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ON A THREE-DIMENSIONAL SYSTEM OF NONLINEAR DIFFERENCE EQUATIONS

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Abstract. We investigate the solutions to the following system of nonlinear difference equations,

$$\begin{cases} x_{n+1} = \frac{f(z_n)}{y_{n-1}}, \\ y_{n+1} = \frac{f(x_n)}{z_{n-1}} \\ z_{n+1} = \frac{f(y_n)}{x_{n-1}} \quad \text{for } n \in N_0, \end{cases}$$

where $x_{-1} = \alpha$, $y_{-1} = \beta$, $z_{-1} = \gamma$, $x_0 = \lambda$, $y_0 = \mu$, and $z_0 = \omega$ are positive real numbers.

1. Introduction

There are various results on systems of difference equations, see [1, 4, 10, 11, 2]. Understanding the theory and dynamics of such systems play a crucial role in mathematics, physics, and biology, see [8, 7, 3].

Consider the following system of difference equations,

$$\begin{cases} x_{n+1} = \frac{f(z_n)}{y_{n-1}}, \\ y_{n+1} = \frac{f(x_n)}{z_{n-1}} \\ z_{n+1} = \frac{f(y_n)}{x_{n-1}} \quad \text{for } n \in N_0, \end{cases} \tag{1}$$

where $x_{-1} = \alpha$, $y_{-1} = \beta$, $z_{-1} = \gamma$, $x_0 = \lambda$, $y_0 = \mu$, and $z_0 = \omega$ are positive numbers.

Next are some papers on periodic and positive solutions to three-dimensional systems of nonlinear difference equations:

Tarek F. Ibrahim studied in [5] the periodic solutions of the following three-dimensional max-type cyclic system of difference equations

$$x_{n+1} = \max\left\{\frac{\alpha}{x_n}, y_n\right\}, y_{n+1} = \max\left\{\frac{\alpha}{y_n}, z_n\right\}, z_{n+1} = \max\left\{\frac{\alpha}{z_n}, x_n\right\}.$$

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M. R. S. Kulenović and Z. Nurkanović in [6] studied the global behavior of the following rational system of nonlinear difference equations

$$x_{n+1} = \frac{a + x_n}{b + y_n}, y_{n+1} = \frac{c + y_n}{d + z_n}, z_{n+1} = \frac{e + z_n}{f + x_n}.$$

Stevo Stević in [9] studied the stability of the following rational system of nonlinear difference equations

$$x_{n+1} = \frac{a_1 x_{n-2}}{b_1 y_n z_{n-1} x_{n-2} + c_1}, y_{n+1} = \frac{a_2 y_{n-2}}{b_2 z_n x_{n-1} y_{n-2} + c_2}, z_{n+1} = \frac{a_3 z_{n-2}}{b_3 x_n y_{n-1} z_{n-2} + c_3}.$$

2. Assumptions

The function f will have one of the following forms:

$$f(t) = 1 \tag{2}$$

$$f(t) = t \tag{3}$$

$$f(t) = \begin{cases} A, & \text{if } t > 0 \\ B, & \text{if } t < 0 \end{cases} \tag{4}$$

$$f(t) = \begin{cases} At, & \text{if } t < 0 \\ Bt, & \text{if } t > 0 \end{cases} \tag{5}$$

where A, $B \in \mathbb{R}$ such that $A^2 + B^2 \neq 0$.

3. Main Results

Theorem 3.1. *Let* (2) *hold and suppose that* x_{-1}, y_{-1}, z_{-1}, x_0, y_0, *and* z_0 *are positive real numbers. Also, let* $\{x_n, y_n, z_n\}$ *be a solution of the system of equations* (1) *with* $x_{-1} = \alpha$, $y_{-1} = \beta$, $z_{-1} = \gamma$, $x_0 = \lambda$, $y_0 = \mu$, *and* $z_0 = \omega$. *Then all solutions of* (1) *are of the following:*

$$\begin{aligned}
x_{12n+1} &= \frac{1}{\beta}, & y_{12n+1} &= \frac{1}{\gamma}, & z_{12n+1} &= \frac{1}{\alpha} \\
x_{12n+2} &= \frac{1}{\mu}, & y_{12n+2} &= \frac{1}{\omega}, & z_{12n+2} &= \frac{1}{\lambda} \\
x_{12n+3} &= \gamma, & y_{12n+3} &= \alpha, & z_{12n+3} &= \beta \\
x_{12n+4} &= \omega, & y_{12n+4} &= \lambda, & z_{12n+4} &= \mu \\
x_{12n+5} &= \frac{1}{\alpha}, & y_{12n+5} &= \frac{1}{\beta}, & z_{12n+5} &= \frac{1}{\gamma} \\
x_{12n+6} &= \frac{1}{\lambda}, & y_{12n+6} &= \frac{1}{\mu}, & z_{12n+6} &= \frac{1}{\omega} \\
x_{12n+7} &= \beta, & y_{12n+7} &= \gamma, & z_{12n+7} &= \alpha \\
x_{12n+8} &= \mu, & y_{12n+8} &= \omega, & z_{12n+8} &= \lambda \\
x_{12n+9} &= \frac{1}{\gamma}, & y_{12n+9} &= \frac{1}{\alpha}, & z_{12n+9} &= \frac{1}{\beta} \\
x_{12n+10} &= \frac{1}{\omega}, & y_{12n+10} &= \frac{1}{\lambda}, & z_{12n+10} &= \frac{1}{\mu} \\
x_{12n+11} &= \alpha, & y_{12n+11} &= \beta, & z_{12n+11} &= \gamma \\
x_{12n+12} &= \lambda, & y_{12n+12} &= \mu, & z_{12n+12} &= \omega.
\end{aligned}$$

Proof. The result holds for $n = 0$. Now suppose the result is true for some $k > 0$, we have the following:

$$x_{12k+1} = \frac{1}{\beta}, \quad y_{12k+1} = \frac{1}{\gamma}, \quad z_{12k+1} = \frac{1}{\alpha}$$
$$x_{12k+2} = \frac{1}{\mu}, \quad y_{12k+2} = \frac{1}{\omega}, \quad z_{12k+2} = \frac{1}{\lambda}$$
$$x_{12k+3} = \gamma, \quad y_{12k+3} = \alpha, \quad z_{12k+3} = \beta$$
$$x_{12k+4} = \omega, \quad y_{12k+4} = \lambda, \quad z_{12k+4} = \mu$$
$$x_{12k+5} = \frac{1}{\alpha}, \quad y_{12k+5} = \frac{1}{\beta}, \quad z_{12k+5} = \frac{1}{\gamma}$$
$$x_{12k+6} = \frac{1}{\lambda}, \quad y_{12k+6} = \frac{1}{\mu}, \quad z_{12k+6} = \frac{1}{\omega}$$
$$x_{12k+7} = \beta, \quad y_{12k+7} = \gamma, \quad z_{12k+7} = \alpha$$
$$x_{12k+8} = \mu, \quad y_{12k+8} = \omega, \quad z_{12k+8} = \lambda$$
$$x_{12k+9} = \frac{1}{\gamma}, \quad y_{12k+9} = \frac{1}{\alpha}, \quad z_{12k+9} = \frac{1}{\beta}$$
$$x_{12k+10} = \frac{1}{\omega}, \quad y_{12k+10} = \frac{1}{\lambda}, \quad z_{12k+10} = \frac{1}{\mu}$$
$$x_{12k+11} = \alpha, \quad y_{12k+11} = \beta, \quad z_{12k+11} = \gamma$$
$$x_{12k+12} = \lambda, \quad y_{12k+12} = \mu, \quad z_{12k+12} = \omega.$$

Also, for $k + 1$ we have the following:

$$x_{12k+13} = \frac{1}{y_{12k+11}} = \frac{1}{\beta}, \quad y_{12k+13} = \frac{1}{z_{12k+11}} = \frac{1}{\gamma}, \quad z_{12k+13} = \frac{1}{x_{12k+11}} = \frac{1}{\alpha}$$
$$x_{12k+14} = \frac{1}{y_{12k+12}} = \frac{1}{\mu}, \quad y_{12k+14} = \frac{1}{z_{12k+12}} = \frac{1}{\omega}, \quad z_{12k+14} = \frac{1}{x_{12k+12}} = \frac{1}{\lambda}$$
$$x_{12k+15} = \frac{1}{y_{12k+13}} = \gamma, \quad y_{12k+15} = \frac{1}{z_{12k+13}} = \alpha, \quad z_{12k+15} = \frac{1}{x_{12k+13}} = \beta$$
$$x_{12k+16} = \frac{1}{y_{12k+14}} = \omega, \quad y_{12k+16} = \frac{1}{z_{12k+14}} = \lambda, \quad z_{12k+16} = \frac{1}{x_{12k+14}} = \mu$$
$$x_{12k+17} = \frac{1}{y_{12k+15}} = \frac{1}{\alpha}, \quad y_{12k+17} = \frac{1}{z_{12k+15}} = \frac{1}{\beta}, \quad z_{12k+17} = \frac{1}{x_{12k+15}} = \frac{1}{\gamma}$$
$$x_{12k+18} = \frac{1}{y_{12k+16}} = \frac{1}{\lambda}, \quad y_{12k+18} = \frac{1}{z_{12k+16}} = \frac{1}{\mu}, \quad z_{12k+18} = \frac{1}{x_{12k+16}} = \frac{1}{\omega}$$
$$x_{12k+19} = \frac{1}{y_{12k+17}} = \beta, \quad y_{12k+19} = \frac{1}{z_{12k+17}} = \gamma, \quad z_{12k+19} = \frac{1}{x_{12k+17}} = \alpha$$
$$x_{12k+20} = \frac{1}{y_{12k+18}} = \mu, \quad y_{12k+20} = \frac{1}{z_{12k+18}} = \omega, \quad z_{12k+20} = \frac{1}{x_{12k+18}} = \lambda$$
$$x_{12k+21} = \frac{1}{y_{12k+19}} = \frac{1}{\gamma}, \quad y_{12k+21} = \frac{1}{z_{12k+19}} = \frac{1}{\alpha}, \quad z_{12k+21} = \frac{1}{x_{12k+19}} = \frac{1}{\beta}$$
$$x_{12k+22} = \frac{1}{y_{12k+20}} = \frac{1}{\omega}, \quad y_{12k+22} = \frac{1}{z_{12k+20}} = \frac{1}{\lambda}, \quad z_{12k+22} = \frac{1}{x_{12k+20}} = \frac{1}{\mu}$$
$$x_{12k+23} = \frac{1}{y_{12k+21}} = \alpha, \quad y_{12k+23} = \frac{1}{z_{12k+21}} = \beta, \quad z_{12k+23} = \frac{1}{x_{12k+21}} = \gamma$$
$$x_{12k+24} = \frac{1}{y_{12k+22}} = \lambda, \quad y_{12k+24} = \frac{1}{z_{12k+22}} = \mu, \quad z_{12k+24} = \frac{1}{x_{12k+22}} = \omega.$$

Therefore the result is true for every $k \in N_0$. This concludes the proof. □

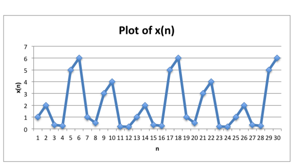
Theorem 3.2. *Suppose that* (2) *hold and let* $\{x_n, y_n, z_n\}$ *be a solution of the system of equations* (1). *Also, assume that* x_{-1}, y_{-1}, z_{-1}, x_0, y_0, *and* z_0 *are positive real numbers. Then all solutions of* (1) *are periodic with period twelve.*

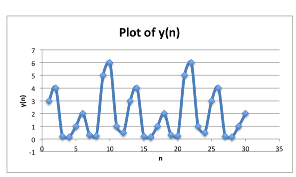
Proof. By (2), we have the following equal:

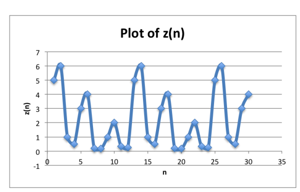
$$
\begin{array}{lll}
x_{n+1} = \dfrac{f(z_n)}{y_{n-1}} = \dfrac{1}{y_{n-1}}, & y_{n+1} = \dfrac{f(x_n)}{z_{n-1}} = \dfrac{1}{z_{n-1}}, & z_{n+1} = \dfrac{f(y_n)}{x_{n-1}} = \dfrac{1}{x_{n-1}} \\
x_{n+2} = \dfrac{f(z_{n+1})}{y_n} = \dfrac{1}{y_n}, & y_{n+2} = \dfrac{f(x_{n+1})}{z_n} = \dfrac{1}{z_n}, & z_{n+2} = \dfrac{f(y_{n+1})}{x_n} = \dfrac{1}{x_n} \\
x_{n+3} = \dfrac{f(z_{n+2})}{y_{n+1}} = z_{n-1}, & y_{n+3} = \dfrac{f(x_{n+2})}{z_{n+1}} = x_{n-1}, & z_{n+3} = \dfrac{f(y_{n+2})}{x_{n+1}} = y_{n-1} \\
x_{n+4} = \dfrac{f(z_{n+3})}{y_{n+2}} = z_n, & y_{n+4} = \dfrac{f(x_{n+3})}{z_{n+2}} = x_n, & z_{n+4} = \dfrac{f(y_{n+3})}{x_{n+2}} = y_n \\
x_{n+4} = \dfrac{f(z_{n+3})}{y_{n+2}} = z_n, & y_{n+4} = \dfrac{f(x_{n+3})}{z_{n+2}} = x_n, & z_{n+4} = \dfrac{f(y_{n+3})}{x_{n+2}} = y_n \\
x_{n+5} = \dfrac{f(z_{n+4})}{y_{n+3}} = \dfrac{1}{x_{n-1}}, & y_{n+5} = \dfrac{f(x_{n+4})}{z_{n+3}} = \dfrac{1}{y_{n-1}}, & z_{n+5} = \dfrac{f(y_{n+4})}{x_{n+3}} = \dfrac{1}{z_{n-1}} \\
x_{n+6} = \dfrac{f(z_{n+5})}{y_{n+4}} = \dfrac{1}{x_n}, & y_{n+6} = \dfrac{f(x_{n+5})}{z_{n+4}} = \dfrac{1}{y_n}, & z_{n+6} = \dfrac{f(y_{n+5})}{x_{n+4}} = \dfrac{1}{z_n} \\
x_{n+7} = \dfrac{f(z_{n+6})}{y_{n+5}} = y_{n-1}, & y_{n+7} = \dfrac{f(x_{n+6})}{z_{n+5}} = z_{n-1}, & z_{n+7} = \dfrac{f(y_{n+6})}{x_{n+5}} = x_{n-1} \\
x_{n+8} = \dfrac{f(z_{n+7})}{y_{n+6}} = y_n, & y_{n+8} = \dfrac{f(x_{n+7})}{z_{n+6}} = z_n, & z_{n+8} = \dfrac{f(y_{n+7})}{x_{n+6}} = x_n \\
x_{n+9} = \dfrac{f(z_{n+8})}{y_{n+7}} = \dfrac{1}{z_{n-1}}, & y_{n+9} = \dfrac{f(x_{n+8})}{z_{n+7}} = \dfrac{1}{x_{n-1}}, & z_{n+9} = \dfrac{f(y_{n+8})}{x_{n+7}} = \dfrac{1}{y_{n-1}} \\
x_{n+10} = \dfrac{f(z_{n+9})}{y_{n+8}} = \dfrac{1}{z_n}, & y_{n+10} = \dfrac{f(x_{n+9})}{z_{n+8}} = \dfrac{1}{x_n}, & z_{n+10} = \dfrac{f(y_{n+9})}{x_{n+8}} = \dfrac{1}{y_n} \\
x_{n+11} = \dfrac{f(z_{n+10})}{y_{n+9}} = x_{n-1}, & y_{n+11} = \dfrac{f(x_{n+10})}{z_{n+9}} = y_{n-1}, & z_{n+11} = \dfrac{f(y_{n+10})}{x_{n+9}} = z_{n-1} \\
x_{n+12} = \dfrac{f(z_{n+11})}{y_{n+10}} = x_n, & y_{n+12} = \dfrac{f(x_{n+11})}{z_{n+10}} = y_n, & z_{n+12} = \dfrac{f(y_{n+11})}{x_{n+10}} = z_n.
\end{array}
$$

This concludes the proof. □

To see the periodic behavior of $\{x_n, y_n, z_n\}$, observe the following three diagrams with $x_1 = 1$, $x_2 = 2$, $y_1 = 3$, $y_2 = 4$, $z_1 = 5$, and $z_2 = 6$:







Theorem 3.3. *Let* (3) *hold and suppose that* x_{-1}, y_{-1}, z_{-1}, x_0, y_0, *and* z_0 *are positive real numbers. Also, let* $\{x_n, y_n, z_n\}$ *be a solution of the system of equations* (1) *with* $x_{-1} = \alpha$, $y_{-1} = \beta$, $z_{-1} = \gamma$, $x_0 = \lambda$, $y_0 = \mu$, *and* $z_0 = \omega$. *Then all solutions of* (1) *are of the following:*

$$\begin{gathered}
x_{6n-5} = \frac{\omega}{\beta}, \quad y_{6n-5} = \frac{\lambda}{\gamma}, \quad z_{6n-5} = \frac{\mu}{\alpha} \\
x_{6n-4} = \frac{1}{\alpha}, \quad y_{6n-4} = \frac{1}{\beta}, \quad z_{6n-4} = \frac{1}{\gamma} \\
x_{6n-3} = \frac{1}{\lambda}, \quad y_{6n-3} = \frac{1}{\mu}, \quad z_{6n-3} = \frac{1}{\omega} \\
x_{6n-2} = \frac{\beta}{\omega}, \quad y_{6n-2} = \frac{\gamma}{\lambda}, \quad z_{6n-2} = \frac{\alpha}{\mu} \\
x_{6n-1} = \alpha, \quad y_{6n-1} = \beta, \quad z_{6n-1} = \gamma \\
x_{6n} = \lambda, \quad y_{6n} = \mu, \quad z_{6n} = \omega.
\end{gathered}$$

Proof. The result holds for $n = 0$. Now suppose the result is true for some $k > 0$, we have the following:

$$\begin{gathered}
x_{6k-5} = \frac{\omega}{\beta}, \quad y_{6k-5} = \frac{\lambda}{\gamma}, \quad z_{6k-5} = \frac{\mu}{\alpha} \\
x_{6k-4} = \frac{1}{\alpha}, \quad y_{6k-4} = \frac{1}{\beta}, \quad z_{6k-4} = \frac{1}{\gamma} \\
x_{6k-3} = \frac{1}{\lambda}, \quad y_{6k-3} = \frac{1}{\mu}, \quad z_{6k-3} = \frac{1}{\omega} \\
x_{6k-2} = \frac{\beta}{\omega}, \quad y_{6k-2} = \frac{\gamma}{\lambda}, \quad z_{6k-2} = \frac{\alpha}{\mu} \\
x_{6k-1} = \alpha, \quad y_{6k-1} = \beta, \quad z_{6k-1} = \gamma \\
x_{6k} = \lambda, \quad y_{6k} = \mu, \quad z_{6k} = \omega.
\end{gathered}$$

Also, for $k + 1$ we have the following:

$$\begin{aligned}
x_{6k+1} &= \frac{f(z_{6k})}{y_{6k-1}} = \frac{z_{6k}}{y_{6k-1}} = \frac{\omega}{\beta} \\
y_{6k+1} &= \frac{f(x_{6k})}{z_{6k-1}} = \frac{x_{6k}}{z_{6k-1}} = \frac{\lambda}{\gamma} \\
z_{6k+1} &= \frac{f(y_{6k})}{x_{6k-1}} = \frac{y_{6k}}{x_{6k-1}} = \frac{\mu}{\alpha} \\
x_{6k+2} &= \frac{f(z_{6k+1})}{y_{6k}} = \frac{z_{6k+1}}{y_{6k}} = \frac{\mu/\alpha}{\mu} = \frac{1}{\alpha} \\
y_{6k+2} &= \frac{f(x_{6k+1})}{z_{6k}} = \frac{x_{6k+1}}{z_{6k}} = \frac{\omega/\beta}{\omega} = \frac{1}{\beta} \\
z_{6k+2} &= \frac{f(y_{6k+1})}{x_{6k}} = \frac{y_{6k+1}}{x_{6k}} = \frac{\lambda/\gamma}{\lambda} = \frac{1}{\gamma} \\
x_{6k+3} &= \frac{f(z_{6k+2})}{y_{6k+1}} = \frac{z_{6k+2}}{y_{6k+1}} = \frac{1/\gamma}{\lambda/\gamma} = \frac{1}{\lambda} \\
y_{6k+3} &= \frac{f(x_{6k+2})}{z_{6k+1}} = \frac{x_{6k+2}}{z_{6k+1}} = \frac{1/\alpha}{\mu/\alpha} = \frac{1}{\mu}
\end{aligned}$$

$$z_{6k+3} = \frac{f(y_{6k+2})}{x_{6k+1}} = \frac{y_{6k+2}}{x_{6k+1}} = \frac{1/\beta}{\omega/\beta} = \frac{1}{\omega}$$

$$x_{6k+4} = \frac{f(z_{6k+3})}{y_{6k+3}} = \frac{z_{6k+3}}{y_{6k+2}} = \frac{1/\omega}{1/\beta} = \frac{\beta}{\omega}$$

$$y_{6k+4} = \frac{f(x_{6k+3})}{z_{6k+3}} = \frac{x_{6k+3}}{z_{6k+2}} = \frac{1/\lambda}{1/\gamma} = \frac{\gamma}{\lambda}$$

$$z_{6k+4} = \frac{f(y_{6k+3})}{x_{6k+3}} = \frac{y_{6k+3}}{x_{6k+2}} = \frac{1/\mu}{1/\alpha} = \frac{\alpha}{\mu}$$

$$x_{6k+5} = \frac{f(z_{6k+4})}{y_{6k+3}} = \frac{z_{6k+4}}{y_{6k+3}} = \frac{\alpha/\mu}{1/\mu} = \alpha$$

$$y_{6k+5} = \frac{f(x_{6k+4})}{z_{6k+3}} = \frac{x_{6k+4}}{z_{6k+3}} = \frac{\beta/\omega}{1/\omega} = \beta$$

$$z_{6k+5} = \frac{f(y_{6k+4})}{x_{6k+3}} = \frac{y_{6k+4}}{x_{6k+3}} = \frac{\gamma/\lambda}{1/\lambda} = \gamma$$

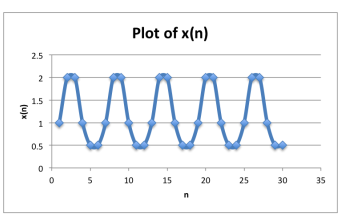
$$x_{6k+6} = \frac{f(z_{6k+5})}{y_{6k+4}} = \frac{z_{6k+5}}{y_{6k+4}} = \frac{\gamma}{\gamma/\lambda} = \lambda$$

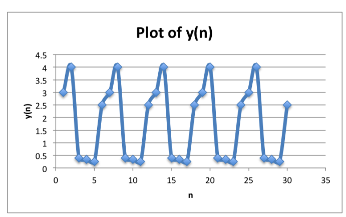
$$y_{6k+6} = \frac{f(x_{6k+5})}{z_{6k+4}} = \frac{x_{6k+5}}{z_{6k+4}} = \frac{\alpha}{\alpha/\mu} = \mu$$

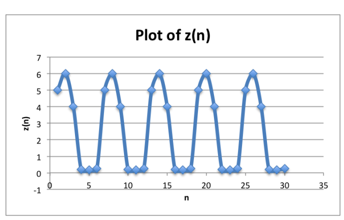
$$z_{6k+6} = \frac{f(y_{6k+5})}{x_{6k+4}} = \frac{y_{6k+5}}{x_{6k+4}} = \frac{\beta}{\beta/\omega} = \omega.$$

Therefore the result is true for every $k \in N_0$. This concludes the proof. □

To see the periodic behavior of $\{x_n, y_n, z_n\}$, observe the following three diagrams with $x_1 = 1$, $x_2 = 2$, $y_1 = 3$, $y_2 = 4$, $z_1 = 5$, and $z_2 = 6$:







Theorem 3.4. *Let* (4) *hold with* $A, B < 0$ *and suppose that* x_{-1}, y_{-1}, z_{-1}, x_0, y_0, *and* z_0 *are positive real numbers. Also, let* $\{x_n, y_n, z_n\}$ *be a solution of the system of equations* (1) *with* $x_{-1} = \alpha$, $y_{-1} = \beta$, $z_{-1} = \gamma$, $x_0 = \lambda$, $y_0 = \mu$, *and* $z_0 = \omega$. *Then all solutions of* (1) *are the following:*

$$x_{12n+1} = \frac{A}{\beta}\left(\frac{A}{B}\right)^{3n}, \quad y_{12n+1} = \frac{A}{\gamma}\left(\frac{A}{B}\right)^{3n}, \quad z_{12n+1} = \frac{A}{\alpha}\left(\frac{A}{B}\right)^{3n}$$

$$x_{12n+2} = \frac{B}{\mu}\left(\frac{B}{A}\right)^{3n}, \quad y_{12n+2} = \frac{B}{\omega}\left(\frac{B}{A}\right)^{3n}, \quad z_{12n+2} = \frac{B}{\lambda}\left(\frac{B}{A}\right)^{3n}$$

$$x_{12n+3} = \gamma\left(\frac{B}{A}\right)^{3n+1}, \quad y_{12n+3} = \alpha\left(\frac{B}{A}\right)^{3n+1}, \quad z_{12n+3} = \beta\left(\frac{B}{A}\right)^{3n+1}$$

$$x_{12n+4} = \omega\left(\frac{A}{B}\right)^{3n+1}, \quad y_{12n+4} = \lambda\left(\frac{A}{B}\right)^{3n+1}, \quad z_{12n+4} = \mu\left(\frac{A}{B}\right)^{3n+1}$$

$$x_{12n+5} = \frac{A}{\alpha}\left(\frac{A}{B}\right)^{3n+1}, \quad y_{12n+5} = \frac{A}{\beta}\left(\frac{A}{B}\right)^{3n+1}, \quad z_{12n+5} = \frac{A}{\gamma}\left(\frac{A}{B}\right)^{3n+1}$$

$$x_{12n+6} = \frac{B}{\lambda}\left(\frac{B}{A}\right)^{3n+1}, \quad y_{12n+6} = \frac{B}{\mu}\left(\frac{B}{A}\right)^{3n+1}, \quad z_{12n+6} = \frac{B}{\omega}\left(\frac{B}{A}\right)^{3n+1}$$

$$x_{12n+7} = \beta\left(\frac{B}{A}\right)^{3n+2}, \quad y_{12n+7} = \gamma\left(\frac{B}{A}\right)^{3n+2}, \quad z_{12n+7} = \alpha\left(\frac{B}{A}\right)^{3n+2}$$

$$x_{12n+8} = \mu\left(\frac{A}{B}\right)^{3n+2}, \quad y_{12n+8} = \omega\left(\frac{A}{B}\right)^{3n+2} \quad z_{12n+8} = \lambda\left(\frac{A}{B}\right)^{3n+2}$$

$$x_{12n+9} = \frac{A}{\gamma}\left(\frac{A}{B}\right)^{3n+2}, \quad y_{12n+9} = \frac{A}{\alpha}\left(\frac{A}{B}\right)^{3n+2}, \quad z_{12n+9} = \frac{A}{\beta}\left(\frac{A}{B}\right)^{3n+2}$$

$$x_{12n+10} = \frac{B}{\omega}\left(\frac{B}{A}\right)^{3n+2}, \quad y_{12n+10} = \frac{B}{\lambda}\left(\frac{B}{A}\right)^{3n+2}, \quad z_{12n+10} = \frac{B}{\mu}\left(\frac{B}{A}\right)^{3n+2}$$

$$x_{12n+11} = \alpha\left(\frac{B}{A}\right)^{3n+3}, \quad y_{12n+11} = \beta\left(\frac{B}{A}\right)^{3n+3}, \quad z_{12n+11} = \gamma\left(\frac{B}{A}\right)^{3n+3}$$

$$x_{12n+12} = \lambda\left(\frac{A}{B}\right)^{3n+3}, \quad y_{12n+12} = \mu\left(\frac{A}{B}\right)^{3n+3}, \quad z_{12n+12} = \omega\left(\frac{A}{B}\right)^{3n+3}.$$

Proof. The result follows by the principle of mathematical induction. □

Corollary 3.1. *If* $A \neq B$ *and* $A, B < 0$, *then the solutions of* (1) *are oscillatory and nonperiodic.*

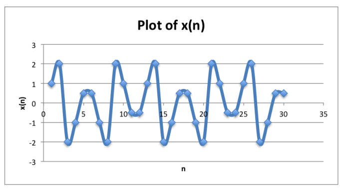
Theorem 3.5. *Suppose* (5) *hold and let* $\{x_n, y_n, z_n\}$ *be a solution of the system of equations* (1). *Also, assume that* x_{-1}, y_{-1}, z_{-1}, x_0, y_0, *and* z_0 *are positive real numbers with* $A > 0$ *and* $B < 0$. *Then all solutions of* (1) *are periodic with period twelve.*

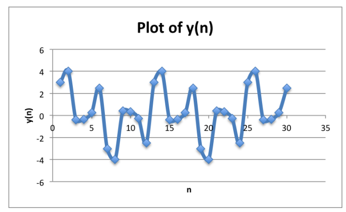
Proof. Let (.,.,.) be the pair of solutions of (1), then the following set

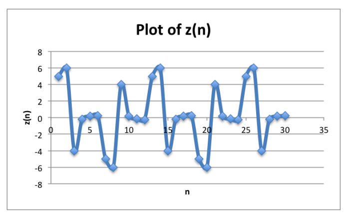
$$\Big\{ (\alpha, \beta.\gamma), (\lambda, \mu, \omega) \left(\frac{B\omega}{\beta}, \frac{B\lambda}{\gamma}, \frac{B\mu}{\alpha}\right), \left(\frac{AB}{\alpha}, \frac{AB}{\beta}, \frac{AB}{\gamma}\right), \left(\frac{A^2}{\lambda}, \frac{A^2}{\mu}, \frac{A^2}{\omega}\right), \left(\frac{\beta A}{\omega}, \frac{\gamma A}{\alpha}, \frac{\alpha A}{\mu}\right),$$
$$\left(\frac{\alpha B}{A}, \frac{\beta B}{A}, \frac{\gamma B}{A}\right), \left(\frac{\lambda B}{A}, \frac{\mu B}{A}, \frac{\omega B}{A}\right), \left(\frac{\omega A}{\beta}, \frac{\lambda A}{\gamma}, \frac{\mu A}{\alpha}\right), \left(\frac{A^2}{\alpha}, \frac{A^2}{\beta}, \frac{A^2}{\gamma}\right), \left(\frac{AB}{\lambda}, \frac{AB}{\mu}, \frac{AB}{\omega}\right), \left(\frac{\beta B}{\omega}, \frac{\gamma B}{\lambda}, \frac{\alpha B}{\mu}\right),$$
$$(\alpha, \beta, \gamma), (\lambda, \mu, \omega), \ldots \Big\}$$

is periodic with period twelve. This concludes the proof. □

To see the periodic and oscillatory behavior of $\{x_n, y_n\}$, observe the following three diagrams with $A = 1$, $B = -1$, $x_1 = 1$, $x_2 = 2$, $y_1 = 3$, $y_2 = 4$, $z_1 = 5$, and $z_2 = 6$:







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