Electronic Journal of Mathematical Analysis and Applications
Vol. 5(2) July 2017, pp. 191-199.
ISSN: 2090-729X(online)
http://fcag-egypt.com/Journals/EJMAA/

DISTRIBUTED CONTROL OF STOCHASTIC ELLIPTIC SYSTEMS INVOLVING LAPLACE OPERATOR

A. S. OKB EL BAB, ABD-ALLAH HYDER AND A. M. ABDALLAH

Abstract. In this paper, new problems concerned with distributed control for Neumann or Dirichlet stochastic elliptic systems are considered. Such systems with a quadratic cost (economic) functional are described. First, the existence and uniqueness of the state process for these systems is proved, then the set of equations and inequalities that characterizes the distributed control is obtained.

1. Introduction

Optimal control deals with the problem of finding a control law for a given system such that a certain optimality criterion is achieved. A control problem includes a cost (economic) functional that is a function of state and control variables. An optimal control is a set of differential equations describing the paths of the control variables that minimize the cost functional.

Lions and others [10-11], [14] have studied the optimal control systems for both elliptic , hyperbolic and parabolic. They have studied the standard cases , which have evolved and got the attention of Sergienko [14] and others. They have studied have the same regulations , but to develop models of the system.

The study of stochastic systems is of significance [1-2], [5-9], since they have applications [3,15] in problems in financial economics, engineering, biological sciences, in physical sciences and other areas of applied Mathematics.

With a deeper look , we thought that we follow the same lines and studied standard random systems and made clear the difference between the two studies to be a new shift in the path that has many applications.

In this paper, we use the idea of the concepts of the difference between the deterministic and stochastic systems to start a new plan in optimal control problems constrained by stochastic partial differential equations and develop it. To improve this deterministic mathematical model, we assume that we replace some deterministic data in the partial differential equations with stochastic input data.

If there are inputs that are random, then the solutions of the systems to the new model problem should also, be including randomness. Then, We need a stochastic

---

2010 *Mathematics Subject Classification.* 65M55, 65N30.
Key words and phrases. Stochastic partial differential equation, Distributed control.
Submitted Oct. 9, 2016.

domain, and may need to use probability theories for the solution to the new model problem.

In this paper, we study the scalar case of state process equation where the model of the system is laplace operator, such that the system of one equation. The plan of the paper is as follows, in section 2, we introduce the definitions and notations. In section 3, we derive the existence and uniqueness for the state process then , we study necessary conditions for optimality. In section 4, we study the Neumann stochastic elliptic systems with constraints. Finally, we show the difference in deterministic systems in section 5.

2. Notations

In this section, we shall consider some definitions introduced in [11-13] concerning the stochastic sobolev space , the embedding and which are necessary to introduce our work.

GivenR^n,define a multi-index$a$as an ordered collection of integers$a = (a_1, ..., a_n)$,such that its length is given by$|a| = \sum_{i=1}^{n} a_i$.If$v$is an$q-$times differentiable function, then for any$a$with$|a| \leq q$the derivative can be expressed as

$$D^a q(x) = \frac{\partial^{|a|} q}{\partial x_1^{a_1}\ \partial x_2^{a_2} ... \partial x_n^{a_n}}$$

By integration by parts formula, for a given an open domain$G \subseteq R^d$ and$v \in C^m(G), \Psi \in C^\infty(G)$ with$|a| \leq m$

$$\int_G m(x) D^a \Psi(x)\ dx = (-1)^{|a|} \int_G \Psi(x) D^a m(x)\ dx$$

For a given$m, n \in L^1(G)$, LetΨbe locally integrable in$G$.Then a locally integrable function$n$is said to be the weak$a-th$derivative of$m = D^a\Psi$ iff

$$\int_G m(x) D^a \Psi(x)\ dx = (-1)^{|a|} \int_G \Psi(x) n(x)\ dx$$

$W^{k,p}(G)$is the Sobolev space of differentiability$k$ and integrability$p$.It consists of functions$u$ which are$k-$weakly differentiable, such that$D^a u \in L^p(G)$for all$|a| \leq k$.

The Sobolev spaces$W^{k,p}(G)$are Banach spaces with the norm

$$\Big\|u\Big\|_{k,p;G} = \Big(\int_G \sum_{|a|\leq k} \Big|D^\alpha f\Big|^p\ dx\Big)^{\frac{1}{p}}.$$

Observe that the space$W^{0,p}(G)$ is just $L^p(G)$. In the case that$p = 2$, we also introduce the notation$H^k(G) = W^{k,2}(G)$. TheseL^2-Sobolev spaces are Hilbert spaces under the inner product

$$\Big\langle u, v\Big\rangle_k = \int_G \sum_{|a|\leq k} D^a u\ D^a v\ dx.$$

We consider a Hilbert space$H$and we will get two important notations for bilinear forms on Hilbert spaces.

Let$b(?, ?) : H \times H \to R$ be a bilinear form. It is said to be

(i) Continuous if there exists a constant$C > 0$such that

$$|b(x, y)| \leq C\|x\|\|y\|\ \|\ \ \forall x, y \in H,$$

(ii) (Eliptic (or Coercive) if there exists a constant$c > 0$such that

$$b(x,x) \geq c\|x\|^2, \forall x \in H.$$

Lax-Milgram states:

Suppose$b(.,.) : H \times H \to R$is a continuous and coercive bilinear form. For every linear form $L \in \acute{H}$, there exists a unique$x \in H$such that:$b(x,y) = L(y), y \in H$. Moreover, if$b(.,.)$is symmetric, then $x \in H$ is characterized by:

$$b(x,x) - 2L(x) = \min_{y \in E} \Big(b(y,y) - 2L(y) \Big)$$

For our stochastic elliptic problems, we use a complete probability space$(\Omega, \mathcal{F}, P)$,where$\Omega$is a sample space (a set of outcomes),$\mathcal{F}$is anσ-algebra of events and$P : \mathcal{F} \to [0,1]$ is a probability measure.

Existence and Uniqueness for Stochastic Elliptic Systems and Dirichlet Conditions

In this section, we study the distributed control problem for stochastic elliptic systems involving Laplace operator. Let us consider the following stochastic elliptic equations:

$$\begin{cases} -\Delta u(x) = W(x) & \text{in } G \\ \\ u(x) = 0 & \text{on } \partial G \end{cases} \tag{2.1}$$

where$G$is a bounded, continous and strictly domain in$\mathbb{R}^n$with boundary∂G While$u(x) \in H_0^1(\Omega, \mathcal{F}, P; G)$is a state process and$W(x)$is a wiener process. We prove the existence and uniqueness of the state process for system (2.1) in the following subsection.

Formulation of the Optimal Control Problem. The main goal of this section is to formulate mixed initial boundary value Dirichlet problem for stochastic elliptic systems . The space$[L^2(\Omega, \mathcal{F}, P; G)]$being the space of controls. For a control$y \in [L^2(\Omega, \mathcal{F}, P; G)]$,the state process of the system$u$is given by the solution of the following system:

$$\begin{cases} -\Delta u(y) = W + y & \text{in } G \\ \\ u(y) = 0 & \text{on } \partial G, \end{cases} \tag{2.2}$$

. The observation equation is given by$\chi(y) \equiv u(y)$,the cost functional is given by:

$$C(y) = \mathbb{E}\Bigg(\int_G (((u(y) - u(0) + (u(0) - \chi_d))^2) dx \Bigg) + \int_\Omega \Bigg(\int_G M(z^2)\ dx \Bigg)\ dp, \tag{2.3}$$

where$\chi_d =$ in $[L^2(\Omega, \mathcal{F}, P; G)]$.

Then, the control problem is defined by:

$$\begin{cases} y \in Y_{ad} & \text{such that} \\ \\ C(z) = \inf C(y) & \forall z \in Y_{ad}, \end{cases}$$

where$Y_{ad}$is a closed convex subset from$[L^2(\Omega, \mathcal{F}, P; G)]$.

Since the cost functional (2.3) can be written as:

$$C(y) = \mathbb{E}\bigg(\int_G (((u(y)-u(0)+(u(0)-\chi_d))^2)dx\bigg) + \int_\Omega \bigg(\int_G M(z^2)\ dx\bigg)\ dp,$$

where

$$\Pi(y,z) = \mathbb{E}\bigg(\int_G \{(u(y)-u(0))^2+(u(z)-u(0))^2\}dx\bigg)+\int_\Omega\int_G \bigg(M(z^2)\bigg)\ dx\ dp, \quad (2.4)$$

$M > 0$ is a positive constant, then

$$L(z) = \mathbb{E}\bigg(\int_G (-u(0)+\chi_d)(u(i)-u(0)\ dx\bigg), \quad (2.5)$$

and$\Pi(y,y)$is a stochatic coercive on$[L^2(\Omega,\mathcal{F},P;G)]$.Since$L(z)$is continuous on$[L^2(\Omega,\mathcal{F},P;G)]$, then there exists a unique optimal control from the general theory in [8]. Moreover, we have the following theorem which gives the characterization of the optimal control.

Theorem 2.2. *If the state$u(y)$is given by (2.1) and if the cost functional is given by (2.3), then there exists a unique optimal control$y \in Y_{ad}$such that$J(y) \leq J(z)\ \forall z \in Y_{ad}$; Moreover, it is characterized by:*

$$\begin{cases} -\Delta h(y) = u(y) - \chi_d & \text{in } G \\ \\ h(y) = 0 & \text{on } \partial G, \end{cases}$$

where$h(y)$is the adjoint state process.

Proof. Since$C(y)$is differentiable and$Y_{ad}$is bounded, then the optimal control$z$is characterized (see e.g [8,9]). Using equations (2.4), (2.5), we get

$$\Pi(y, z-y) \geq L(z-y), \quad (2.6)$$

and

$$\begin{aligned} \Pi(y,z-y) &- L(z-y) \\ &= \mathbb{E}\bigg(\int_G ((u(y)-u(0))((u(z-y)-u(0))))dx\bigg) \\ &- \mathbb{E}\bigg(\int_G (((u(0)-\chi_d)((u(z-y)-u(0)))))dx\bigg) \\ &+ \int_\Omega\bigg(\int_G My(z-y)\ dx\bigg)dp \\ &= \int_\Omega\bigg(\int_G My(z-y))\ dx\bigg)dp \\ &+ \mathbb{E}\bigg(\int_G ((u(y)-\chi_d)(u(z)-u(y))dx\bigg) \geq 0, \end{aligned}$$

with$(B^*h(y),u(y)) = (h(y),Bu(y))$,and B is defined by:

$$B\ \Phi = B\ \{u(y)\} = (-\Delta u(y)).$$

Applying the derivative in the sense of distribution, we get

$$B^*h(y) = u(y) - \chi_d,$$

where$B=-\Delta$ and . So,

$$\begin{aligned}\Pi(y,z-y) \quad - \quad & L(z-y)\\ = \quad & \int_\Omega\left(\int_G (My,z-y)dx\right)dp+\mathbb{E}\left(\int_G(h(-\Delta u(z)))dx\right)\geq 0\end{aligned}$$

Hence, from (2.6) we obtain$\mathbb{E}\left(\int_G((h+My)(z-y)dx\right)\geq 0$ □

Remark 2.1

If constraints are absent, i.e. when$Y_{ad}=Y$,then$h(z)+My=0, z_j\neq y_j$or$y=-\frac{h(z)}{M}$the differential problem of finding the vector-function satisfies the the following relations.

For the state process

$$\begin{cases} Bu+\dfrac{h(z)}{M}=W & \text{in } G\\ u=0 & \text{on } \partial G.\end{cases}$$

For the adjoint state process

$$\begin{cases} Bh(y)-u(y)=-\chi_d & \text{in } G\\ h(y)=0 & \text{on } \partial G.\end{cases}$$

Neumann Stochastic Elliptic Systems

In this section, we study the optimal control problem for stochastic elliptic system with Neumann conditions.

$$\begin{cases} -\Delta u(x)=W(x) & \text{in } G_1\\ \dfrac{\partial u(x)}{\partial V_A}=g & \text{on } \partial G,\end{cases} \tag{3.1}$$

where$g\in H^{\frac{1}{2}}(\Omega,\mathcal{F},P;G)$.

Existence and Uniqueness of Solution. In this subsection, we study the existence and uniqueness of solutions for stochastic systems governed by Neumann problems. Since

$$[H_0^1(\Omega,\mathcal{F},P;G)]^2\subseteq[H^1(\Omega,\mathcal{F},P;G)]^2,$$

then

$$\|u\|^2_{[H_0^1(\Omega,\mathcal{F},P;G)]^2}\subseteq\|u\|^2_{[H^1(\Omega,\mathcal{F},P;G)]^2},$$

which proves the coerciveness of bilinear form$a(u,u)$on$[H^1(\Omega,\mathcal{F},P;G)]^2$

$$b(u,u)\geq c\|u\|^2_{[H^1(\Omega,\mathcal{F},P;G)]^2}\ \text{(Stochastic coerciveness)} \tag{3.2}$$

Theorem 3.1. *Assume that (3.2) holds, and then there exists a unique solution*u*of system (3.1).*

Proof. Since the bilinear form$b(u,\Psi)$is continuous and stochastic coercive on$[H^1(\Omega,\mathcal{F},P;G)]^2$, then by Lax Milgram lemma there exist a unique solution of:

$$b(u,\Psi)=L(\Psi),\forall u\in[H^1(\Omega,F,\mathbf{P};G)]^2, \tag{3.3}$$

where$L(\Psi)$is continuous linear form defined on $[H^1(\Omega,\mathcal{F},P;G)]^2$by using Green's formula, we obtain (3.1):

$$L(\Psi)=\mathbb{E}\Big(\int_G (W\ \Psi)\ dx+\int_{\partial G}(g\ \Psi)\ d\partial G\Big),$$

then (3.3) is equivalent to

$$\begin{aligned} b(u,\Psi) &= \mathbb{E}\Big(\int_G (\nabla u\ \nabla\Psi)\ dx\Big) \\ &+ \mathbb{E}\Big(\int_{\partial G}\frac{\partial u(x)}{\partial V_A}\Psi\Big) \\ &= \mathbb{E}\Big(\int_G (W\ \Psi)\ dx+\int_{\partial G}(g\ \Psi)\ d\partial G\Big) \end{aligned}$$

Hence (3.3) is equivalent to (3.1) and there exists a unique solution of (3.1).

Formulation of the Optimal Control Problem with Neumann Conditions.

Here, we formulate the problem and establish necessary and sufficient conditions for the optimal control of distributed type. The space$[L^2(\Omega,\mathcal{F},P;G)]^2$is the space of controls. For a control$y\in[L^2(\Omega,\mathcal{F},P;G)]^2$,the state$u(y)$of the system is given by the solution of

$$\begin{cases} -\Delta u(y)=W(y)+y & \text{in } G_1 \\ \dfrac{\partial u(y)}{\partial V_A}=g & \text{on } \partial G. \end{cases} \tag{3.4}$$

The observation is given by$\chi(y)=u(y)$,the cost functional is given again by (3.4). The optimal control is characterize by the following theorem:

Theorem 3.2. *Assume that (3.2) holds, if the cost functional is given by (2.7), then there exists an optimal control*$y=(y_1,y_2)\in[L^2(\Omega,\mathcal{F},P;G)]^2$.*Moreover, it is characterized by the following equations and inequalities:*

$$\begin{cases} -\Delta h(y)=M\dfrac{\partial u(y)}{\partial V_A}-\chi_d & \text{in } G \\ \dfrac{\partial h(y)}{\partial V_A^*}=0. \end{cases}$$

Together with (3.4), where$p(u)$is the adjoint state

$$\mathbb{E}\Big(\int_G\Big(Ny(z-y)+Ny(z-y)+h(z-y)+h(z-y)\Big)dx\Big)\geq 0. \tag{3.5}$$

Remark 3.1

If constraints are absent, i.e. when$Y_{ad}=Y$,then$h(y)+Ny=0$or$y=-\dfrac{h(y)}{N}$the differential problem of finding the vector-function satisfies the following relations:

For the state process

$$\begin{cases} AU=W & \text{in } G \\ \dfrac{\partial U(y)}{\partial V_A}+\dfrac{h(u)}{N}=g & \text{on } \partial G. \end{cases}$$

For the adjoint state process

$$\begin{cases} Ah(y) - M\dfrac{\partial h(y)}{\partial V_A^*} = -\chi_d & \text{in } G \\ \\ \dfrac{\partial h(y)}{\partial V_A^*} = 0, & \text{on } \partial G. \end{cases}$$

Dirichlet and Neumann Elliptic Systems

In this section, we study the distributed control problem for elliptic systems involving Laplace operator. We consider the following elliptic equations:

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } G \\ \\ u(x) = 0 & \text{on } \partial G \end{cases} \tag{4.1}$$

where$G$is a bounded, continous and strictly domain in$\mathbb{R}^n$with boundary∂G While$u(x) \in H_0^1(G), f \in L^2(G)$is a state process and$W(x)$is a wiener process. We derive the existence and uniqueness of state of the system (4.1) in the following subsection.

The space$[L^2(G)]$being the space of controls. For a control$y \in [L^2(G)]$,the state process of the system$u$is given by the solution of the following system:

$$\begin{cases} -\Delta u(y) = f + y & \text{in } G \\ \\ u(y) = 0 & \text{on } \partial G, \end{cases} \tag{4.2}$$

. The observation equation is given by$\chi(y) \equiv u(y)$,the cost functional is given by:

$$C(y) = \bigg(\int_G (((u(y) - u(0) + (u(0) - \chi_d))^2)dx\bigg) + \bigg(\int_G M(z^2)\ dx\bigg), \tag{4.3}$$

whereχ_d in $[L^2(G)]$.

Then, the control problem is defined by:

$$\begin{cases} y \in Y_{ad} & \text{such that} \\ \\ C(z) = \inf C(y) & \forall z \in Y_{ad}, \end{cases}$$

where$Y_{ad}$is a closed convex subset from$[L^2(G)]$.

Since the cost functional (4.3) can be written as:

$$C(y) \quad = \quad \bigg(\int_G (((u(y) - u(0) + (u(0) - \chi_d))^2)dx\bigg) + \bigg(\int_G M(z^2)\ dx\bigg),$$

where

$$\Pi(y, z) = \bigg(\int_G \{(u(y) - u(0))^2 + (u(z) - u(0))^2\}dx\bigg) + \int_G \bigg(M(z^2)\bigg)\ dx, \tag{4.4}$$

$M > 0$ is a positive constant, then

$$L(z) = \mathbb{E}\bigg(\int_G (-u(0) + \chi_d)(u(i) - u(0)\ dx\bigg), \tag{4.5}$$

and$\Pi(y,y)$is a coercive on$[L^2(G)]$.Since$L(z)$is continuous on$[L^2(G)]$, then there exists a unique optimal control from the general theory in [8].

Since$C(y)$is differentiable and$Y_{ad}$is bounded, then the optimal control$z$is characterized (see e.g [8,9]). Using equations (4.4), (4.5), we get

$$\Pi(y, z-y) \geq L(z-y), \tag{4.6}$$

$$\left(\int_G ((h+My)(z-y)dx\right) \geq 0$$

If constraints are absent, i.e. when$Y_{ad} = Y$,then$h(z) + My = 0, z_j \neq y_jory = -\dfrac{h(z)}{M}$the differential problem of finding the vector-function satisfies the the following relations.

For the state equation

$$\begin{cases} Bu + \dfrac{h(z)}{M} = f & \text{in } G \\ \\ u = 0 & \text{on } \partial G. \end{cases}$$

For the adjoint state equation

$$\begin{cases} Bh(y) - u(y) = -\chi_d & \text{in } G \\ \\ h(y) = 0 & \text{on } \partial G. \end{cases}$$

There is no change in Neumann, where the difference are also in bilinear form, linear form and the cost functional.

Conclusion

This paper is the real development for works of Lions [10]. Also, we make a new vision for study of the optimal control, there are difference data. White noise , expectation in (linear and bilinear forms) and stochastic cost functional are the difference about [10]. This paper is scalar case that ensure [2].

References

[1] A.Debussche , M.Fuhrman and G.Tessitore, Optimal control of a stochastic heat equation with boundarynoise and boundary-control. ESAIM Control Optim Calc Var.13, 1, (2007), 178-205.

[2] A.S.Okb El Bab, Abd-Allah Hyder and A.M.Abdallah, OPTIMAL DISTRIBUTED CONTROL OF STOCHASTIC ELLIPTIC SYSTEMS WITH CONSTRAINTS, Journal of Advances in Mathematics ,12, 6, (2016),6347-6360 .

[3] Cla?udio Roberto A?vila da Silva Ju?nior, Milton Kist, and Marcelo Borges dos Santos, Application of Galerkin Method to Kirchhoff Plates Stochastic Bending Problem ,ISRN Applied Mathematics , 2014, (2014).

[4] H.A.Ghany and Abd-Allah Hyder, White noise functional solutions for the Wick-type two-dimensional stochastic Zakharov-Kuznetsov equations, International Review of Physics, 6, 2, (2012), 153-157.

[5] H.A.Ghany and Abd-Allah Hyder, Local and global well-posedness of stochastic Zakharov-Kuznetsov equation, Journal of Computational Analysis and Applications, 15, 3, (2013), 1332-1343.

[6] H.A.Ghany and Abd-Allah Hyder, Soliton solutions for Wick-type stochastic fractional KdV equations, International Journal of Mathematical Analysis, 7, 4, (2013), 2199-2208.

[7] H.A.Ghany and Abd-Allah Hyder, Exact solutions for the Wick-type stochastic time-fractional KdV equations, Kuwait Journal of Science, 41, 5, (2014), 75-84.

[8] H.A.Ghany, A.S.Okb El Bab, A.M.Zabal and Abd-Allah Hyder, The fractional coupled KdV equations: Exact solutions and white noise functional approach, Chinese Physics B, 22, 6, (2013), 080501.
[9] H.A.Ghany and Abd-Allah Hyder, Abundant solutions of Wick-type stochastic fractional 2D KdV equations, Chinese Physics B, 23, (2014).
[10] Lions, J.L., Optimal control of a system governed by partial differential equations, Springer -Verlag, New York 170, (1971).
[11] Lions, J.L., Some methods in the mathematical analysis of systems and their control, Sc. Press, Beijing, China, (1981).
[12] Pontryagin, L.S., et al., the Mathematical theory of optimal processes, Intersc. (1962).
[13] P.L.Chow; Turbulence and Related Problems,in Probabilistic Analysis and Related Topics, 1,edited by A.T.Bharucha-Reid, Academic Press,New York, 1987.
[14] Sergienko, I.V., Deineka, V.S., Optimal control of an elliptic system with conjugation conditions and Neumann boundary conditions, Cybernetic and Systems Analysis, 40, 6, (2004).
[15] X.Y.Zhou, On the necessary conditions of optimal controls for stochastic partial differential equations, SIAM J. Control Optim., 31, 6, (1993), 1462-1478.

A. S. Okb El Bab
Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City, Cairo, Egypt
E-mail address: ahmedokbelbab@yahoo.com

Abd-Allah Hyder
Department of Engineering Mathematics and Physics, Faculty of Engineering, Al-Azhar University, Cairo, Egypt
E-mail address: abdallah.hyder@yahoo.com

A. M. Abdallah
Department of Basic Science, Higher Technological Institute, Tenth of Ramadan, Egypt.
E-mail address: ahmedabdallah6236@yahoo.com