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EXISTENCE OF POSITIVE SOLUTIONS FOR ITERATIVE SYSTEMS OF NONLINEAR m-POINT BOUNDARY VALUE PROBLEMS ON TIME SCALES

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Abstract. In this paper, we establish the existence of positive solutions for the iterative system of nonlinear dynamic equations on time scales

$$y_i^{\Delta\Delta}(t) + p_i(t)f_i(y_{i+1}(t)) = 0, \quad 1 \le i \le n, \quad t \in [t_1, \sigma(t_m)]_{\mathbb{T}},$$
$$y_{n+1}(t) = y_1(t), \quad t \in [t_1, \sigma(t_m)]_{\mathbb{T}},$$

satisfying the m-point boundary conditions

$$y_i(t_1) = 0,$$
$$\alpha y_i(\sigma(t_m)) + \beta y_i^{\Delta}(\sigma(t_m)) = \sum_{k=2}^{m-1} y_i^{\Delta}(t_k), \quad 1 \le i \le n,$$

by applying Guo–Krasnosel'skii fixed point theorem.

1. Introduction

The theory of time scales was introduced by Hilger [18] not only to unify continuous and discrete theory, but also to provide an accurate information of phenomena that manifest themselves partly in continuous time and partly in discrete time. This theory [1, 5, 6] can be applied to various real life situations like epidemic models, stock markets and mathematical modeling of physical and biological systems.

The existence of positive solutions of boundary value problems have created a great deal of interest due to wide applicability in both theory and applications. By using fixed point theorems in cones, Fink and Gatica [7], Wang [27], Zhou and Xu [28], Henderson et al. [9, 10, 11, 12] have studied existence of positive solutions for system of nonlinear boundary value problems associated with ordinary differential equations. Agarwal and O'Regan [2], Sun et al. [25, 26], Henderson et al. [13, 14, 15, 16, 17] considered the system of nonlinear boundary value problems associated with difference equations and established the existence of positive solutions to boundary value problems by using various techniques. In recent years, much attention is paid in establishing the existence of positive solutions for iterative

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systems of nonlinear boundary value problems on time scales by the researchers. For some recent contributions, we refer to [3, 4, 19, 21, 22, 23, 24].

Motivated by the papers mentioned above, in this paper, we establish the existence of positive solutions for the iterative systems of second order nonlinear dynamic equations on time scales

$$\left.\begin{aligned} y_i^{\Delta\Delta}(t)+p_i(t)f_i(y_{i+1}(t))=0, \quad 1\le i\le n, \quad t\in[t_1,\sigma(t_m)]_{\mathbb{T}},\\ y_{n+1}(t)=y_1(t), \quad t\in[t_1,\sigma(t_m)]_{\mathbb{T}}, \end{aligned}\right\} \tag{1}$$

satisfying the m-point boundary conditions

$$\left.\begin{aligned} y_i(t_1)=0,\\ \alpha y_i(\sigma(t_m))+\beta y_i^{\Delta}(\sigma(t_m))=\sum_{k=2}^{m-1} y_i^{\Delta}(t_k), \quad 1\le i\le n, \end{aligned}\right\} \tag{2}$$

where $\mathbb{T}$ is the time scale with $t_1, t_2, \cdots, t_{m-1}, \sigma(t_m), \sigma^2(t_m)\in\mathbb{T}$, $0\le t_1<t_2<\cdots<t_{m-1}<\sigma(t_m)$, $\alpha>0$, $\beta>m-2$ are real numbers and $m\ge 3$. We assume the following conditions hold through out the paper:

($A1$) $f_i:\mathbb{R}^+\to\mathbb{R}^+$ is continuous for $1\le i\le n$,

($A2$) $p_i:[t_1,\sigma(t_m)]_{\mathbb{T}}\to\mathbb{R}^+$ is continuous and p_i does not vanish identically on any closed subinterval of $[t_1,\sigma(t_m)]_{\mathbb{T}}$ for $1\le i\le n$,

($A3$) α and β are positive constants such that $\alpha>\frac{\beta}{t_2-t_1}$ and $\beta>m-2$.

We define the nonnegative extended real numbers f_{i0} and $f_{i\infty}$ by

$$f_{i0}=\lim_{x\to 0^+}\frac{f_i(x)}{x} \text{ and } f_{i\infty}=\lim_{x\to\infty}\frac{f_i(x)}{x}, \text{ for } 1\le i\le n,$$

and assume that they will exist. When $f_{i0}=0$ and $f_{i\infty}=\infty$ for $1\le i\le n$ is the called super linear case and $f_{i0}=\infty$ and $f_{i\infty}=0$ for $1\le i\le n$ is called the sub linear case.

The rest of the paper is organized as follows. In Section 2, we construct the Green's function for the homogeneous problem corresponding to (1)-(2) and estimate bounds for the Green's function. In Section 3, we establish the existence of at least one positive solution of the boundary value problem (1)-(2) by using the Guo–Krasnosel'skii fixed point theorem for operators on a cone in a Banach space. Finally as an application, we demonstrate our results with an example.

2. Green's Function and Bounds

In this section, we construct the Green's function for the homogeneous problem corresponding to (1)-(2) and estimate bounds for the Green's function.

Let $G(t,s)$ be the Green's function for the homogeneous boundary value problem

$$-y_1^{\Delta\Delta}(t)=0, \quad t\in[t_1,\sigma(t_m)]_{\mathbb{T}}, \tag{3}$$

$$\left.\begin{aligned} y_1(t_1)=0,\\ \alpha y_1(\sigma(t_m))+\beta y_1^{\Delta}(\sigma(t_m))=\sum_{k=2}^{m-1} y_1^{\Delta}(t_k), \ m\ge 3. \end{aligned}\right\} \tag{4}$$

Lemma 1. [22, 24] *Let $d = \alpha(\sigma(t_m) - t_1) + \beta - m + 2 \neq 0$. Then the Green's function $G(t,s)$ for the homogeneous boundary value problem* (3)-(4) *is given by*

$$G(t,s) = \begin{cases} G_1(t,s), & t_1 \le s \le \sigma(s) \le t_2, \\ G_2(t,s), & t_2 \le s \le \sigma(s) \le t_3, \\ \cdot & \\ \cdot & \\ \cdot & \\ G_{m-2}(t,s), & t_{m-2} \le s \le \sigma(s) \le t_{m-1}, \\ G_{m-1}(t,s), & t_{m-1} \le s \le \sigma(s) \le \sigma(t_m), \end{cases} \tag{5}$$

where

$$G_j(t,s) = \begin{cases} \frac{1}{d}[(\alpha_1(\sigma(t_m) - t) + \beta_1 - m + j + 1)(\sigma(s) - t_1) + (j-1)(t - \sigma(s))], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \sigma(s) \le t, \\ \frac{1}{d}(t - t_1)[\alpha_1(\sigma(t_m) - \sigma(s)) + \beta_1 - m + j + 1], \quad t \le s, \end{cases}$$

for $j = 1, 2, \cdots, m-1$.

Lemma 2. [22, 24] *Assume that the condition $(A3)$ is satisfied. Then the Green's function $G(t,s)$ of* (3)-(4) *is positive, for all $(t,s) \in (t_1, \sigma(t_m))_{\mathbb{T}} \times (t_1, t_m)_{\mathbb{T}}$.*

Theorem 3. [22, 24] *Assume that the condition $(A3)$ is satisfied. Then the Green's function $G(t,s)$ in* (5) *satisfies the following inequality,*

$$g(t)G(\sigma(s),s) \le G(t,s) \le G(\sigma(s),s), \text{ for all } (t,s) \in [t_1, \sigma(t_m)]_{\mathbb{T}} \times [t_1, t_m]_{\mathbb{T}}, \tag{6}$$

where

$$g(t) = min\left\{ \frac{\sigma(t_m) - t}{\sigma(t_m) - t_1}, \ \frac{t - t_1}{\sigma(t_m) - t_1} \right\}.$$

Lemma 4. [22, 24] *Assume that the condition $(A3)$ is satisfied and $s \in [t_1, t_m]_{\mathbb{T}}$. Then the Green's function $G(t,s)$ in* (5) *satisfies*

$$\min_{t \in [t_{m-1}, \sigma(t_m)]_{\mathbb{T}}} G(t,s) \ge kG(\sigma(s), s),$$

where

$$k = \frac{\beta - m + 2}{\alpha(\sigma(t_m) - t_1) + \beta - m + 2} < 1. \tag{7}$$

Now, we express the solution of the boundary value problem (1)-(2) in to an equivalent integral equation, see [3]. Therefore, an n-tuple $(y_1(t), y_2(t), \cdots, y_n(t))$ is a solution of the boundary value problem (1)-(2) if and only if

$$y_i(t) = \int_{t_1}^{\sigma(t_m)} G(t,s) p_i(s) f_i(y_{i+1}(s)) \Delta s, \ 1 \le i \le n, \ t \in [t_1, \sigma(t_m)]_{\mathbb{T}},$$

where

$$y_{n+1}(t) = y_1(t), \ t \in [t_1, \sigma(t_m)]_{\mathbb{T}}.$$

In particular

$$y_1(t) = \int_{t_1}^{\sigma(t_m)} G(t,s_1)p_1(s_1)f_1\Bigg(\int_{t_1}^{\sigma(t_m)} G(s_1,s_2)p_2(s_2)\cdots$$
$$f_{n-1}\Bigg(\int_{t_1}^{\sigma(t_m)} G(s_{n-1},s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n\Bigg)\cdots\Delta s_2\Bigg)\Delta s_1,$$

$t \in [t_1, \sigma(t_m)]_{\mathbb{T}}$.

To establish the existence of positive solutions for the boundary value problem (1)-(2), we will employ the following Guo–Krasnosel'skii fixed point theorem [8, 20].

Theorem 5. [8, 20] *Let X be a Banach Space, $\kappa \subseteq X$ be a cone and suppose that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $T : \kappa \cap (\overline{\Omega}_2 \backslash \Omega_1) \to \kappa$ is completely continuous operator such that either*

(i) $\|Tu\| \le \|u\|,\ u \in \kappa \cap \partial\Omega_1$ and $\|Tu\| \ge \|u\|,\ u \in \kappa \cap \partial\Omega_2$, or
(ii) $\|Tu\| \ge \|u\|,\ u \in \kappa \cap \partial\Omega_1$ and $\|Tu\| \le \|u\|,\ u \in \kappa \cap \partial\Omega_2$ holds.

Then T has a fixed point in $\kappa \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Positive Solutions in a Cone

In this section, we establish criteria for the existence of at least one positive solution of the boundary value problem (1)-(2).

For our construction, let $B = \{x \mid x \in C[t_1, \sigma(t_m)]_{\mathbb{T}}\}$ with the norm

$$\|x\| = \sup_{t \in [t_1, \sigma(t_m)]_{\mathbb{T}}} |x(t)|.$$

Then $(B, \|\cdot\|)$ is a Banach space, we refer [8, 13]. Define a cone $P \subset B$ by

$$P = \Big\{x \in B \mid x(t) \ge 0 \text{ on } [t_1, \sigma(t_m)]_{\mathbb{T}} \text{ and } \min_{t \in [t_{m-1}, \sigma(t_m)]_{\mathbb{T}}} x(t) \ge k\|x\|\Big\},$$

where k is given in (7).

Now, we define an integral operator $T : P \to B$, for $y_1 \in P$, by

$$Ty_1(t) = \int_{t_1}^{\sigma(t_m)} G(t,s_1)p_1(s_1)f_1\Bigg(\int_{t_1}^{\sigma(t_m)} G(s_1,s_2)p_2(s_2)\cdots$$
$$f_{n-1}\Bigg(\int_{t_1}^{\sigma(t_m)} G(s_{n-1},s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n\Bigg)\cdots\Delta s_2\Bigg)\Delta s_1. \qquad (8)$$

Notice from $(A1)$, $(A2)$ and Lemma 2 that, for $y_1 \in P$, $Ty_1(t) \ge 0$ on $[t_1, \sigma(t_m)]_{\mathbb{T}}$. Also, for $y_1 \in P$, we have from Theorem 3, that

$$Ty_1(t) \le \int_{t_1}^{\sigma(t_m)} G(\sigma(s_1),s_1)p_1(s_1)f_1\Bigg(\int_{t_1}^{\sigma(t_m)} G(s_1,s_2)p_2(s_2)\cdots$$
$$f_{n-1}\Bigg(\int_{t_1}^{\sigma(t_m)} G(s_{n-1},s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n\Bigg)\cdots\Delta s_2\Bigg)\Delta s_1$$

so that

$$\begin{aligned}\|Ty_1\| \leq &\int_{t_1}^{\sigma(t_m)} G(\sigma(s_1), s_1)p_1(s_1)f_1\Bigg(\int_{t_1}^{\sigma(t_m)} G(s_1, s_2)p_2(s_2)\cdots \\ &f_{n-1}\Bigg(\int_{t_1}^{\sigma(t_m)} G(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n\Bigg)\cdots\Delta s_2\Bigg)\Delta s_1.\end{aligned} \tag{9}$$

Next, if $y_1 \in P$, we have from Lemma 4 and (9) that

$$\begin{aligned}&\min_{t\in[t_{m-1},\sigma(t_m)]_{\mathbb{T}}} Ty_1(t) = \\ &\min_{t\in[t_{m-1},\sigma(t_m)]_{\mathbb{T}}}\Bigg\{\int_{t_1}^{\sigma(t_m)} G(t, s_1)p_1(s_1)f_1\Bigg(\int_{t_1}^{\sigma(t_m)} G(s_1, s_2)p_2(s_2)\cdots \\ &f_{n-1}\Bigg(\int_{t_1}^{\sigma(t_m)} G(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n\Bigg)\cdots\Delta s_2\Bigg)\Delta s_1\Bigg\} \\ &\geq k\int_{t_1}^{\sigma(t_m)} G(\sigma(s_1), s_1)p_1(s_1)f_1\Bigg(\int_{t_1}^{\sigma(t_m)} G(s_1, s_2)p_2(s_2)\cdots \\ &f_{n-1}\Bigg(\int_{t_1}^{\sigma(t_m)} G(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n\Bigg)\cdots\Delta s_2\Bigg)\Delta s_1 \\ &\geq k\|Ty_1\|.\end{aligned}$$

Hence, $Ty_1 \in P$ and so $T : P \to P$. Further, the operator T is completely continuous operator by an application of the Ascoli-Arzela Theorem.

Theorem 6. *Assume that the conditions (A1)-(A3) are satisfied. If $f_{i0} = 0$ and $f_{i\infty} = \infty$, for $1 \leq i \leq n$ hold, then the boundary value problem (1)-(2) has at least one positive solution.*

Proof. Let T be the cone preserving, completely continuous operator that was defined in (8). From the definitions of $f_{i0} = 0$, $1 \leq i \leq n$, there exist $\eta_1 > 0$ and $H_1 > 0$ such that, for each $1 \leq i \leq n$,

$$f_i(x) \leq \eta_1 x,\ 0 < x \leq H_1,$$

where η_1 satisfies

$$\eta_1 \int_{t_1}^{\sigma(t_m)} G(\sigma(s), s)p_i(s)\Delta s \leq 1. \tag{10}$$

Let $y_1 \in P$ with $\|y_1\| = H_1$. Then from Theorem 3, for $t_1 \le s_{n-1} \le \sigma(t_m)$, we have

$$\begin{aligned}
\int_{t_1}^{\sigma(t_m)} & G(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n \\
&\le \int_{t_1}^{\sigma(t_m)} G(\sigma(s_n), s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n \\
&\le \int_{t_1}^{\sigma(t_m)} G(\sigma(s_n), s_n)p_n(s_n)\eta_1 y_1(s_n)\Delta s_n \\
&\le \eta_1 \int_{t_1}^{\sigma(t_m)} G(\sigma(s_n), s_n)p_n(s_n)\|y_1\|\Delta s_n \\
&\le \|y_1\| = H_1.
\end{aligned}$$

It follows in a similar manner from Theorem 3, for $t_1 \le s_{n-2} \le \sigma(t_m)$,

$$\begin{aligned}
&\int_{t_1}^{\sigma(t_m)} G(s_{n-2}, s_{n-1})p_{n-1}(s_{n-1}) \\
&f_{n-1}\left(\int_{t_1}^{\sigma(t_m)} G(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n \right)\Delta s_{n-1} \\
&\le \int_{t_1}^{\sigma(t_m)} G(\sigma(s_{n-1}), s_{n-1})p_{n-1}(s_{n-1})\eta_1 H_1 \Delta s_{n-1} \le H_1.
\end{aligned}$$

Continuing with this bootstrapping argument, we have, for $t_1 \le t \le \sigma(t_m)$,

$$\begin{aligned}
\int_{t_1}^{\sigma(t_m)} G(t, s_1)p_1(s_1)f_1\Bigg(\int_{t_1}^{\sigma(t_m)} & G(s_1, s_2)p_2(s_2) \cdots \\
& f_n(y_1(s_n))\Delta s_n \cdots \Delta s_2 \Bigg)\Delta s_1 \le H_1,
\end{aligned}$$

so that, for $t_1 \le t \le \sigma(t_m)$,

$$Ty_1(t) \le H_1.$$

Hence, $\|Ty_1\| \le H_1 = \|y_1\|$. If we set

$$\Omega_1 = \{x \in B \mid \|x\| < H_1\},$$

then

$$\|Ty_1\| \le \|y_1\|, \text{ for } y_1 \in P \cap \partial\Omega_1. \tag{11}$$

Further, since $f_{i\infty} = \infty$, $1 \le i \le n$, there exist $\eta_2 > 0$ and $\overline{H}_2 > 0$ such that, for each $1 \le i \le n$,

$$f_i(x) \ge \eta_2 x, \ x \ge \overline{H}_2,$$

where η_2 satisfies

$$k^2\eta_2 \int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s), s)p_i(s)\Delta s \ge 1. \tag{12}$$

Let

$$H_2 = \max\left\{ 2H_1, \frac{\overline{H}_2}{k} \right\}.$$

Choose $y_1 \in P$ and $\|y_1\| = H_2$. Then,

$$\min_{t\in[t_{m-1},\sigma(t_m)]_{\mathbb{T}}} y_1(t) \geq k\|y_1\| \geq \overline{H}_2.$$

From Lemma 4, for $t_1 \leq s_{n-1} \leq \sigma(t_m)$, we have

$$\begin{aligned}
&\int_{t_1}^{\sigma(t_m)} G(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n \\
&\qquad\qquad \geq \int_{t_{m-1}}^{\sigma(t_m)} G(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n \\
&\qquad\qquad \geq k\int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s_n), s_n)p_n(s_n)\eta_2 y_1(s_n)\Delta s_n \\
&\qquad\qquad \geq k^2\eta_2\int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s_n), s_n)p_n(s_n)\|y_1\|\Delta s_n \\
&\qquad\qquad \geq \|y_1\| = H_2.
\end{aligned}$$

It follows in a similar manner from Lemma 4, for $t_1 \leq s_{n-2} \leq \sigma(t_m)$,

$$\begin{aligned}
&\int_{t_1}^{\sigma(t_m)} G(s_{n-2}, s_{n-1})p_{n-1}(s_{n-1}) \\
&f_{n-1}\left(\int_{t_1}^{\sigma(t_m)} G(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n\right)\Delta s_{n-1} \\
&\geq k\int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s_{n-1}), s_{n-1})p_{n-1}(s_{n-1})\eta_2 H_2\Delta s_{n-1} \\
&\geq k^2\eta_2 H_2\int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s_{n-1}), s_{n-1})p_{n-1}(s_{n-1})\Delta s_{n-1} \geq H_2.
\end{aligned}$$

Again, using a bootstrapping argument, we have

$$\begin{aligned}
\int_{t_1}^{\sigma(t_m)} G(t, s_1)p_1(s_1)f_1\Bigg(\int_{t_1}^{\sigma(t_m)} &G(s_1, s_2)p_2(s_2)\cdots \\
&f_n(y_1(s_n))\Delta s_n\cdots\Delta s_2\Bigg)\Delta s_1 \geq H_2,
\end{aligned}$$

so that

$$Ty_1(t) \geq H_2 = \|y_1\|.$$

Hence, $\|Ty_1\| \geq \|y_1\|$. So, if we set

$$\Omega_2 = \{x \in B \mid \|x\| < H_2\},$$

then

$$\|Ty_1\| \geq \|y_1\|, \text{ for } y_1 \in P \cap \partial\Omega_2. \tag{13}$$

Applying Theorem 5 to (11) and (13), it follows that T has a fixed point $y_1 \in P\cap(\overline{\Omega}_2\backslash\Omega_1)$. As such, setting $y_{n+1} = y_1$, we obtain a positive solution $(y_1, y_2, \cdots, y_n)$ of (1)-(2). □

Theorem 7. *Assume that the conditions* (A1)-(A3) *are satisfied. If* $f_{i0} = \infty$ *and* $f_{i\infty} = 0$*, for* $1 \leq i \leq n$ *hold, then the boundary value problem* (1)-(2) *has at least one positive solution.*

Proof. Let T be the cone preserving, completely continuous operator that was defined in (8). Since $f_{i0} = \infty$, $1 \leq i \leq n$, there exist $\eta_3 > 0$ and $H_3 > 0$ such that, for each $1 \leq i \leq n$,

$$f_i(x) \geq \eta_3 x, \ 0 < x \leq H_3,$$

where $\eta_3 \geq \eta_2$ and η_2 is given in (12).

Let $y_1 \in P$ with $\|y_1\| = H_3$. Then from Lemma 4, for $t_1 \leq s_{n-1} \leq \sigma(t_m)$, we have

$$\begin{aligned}
&\int_{t_1}^{\sigma(t_m)} G(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n \\
&\qquad \geq \int_{t_{m-1}}^{\sigma(t_m)} G(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n \\
&\qquad \geq k\int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s_n), s_n)p_n(s_n)\eta_3 y_1(s_n)\Delta s_n \\
&\qquad \geq k^2\eta_3\int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s_n), s_n)p_n(s_n)\|y_1\|\Delta s_n \\
&\qquad \geq \|y_1\| = H_3.
\end{aligned}$$

It follows in a similar manner from Lemma 4, for $t_1 \leq s_{n-2} \leq \sigma(t_m)$,

$$\begin{aligned}
&\int_{t_1}^{\sigma(t_m)} G(s_{n-2}, s_{n-1})p_{n-1}(s_{n-1}) \\
&f_{n-1}\left(\int_{t_1}^{\sigma(t_m)} G(s_{n-1}, s_n)p_n(s_n)f_n(y_1(s_n))\Delta s_n\right)\Delta s_{n-1} \\
&\geq k\int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s_{n-1}), s_{n-1})p_{n-1}(s_{n-1})\eta_3 H_3\Delta s_{n-1} \\
&\geq k^2\eta_3 H_3\int_{t_{m-1}}^{\sigma(t_m)} G(\sigma(s_{n-1}), s_{n-1})p_{n-1}(s_{n-1})\Delta s_{n-1} \geq H_3.
\end{aligned}$$

Continuing with this bootstrapping argument, it follows that

$$\begin{aligned}
\int_{t_1}^{\sigma(t_m)} G(t, s_1)p_1(s_1)f_1\Bigg(\int_{t_1}^{\sigma(t_m)} G(s_1, s_2)p_2(s_2)\cdots& \\
f_n(y_1(s_n))\Delta s_n\cdots\Delta s_2\Bigg)\Delta s_1 \geq H_3&,
\end{aligned}$$

so that

$$Ty_1(t) \geq H_3 = \|y_1\|.$$

Hence, $\|Ty_1\| \geq \|y_1\|$. So, if we set

$$\Omega_3 = \{x \in B \mid \|x\| < H_3\},$$

then

$$\|Ty_1\| \geq \|y_1\|, \text{ for } y_1 \in P \cap \partial\Omega_3. \tag{14}$$

Next, since $f_{i\infty} = 0$, $1 \leq i \leq n$, there exist $\eta_4 > 0$ and $\overline{H}_4 > 0$ such that, for each $1 \leq i \leq n$,

$$f_i(x) \leq \eta_4 x, \ x \geq \overline{H}_4,$$

where $\eta_4 \leq \eta_1$ and η_1 is given in (10).

For each $1 \leq i \leq n$, set

$$f_i^*(x) = \sup_{0 \leq s \leq x} f_i(s).$$

Then, it is straightforward that, for each $1 \leq i \leq n$, f_i^* is a nondecreasing real-valued function, $f_i \leq f_i^*$ and

$$\lim_{x \to \infty} \frac{f_i^*(x)}{x} = 0.$$

It follows that there exists $H_4 > \max\{2H_3, \overline{H}_4\}$ such that, for each $1 \leq i \leq n$,

$$f_i^*(x) \leq f_i^*(H_4), \ 0 < x \leq H_4.$$

Choose $y_1 \in P$ with $\|y_1\| = H_4$. Then, using the usual bootstrapping argument, we have

$$\begin{aligned}
Ty_1(t) &= \int_{t_1}^{\sigma(t_m)} G(t, s_1)p_1(s_1)f_1\left(\int_{t_1}^{\sigma(t_m)} G(s_1, s_2)p_2(s_2) \cdots \Delta s_2\right)\Delta s_1 \\
&\leq \int_{t_1}^{\sigma(t_m)} G(t, s_1)p_1(s_1)f_1^*\left(\int_{t_1}^{\sigma(t_m)} G(s_1, s_2)p_2(s_2) \cdots \Delta s_2\right)\Delta s_1 \\
&\leq \int_{t_1}^{\sigma(t_m)} G(\sigma(s_1), s_1)p_1(s_1)f_1^*(H_4)\Delta s_1 \\
&\leq \int_{t_1}^{\sigma(t_m)} G(\sigma(s_1), s_1)p_1(s_1)\eta_4 H_4 \Delta s_1 \\
&\leq H_4 = \|y_1\|.
\end{aligned}$$

Hence, $\|Ty_1\| \leq \|y_1\|$. So, if we set

$$\Omega_4 = \{x \in B \mid \|x\| < H_4\},$$

then

$$\|Ty_1\| \leq \|y_1\|, \text{ for } y_1 \in P \cap \partial\Omega_4. \tag{15}$$

Applying Theorem 5 to (14) and (15), we obtain that T has a fixed point $y_1 \in P \cap (\overline{\Omega}_4 \backslash \Omega_3)$, which in turn with $y_{n+1} = y_1$, yields an n-tuple $(y_1, y_2, \cdots, y_n)$ satisfying (1)-(2). The proof is completed. $\square$

4. Examples

Let us consider the examples to illustrate our results.

Example 1. Let $\mathbb{T} = \{(\frac{1}{2})^p : p \in \mathbb{N}_0\} \cup [1, 3]$, $n = 2$, $m = 3$, $\alpha = 5$, $\beta = 2$, $t_1 = \frac{1}{2}$, $t_2 = 1$ and $t_3 = 2$. Now, consider the iterative systems of nonlinear dynamic equations on time scales

$$\left.\begin{aligned}
y_1^{\Delta\Delta}(t) + p_1(t)f_1(y_2(t)) = 0, \quad t \in \left[\frac{1}{2}, \sigma(2)\right]_{\mathbb{T}}, \\
y_2^{\Delta\Delta}(t) + p_2(t)f_2(y_1(t)) = 0, \quad t \in \left[\frac{1}{2}, \sigma(2)\right]_{\mathbb{T}},
\end{aligned}\right\} \tag{16}$$

satisfying three-point boundary conditions

$$\left.\begin{aligned} y_1\left(\frac{1}{2}\right) = 0,\ 5y_1(\sigma(2)) + 2y_1^{\Delta}(\sigma(2)) = y_1^{\Delta}(1),\\ y_2\left(\frac{1}{2}\right) = 0,\ 5y_2(\sigma(2)) + 2y_2^{\Delta}(\sigma(2)) = y_2^{\Delta}(1),\end{aligned}\right\} \tag{17}$$

where $p_1(t) = p_2(t) = t$, $f_1(y_2) = y_2^2(1 + e^{-2y_2})$ and $f_2(y_1) = y_1^2(1 - 3e^{-y_1})$.
Then all the conditions of Theorem 6 are satisfied and hence, the boundary value problem (16)-(17) has at least one positive solution.

Example 2. Let Let $\mathbb{T} = \{(\frac{1}{2})^p : p \in \mathbb{N}_0\} \cup [1, 3]$, $n = 2$, $m = 3$, $\alpha = 5$, $\beta = 2$, $t_1 = \frac{1}{2}$, $t_2 = 1$ and $t_3 = 2$. Now, consider the iterative systems of nonlinear dynamic equations on time scales

$$\left.\begin{aligned} y_1^{\Delta\Delta}(t) + p_1(t)f_1(y_2(t)) = 0, \quad t \in \left[\frac{1}{2}, \sigma(2)\right]_{\mathbb{T}},\\ y_2^{\Delta\Delta}(t) + p_2(t)f_2(y_1(t)) = 0, \quad t \in \left[\frac{1}{2}, \sigma(2)\right]_{\mathbb{T}},\end{aligned}\right\} \tag{18}$$

satisfying three-point boundary conditions

$$\left.\begin{aligned} y_1\left(\frac{1}{2}\right) = 0,\ 5y_1(\sigma(2)) + 2y_1^{\Delta}(\sigma(2)) = y_1^{\Delta}(1),\\ y_2\left(\frac{1}{2}\right) = 0,\ 5y_2(\sigma(2)) + 2y_2^{\Delta}(\sigma(2)) = y_2^{\Delta}(1),\end{aligned}\right\} \tag{19}$$

where $p_1(t) = p_2(t) = \frac{t}{2}$, $f_1(y_2) = y_2^{\frac{2}{3}}$ and $f_2(y_1) = y_1^{\frac{3}{4}}$.
Then all the conditions of Theorem 7 are satisfied and hence, the boundary value problem (18)-(19) has at least one positive solution.

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References

[1] R. P. Agarwal and M. Bohner, Basic calculus on time scales and some of its applications, Results Math., 35, 3-22, 1999.

[2] R. P. Agarwal and D. O'Regan, A coupled system of difference equations, Appl. Math. Comput., 114, 39-49, 2000.

[3] M. Benchohra, F. Berhoun, S. Hamani, J. Henderson, S. K. Ntouyas, A. Ouahab and I. K. Purnaras, Eigenvalues for iterative systems of nonlinear boundary value problems on time scales, Nonlinear Dyn. Sys. Theory, 9(1), 11-22, 2009.

[4] M. Benchohra, J. Henderson and S. K. Ntouyas, Eigenvalue problems for systems of nonlinear boundary value problems on time scales, Adv. Difference Equ., 2007, Article ID: 31640, 1-10, 2007.

[5] M. Bohner and A. Peterson, Dynamic Equations on Time Scales, An Introduction with Applications, Birkh?auser, Boston, MA, 2001.

[6] M. Bohner and A. C. Peterson, Advances in Dynamic Equations on Time Scales, Birkhauser, Boston, 2003.

[7] A. M. Fink and J. A. Gatica, Positive solutions of second order systems of boundary value problem, J. Math. Anal. Appl., 180, 93-108, 1993.

[8] D. Guo and V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, Orlando, 1988.

[9] J. Henderson and S. K. Ntouyas, Positive solutions for systems of nth order three-point nonlocal boundary value problems, Electron. J. Qual. Theory Differ. Equ., 2007(18), 1-12, 2007.
[10] J. Henderson and S. K. Ntouyas, Positive solutions for systems of three-point nonlinear boundary value problems, Austr. J. Math. Anal. Appl., 5(1), Artticle: 11, 1-9, 2008.
[11] J. Henderson, S. K. Ntouyas and I. K. Purnaras, Positive solutions for systems of generalized three-point nonlinear boundary value problems, Comment. Math. Univ. Carolin., 49, 79-91, 2008.
[12] J. Henderson, S. K. Ntouyas and I. K. Purnaras, Positive solutions for systems of second order four-point nonlinear boundary value problems, Comm. Appl. Anal., 12(1), 29-40, 2008.
[13] J. Henderson, S. K. Ntouyas and I. K. Purnaras, Positive solutions for systems of nonlinear discrete boundary value problems, J. Difference Equ. Appl., 15(10), 895-912, 2009.
[14] J. Henderson and R. Luca, Existence of positive solutions for a system of second-order multi-point discrete boundary value problems, J. Difference Equ. Appl., 19(11), 1889-1906, 2013.
[15] J. Henderson and R. Luca, Positive solutions for systems of second-order difference equations, Discrete Dyn. Nature Soc., 2015, Article ID: 468648, 1-8, 2015.
[16] J. Henderson and R. Luca, Positive solutions for a system of difference equations with coupled multi-point boundary conditions, J. Difference Equ. Appl., 22(2), 188-216, 2016.
[17] J. Henderson and R. Luca, Existence and multiplicity of positive solutions for a system of difference equations with coupled boundary conditions, J. Appl. Anal. Comput., 7(1), 134-146, 2017.
[18] S. Hilger, Analysis on measure chains - A unified approach to continuous and discrete calculus, Results Math., 18(4), 18-56, 1990.
[19] I. Y. Karaca and F. Tokmak, Eigenvalues for iterative systems of nonlinear m-point boundary value problems on time scales, Boundary Value Problems, 2014(63), 1-17, 2013.
[20] M. A. Krasnosel'skii, Positive Solutions of Operator Equations, P. Noordhoff Ltd, Groningen, The Netherlands, 1964.
[21] K. R. Prasad and N. Sreedhar, Solvability of iterative systems of nonlinear m-point boundary value problems on time scales, Func. Anal.-TMA, 2, 56-68, 2016.
[22] K. R. Prasad, N. Sreedhar and Y. Narasimhulu, Eigenvalue intervals for iterative systems of nonlinear m-point boundary value problems on time scales, Differ. Equ. Dyn. Syst., 22(4), 353-368, 2014.
[23] K. R. Prasad, N. Sreedhar and M. A. S. Srinivas, Eigenvalue intervals for iterative systems of nonlinear three-point boundary value problems on time scales, Nonlinear Studies, 22(3), 419-431, 2015.
[24] K. R. Prasad, N. Sreedhar and M. A. S. Srinivas, Existence of positive solutions for systems of second order multi-point boundary value problems on time scales, Proc. Indian Acad. Sci., 125(3), 353-370, 2015.
[25] J. P. Sun and W. T. Li, Multiple positive solutions of a discrete difference system, Appl. Math. Comput., 143, 213-221, 2003.
[26] J. P. Sun, Y. H. Zhao and W. T. Li, Existence of positive solution for second-order nonlinear discrete system with parameter, Math. Comp. Modelling, 41, 493-499, 2005.
[27] H. Wang, On the number of positive solutions of nonlinear systems, J. Math. Anal. Appl., 281, 287-306, 2003.
[28] Y. Zhou and Y. Xu, Positive solutions of three-point boundary value problems for systems of nonlinear second order ordinary differential equations, J. Math. Anal. Appl., 320, 578-590, 2006.

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