}\left(\sum_{n=0}^{\infty} S_n(B)p_1^nt^n\right)\\
&= \sum_{n=0}^{\infty} S_n(B)\,\delta_{p_1p_2}^{k+n-1}(p_1)t^n,
\end{aligned}$$

which is the right-hand side of (6). On the other part, since

$$g(p_1)=\frac{1}{\sum\limits_{n=0}^{\infty} S_n(-B)p_1^nt^n},$$

we have

$$\begin{aligned}
\partial_{p_1p_2}g(p_1) &= \frac{1}{p_1-p_2}\left(\frac{1}{\sum\limits_{n=0}^{\infty} S_n(-B)p_1^nt^n}-\frac{1}{\sum\limits_{n=0}^{\infty} S_n(-B)p_2^nt^n}\right)\\
&= \frac{1}{p_1-p_2}\left(\frac{\sum\limits_{n=0}^{\infty} S_n(-B)p_2^nt^n-\sum\limits_{n=0}^{\infty} S_n(-B)p_1^nt^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_1^nt^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_2^nt^n\right)}\right)\\
&= \frac{\sum\limits_{n=0}^{\infty} S_n(-B)\frac{p_2^n-p_1^n}{p_1-p_2}t^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_1^nt^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_2^nt^n\right)}\\
&= -\frac{\sum\limits_{n=0}^{\infty} S_n(-B)\,\delta_{p_1p_2}^{n-1}(p_1)t^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_1^nt^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_2^nt^n\right)}.
\end{aligned}$$

By virtue of Proposition 1, it follows that

$$\begin{aligned}
\delta^k_{p_1p_2} g\ (p_1) &= S_{k-1}(p_1+p_2)g(p_1) + p_2^k \partial_{p_1p_2} g(p_1) \\
&= \frac{S_{k-1}(p_1+p_2)}{\sum\limits_{n=0}^{\infty} S_n(-B)p_1^n t^n} - p_2^k \frac{\sum\limits_{n=0}^{\infty} S_n(-B)\delta^{n-1}_{p_1p_2}(p_1)t^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_1^n t^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_2^n t^n\right)} \\
&= \frac{\sum\limits_{n=0}^{\infty} S_n(-B)\left[p_2^n\ \delta^{k-1}_{p_1p_2}(p_1) - p_2^k\ \delta^{n-1}_{p_1p_2}(p_1)\right]t^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_1^n t^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_2^n t^n\right)}.
\end{aligned}$$

Hence, we have

$$\delta^k_{p_1p_2} g(p_1) = \frac{\sum\limits_{n=0}^{k-1} S_n(-B)\left[p_2^n\ \delta^{k-1}_{p_1p_2}(p_1) - p_2^k\ \delta^{n-1}_{p_1p_2}(p_1)\right]t^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_1^n t^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_2^n t^n\right)} + \frac{\sum\limits_{n=k+1}^{\infty} S_n(-B)\left[p_2^n\ \delta^{k-1}_{p_1p_2}(p_1) - p_2^k\ \delta^{n-1}_{p_1p_2}(p_1)\right]t^n}{\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_1^n t^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_2^n t^n\right)},$$

$$= \frac{\sum\limits_{n=0}^{k-1} S_n(-B)(p_1p_2)^n\ \delta^{k-n-1}_{p_1p_2}(p_1)t^n - (p_1p_2t)^k \sum\limits_{n=0}^{\infty} S_{n+k+1}(-B)\ \delta^{n}_{p_1p_2}(p_1)t^{n+1}}{\left(\sum\limits_{n=0}^{\infty} S_n(-B)(p_1t)^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-B)(p_2t)^n\right)}.$$

This completes the proof.

3. On the Generating Functions of Some Polynomials

We now derive new generating functions of the products of some well-known polynomials. Indeed, we consider Theorem 1 in order to derive Fibonacci numbers and Tchebychev polynomials of second kind and the symmetric functions for $k = 1$.

Theorem 2 Given two alphabets $P = \{p_1, p_2\}$ and $B = \{b_1, b_2, b_3\}$, we have

$$\sum_{n=0}^{\infty} S_n(B)S_n(p_1+p_2)t^n = \frac{1 - p_1p_2\,(b_1b_2 + b_1b_3 + b_2b_3)\,t^2 - p_1p_2b_1b_2b_3\,(p_1+p_2)\,t^3}{\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_1^n t^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-B)p_2^n t^n\right)}. \tag{7}$$

Case 1: For $p_1 = b_1 = 1$, $b_2 = y$ and $p_2 = x$, $b_3 = \alpha$ in Theorem 2, we propose the following new generating function

$$\sum_{n=0}^{\infty} S_n(1+x)S_n(1+y+\alpha)t^n = \frac{1 - x(y+\alpha+\alpha y)t^2 - xy\alpha(1+x)t^3}{(1-t)\,(1-zt)\,(1-yt)\,(1-xyt)\,(1-\alpha t)(1-\alpha xt)}. \tag{8}$$

For $\alpha = 0$, we obtain the following identity of Ramanujan [6, 7, 10]:

$$\sum_{n=0}^{\infty} S_n(1+x)S_n(1+y)t^n = \frac{1 - xyt^2}{(1-t)\,(1-xt)\,(1-yt)\,(1-xyt)}.$$

Case 2: Replacing p_2 by $(-p_2)$ and assuming that $p_1p_2 = 1,\ p_1 - p_2 = 1$ in Theorem 2, we derive a new generating function of both Fibonacci numbers and symmetric functions in several variables as follows

$$\sum_{n\ =0}^{\infty} S_n(b_1+b_2+b_3)F_nt^n = \frac{1-(b_1b_2+b_1b_3+b_2b_3)\,t^2-b_1b_2b_3t^3}{(1-b_1z-b_1^2t^2)\,(1-b_2z-b_2^2t^2)\,(1-b_3z-b_3^2t^2)}.$$

Remark 1. *Let $b_3 = 0$ and by replacing b_2 by $(-b_2)in(7)$,and doing the following specialization $b_1b_2 = 1,\ b_1-b_2 = 1$, we obtain the generating function for Fibonacci numbers of second order* [6, 7]*:*

$$\sum_{n\ =0}^{\infty} F_n^2t^n\frac{1-t^2}{1-t-4t^2-t^3+t^4}.$$

Case 3: Replacing p_1 by $2p_1$ and p_2 by $(-2p_2)$, and assuming that $4p_1p_2 = -1$ in Theorem 2 allow us to deduce the Tchebychev polynomials of second kind and the symmetric functions in several variables, for $y = p_1 - p_2$, as follows

$$\sum_{n=0}^{\infty} S_n(b_1+b_2+b_3)U_n(y)t^n = \frac{1-(b_1b_2+b_1b_3+b_2b_3)\,t^2-b_1b_2b_3t^3}{(1-2b_1yt-b_1^2t^2)\,(1-2b_2yt-b_2^2t^2)\,(1-2b_3yt-b_3^2t^2)}.$$

Remark 2. *1- Let $b_3 = 0$ and by replacing b_2 by $(-b_2)$ $in(7)$,and doing the following specialization $b_1b_2 = 1,\ b_1 - b_2 = 1$, we obtain the generating function for the combined Fibonacci numbers and Tchebychev polynomials of the second kind* [6, 7]*:*

$$\sum_{n\ =0}^{\infty} F_nU_n(y)t^n = \frac{1+t^2}{1-2yt+(3-4y^2)t^2+2yt^3+t^4}.$$

2- Let $b_3 = 0$ and by replacing $2b_2$ by $(-2b_2)$ $in(7)$,making the following specialization $4b_1b_2 = -1$,recall that for $x = b_1 - b_2$, we obtain the generating function for Tchebychev polynomials of the second kind [6, 7]*:*

$$\sum_{n\ =0}^{\infty} U_n(x)U_n(y)t^n = \frac{1-t^2}{1-4yxt+(4x^2+4y^2-2)t^2-4yxt^3+t^4}.$$

Case 4: If $B = \{1, 0, 0\}$ in Theorem 2, then we deduce the following corollary.
Corollary 1 Given an alphabet $P = \{p_1, p_2\}$, we have

$$\sum_{n=0}^{\infty} S_n(p_1+[-p_2])t^n = \frac{S_0(p_1+[-p_2])}{S_0(p_1+[-p_2])-S_1(p_1+[-p_2])t-p_1p_2t^2}. \quad (9)$$

Corollary 2 Given an alphabet $P = \{p_1, p_2\}$, we have

$$\sum_{n=0}^{\infty} S_{n+1}(p_1+[-p_2])t^n = \frac{S_1(p_1+[-p_2])+p_1p_2t}{S_0(p_1+[-p_2])-S_1(p_1+[-p_2])t-p_1p_2t^2}. \quad (10)$$

Assuming that $p_1-p_2 = 1$ and $p_1p_2 = 1$ in (9) and (10) , we obtain the generating functions given by Boussayoud et al [5] which represent:

(1) The generating function of the Fibonacci numbers F_n.
(2) The generating function of the Lucas numbers L_n.

Assuming that $p_1-p_2 = 2$ and $p_1p_2 = 1$ in (9) and (10) , we obtain the generating functions given by Boussayoud et al [5] which represent:

(1) The generating function of the Pell numbers P_n.
(2) The generating function of the Pell-Lucas numbers Q_n.

4. **Hadamard Product**

In this section, we show the efficiency of the proposed method by determining the Hadamard product. In fact, by taking $P = 0$ in (2), we obtain

$$\sum_{n=0}^{\infty} S_n(b_1 + b_2)t^n = \frac{1}{(1 - b_1t)(1 - b_2t)}. \tag{11}$$

For the special case where $b_1 = b_2 = 1$ in (11), we have

$$\sum_{n=0}^{\infty} \binom{n+1}{n} t^n = \frac{1}{(1 - t)^2}, \tag{12}$$

which is considered in [3].

By replacing t by p_1t in (12), we get

$$\sum_{n=0}^{\infty} \binom{n+1}{n} p_1^n t^n = \frac{1}{(1 - p_1t)^2}. \tag{13}$$

Using Theorem 1 with the action of the operator $\delta_{p_1p_2}$ on both sides of identity (13) we obtain

$$\sum_{n=0}^{\infty} \binom{n+1}{n} S_n(p_1 + p_2)t^n = \frac{1 - p_1p_2t^2}{(1 - p_1t)^2(1 - p_2t)^2}. \tag{14}$$

By taking $p_1 = 1$ and $p_2 = 1$, we have

$$\sum_{n=0}^{\infty} \left[\binom{n+1}{n}\right]^2 t^n = \frac{1 + t}{(1 - t)^3}, \tag{15}$$

which is also considered in [10].

On the other hand, using formula (7) with the action of the operator $\delta_{p_1p_2}$ on both sides of (15), and replacing t by p_1t lead to

$$\sum_{n=0}^{\infty} \left[\binom{n+1}{n}\right]^2 S_n(p_1 + p_2)t^n = \delta_{p_1p_2} \frac{1}{(1 - p_1t)^3} + t \times \delta_{p_1p_2} \frac{p_1}{(1 - p_1t)^3}.$$

By using formulas (4), (5) and (6), it follows that

$$\delta_{p_1p_2} \frac{1}{(1 - p_1t)^3} = \frac{1 - p_1p_2t^2 \sum\limits_{n=0}^{1} (-1)^{n+2} \binom{3}{n+2} S_n(p_1 + p_2)t^n}{(1 - p_1t)^3(1 - p_2t)^3},$$

$$\delta_{p_1p_2} \frac{p_1}{(1 - p_1t)^3} = \frac{\left[\sum\limits_{n=0}^{1} (-1)^n \binom{3}{n} p_1^n p_2^n S_{1-n}(p_1 + p_2)t^n - p_1^2p_2^2t^3 \sum\limits_{n=0}^{1} (-1)^{n+3} \binom{3}{n+2} S_n(p_1 + p_2)t^n\right]}{(1 - p_1t)^3(1 - p_2t)^3}.$$

Notice that, for $p_1 = 1$ and $p_2 = 1$, we have

$$\sum_{n=0}^{\infty} \left[\binom{n+1}{n}\right]^3 t^n = \frac{1 + 4t + t^2}{(1 - t)^4}.$$

Using the same procedure, we obtain the following new results

- $\sum\limits_{n=0}^{\infty} \left[\binom{n+1}{n}\right]^4 t^n = \frac{1+11t+11t^2+t^3}{(1-t)^5}$,

- $\sum\limits_{n=0}^{\infty}\left[\binom{n+1}{n}\right]^{5}t^{n}=\frac{1+26t+66t^{2}+26t^{3}+t^{4}}{(1-t)^{6}}$,
- $\sum\limits_{n=0}^{\infty}\left[\binom{n+1}{n}\right]^{6}t^{n}=\frac{1+57t+302t^{2}+302t^{3}+57t^{4}+t^{5}}{(1-t)^{7}}$,
- $\sum\limits_{n=0}^{\infty}\left[\binom{n+1}{n}\right]^{7}z^{n}=\frac{1+120t+1191t^{2}+2416t^{3}+1191t^{4}+120t^{5+}t^{6}}{(1-t)^{8}}$,
- $\sum\limits_{n=0}^{\infty}\left[\binom{n+1}{n}\right]^{8}t^{n}=\frac{1+247t+4293t^{2}+15619t^{3}+15619t^{4}+4293t^{5}+247t^{6}+t^{7}}{(1-t)^{9}}$,
- $\sum\limits_{n=0}^{\infty}\left[\binom{n+1}{n}\right]^{9}t^{n}=\frac{1+502t+14608t^{2}+88234t^{3}+156190t^{4}+88234t^{5}+14608t^{6}+502t^{7}+t^{8}}{(1-t)^{10}}$,
- $\sum\limits_{n=0}^{\infty}\left[\binom{n+1}{n}\right]^{10}t^{n}=\frac{1+1013t+47840t^{2}+455192t^{3}+1310354t^{4}+1310354t^{5}+455192t^{6}+47840t^{7}+1013t^{8}+t^{9}}{(1-t)^{11}}$.

5. Conclusion

In this paper, we have derived new theorems in order to determine generating functions of Fibonacci and Lucas numbers and Tchebychev polynomials of the second kinds. The derived theorems and corollaries are based on symmetric functions and products of these numbers.

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