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ON THE MAXIMAL AND MINIMAL SOLUTIONS OF A STOCHASTIC DIFFERENTIAL EQUATION

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Abstract. In this paper we are concerned with a problem of stochastic differential equation with nonlocal condition, the solution is represented as a stochastic integral equation that contains stochastic Riemann integral and stochastic Riemann-Stieltjes integral. We study the existence of at least one mean square continuous solution for this type. The existence of the maximal and minimal solutions will be proved.

1. Introduction

Stochastic differential equations have been extensively studied by several authors (see [1]-[13] and references therein).
Let $\{W(t), t \in [0,T]\}$ be a Brownian motion, let X_0 be a second order random variable independent of the Brownian motion $\{W(t), t \in [0,T]\}$.
Let $g : [0,T] \to R^+$ is a continuous deterministic function.
Consider the stochastic differential equation

$$dX(t) = f(t, X(t))dt + g(t)dW(t), \ t \in (0,T] \tag{1}$$

with the random nonlocal initial condition

$$X(0) + \sum_{k=1}^{m} a_k X(\tau_k) = X_0, \ \tau_k \in (0,T) \tag{2}$$

where a_k are positive real numbers. The existence of unique mean square continuous solution of the problem (1)-(2), continuous dependence of random initial value X_0 and continuous dependence of nonrandom initial coefficients a_k have been proved in [4].
Here, the existence of at least one mean square continuous solution for the nonlocal problem will be studied. The existence of the maximal and minimal solutions will be proved.

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2. Preliminaries

Here we give some preliminaries which will be needed in our work.

Definition 1 [12][Random Caratheodory function] Let X be a stochastic process and let $t \in I = [a, b]$, a and b are real numbers. A stochastic function $f(t, X(\omega))$ is called a Caratheodory function if it satisfies the following conditions

(1) $f(t, X(.))$ is measurable for every t,
(2) $f(., X(\omega))$ is continuous for a.e. stochastic process X.

Theorem 1[11][Schauder and Tychonoff theorem]
Let Q be a closed bounded convex set in a Banach space and Let T be a completely continuous operator on Q such that $T(Q) \subset Q$. Then T has at least one fixed point in Q. That is, there is at least one $x^* \in Q$ such that $T(x^*) = x^*$.

Definition 2 [9] A family of real random functions $(X_1(t), X_2(t), ..., X_k(t))$ is uniformly bounded in mean square sense if there exist a $\beta \in R$ (β is finite) such that $E(X_n^2(t)) < \beta$ for all $n \geq 1$ and all $t \in I = [a, b]$, where a, b are real numbers.

Definition 3 [9] A family of real random functions $(X_1(t), X_2(t), ..., X_k(t))$ is equicontinuous in mean square sense if for each $t \in I = [a, b]$, where a, b are real numbers and $\epsilon > 0$, there exist a $\delta > 0$ such that

$$E([X_n(t_2) - X_n(t_1)]^2) < \epsilon, \ \forall \ n \geq 1 \text{ when ever } \mid t_2 - t_1 \mid < \delta.$$

Theorem 2[9][Arzela theorem]
Every uniformly bounded equicontinuous family (sequence) of functions $(f_1(x), f_2(x), ..., f_k(x))$ has at least one subsequence which converges uniformly on the $I = [a, b]$, where a, b are real numbers

Theorem 3[10][Stochastic Lebesgue dominated convergence theorem]
Let $X_n(t)$ be a sequence of random vectors (or functions) is converging to $X(t)$ such that

$$X(t) = \lim_{n\to\infty} X_n(t), \ a.s.,$$

and $X_n(t)$ is dominated by an integrable function $a(t)$ such that $\parallel X_n(t) \parallel_2 \leq a(t)$. Then

(1) $E[\lim_{n\to\infty} X_n] = \lim_{n\to\infty} E[X_n]$ and
(2) $E[X_n(t) - X(t)] \to 0$ as $n \to \infty$

where $a.s.$ means that it happens with probability one.

Lemma 1[6] [Properties of Itô Integral] For all constants $a, b \in R$ and for all step processes $G, H \in L_2(\Omega)$:

(1) $\int_0^T (aG + bH)dW = a\int_0^T GdW + b\int_0^T HdW.$
(2) $E\left(\int_0^T GdW\right) = 0.$
(3) $E\left(\int_0^T GdW\right)^2 = E\left(\int_0^T G^2 dt\right).$
(4) $E\left(\int_0^T GdW \int_0^T HdW\right) = E\left(\int_0^T GHdt\right).$

3. Existence of at least one solution

Let $I = [0, T]$, (Ω, F, P) be a fixed probability space, where Ω is a sample space, F is a $\sigma-$algebra and P is a probability measure. We denote by $L_2(\Omega)$ the Banach space of random variables $X : \Omega \to R$ such that

$$\int_{\Omega} X^2 dP < \infty.$$

Let $X(t;\omega) = \{X(t), t \in I, \omega \in \Omega\}$ be a second order stochastic process, i.e.,

$$E(X^2(t)) < \infty,\ t \in I.$$

Now let $C = C(I, L_2(\Omega))$ be the class of all mean square continuous second order stochastic processes with the norm

$$\| X \|_C = \sup_{t\in[0,T]} \| X(t) \|_2 = \sup_{t\in[0,T]} \sqrt{E(X(t))^2}.$$

Consider the following assumptions
(i) The functions $f : [0, T] \times L_2(\Omega) \to L_2(\Omega)$ is Caratheodory function.
(ii)There exists an integrable function $l(t) \in L^1$ such that

$$\| f(t, X) \|_2 \le l(t),\ \forall (t, X) \in I \times L_2(\Omega)$$

and

$$\int_{t_1}^{t_2} l(t) \le k_1.$$

(iii) The function $g : I \to R^+$ is a continuous deterministic function such that

$$\int_{t_1}^{t_2} g^2(t) \le k_2^2,\ \forall t \in I.$$

Now we have the following lemmas.
Lemma 2[6] For a deterministic function $g(t) : I \to R^+$ and a Brownian motion $W(t)$

$$\left\|\int_0^t g(s)dW(s)\right\|^2 = \int_0^t g^2(s)ds.$$

Lemma 3 The solution of the stochastic nonlocal problem (1) and (2) can be expressed by the stochastic integral equation

$$\begin{aligned} X(t) &= a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{m} a_k \int_0^{\tau_k} g(s)dW(s)\right) \\ &+ \int_0^t f(s, X(s))ds + \int_0^t g(s)dW(s) \end{aligned} \quad (3)$$

where $a = \left(1 + \sum_{k=1}^{m} a_k\right)^{-1}$.

Proof. Integrating equation (1), we obtain

$$X(t) = X(0) + \int_0^t f(s, X(s))ds + \int_0^t g(s)dW(s),$$

then

$$\sum_{k=1}^{m} a_k X(\tau_k) = \sum_{k=1}^{m} a_k X(0) + \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(s))ds + \sum_{k=1}^{m} \int_0^{\tau_k} g(s)dW(s),$$

$$X_0 - X(0) = \sum_{k=1}^{m} a_k X(0) + \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(s))ds + \sum_{k=1}^{m} \int_0^{\tau_k} g(s)dW(s)$$

and

$$\left(1 + \sum_{k=1}^{m} a_k\right) X(0) = X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{m} \int_0^{\tau_k} g(s)dW(s),$$

then

$$X(0) = \left(1 + \sum_{k=1}^{m} a_k\right)^{-1} \left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{m} a_k \int_0^{\tau_k} g(s)dW(s)\right).$$

Hence

$$\begin{aligned} X(t) &= a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{m} a_k \int_0^{\tau_k} g(s)dW(s)\right) \\ &+ \int_0^t f(s, X(s))ds + \int_0^t g(s)dW(s), \end{aligned}$$

where $a = \left(1 + \sum_{k=1}^{m} a_k\right)^{-1}$.

Now for the existence of at least continuous solution $X \in C$ of the stochastic nonlocal problem (1)-(2), we have the following theorem.

Theorem 4 Let the assumptions (i)-(iii) be satisfied, then the problem (1)-(2) has at least one solution $X \in C$ given by the stochastic integral equation (3).

Proof. Define the set Q,

$$Q = \{X \in C : \| X \|_C \leq \beta\} \subset C.$$

Now for each $X \in Q$, we can define the operator H by

$$\begin{aligned} HX(t) &= a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, X(s))ds - \sum_{k=1}^{m} a_k \int_0^{\tau_k} g(s)dW(s)\right) \\ &+ \int_0^t f(s, X(s))ds + \int_0^t g(s)dW(s). \end{aligned}$$

We can prove that $HQ \subset Q$, f or this let $X(t) \in Q$, then

$$\begin{aligned}
\| HX(t) \|_2 &\leq a \| X_0 \|_2 + a\sum_{k=1}^{m} a_k \int_0^{\tau_k} \| f(s, X(s)) \|_2 \, ds + a\sum_{k=1}^{m} a_k \left\| \int_0^{\tau_k} g(s)dW(s) \right\|_2 \\
&+ \int_0^t \| f(s, X(s)) \|_2 \, ds + \left\| \int_0^t g(s)dW(s) \right\|_2 \\
&\leq a \| X_0 \|_2 + a\sum_{k=1}^{m} a_k \int_0^{\tau_k} l(s)ds + a\sum_{k=1}^{m} a_k \sqrt{\int_0^{\tau_k} g^2(s)ds} \\
&+ \int_0^t l(s)ds + \sqrt{\int_0^t g^2(s)ds} \\
&\leq a \| X_0 \|_2 + a\sum_{k=1}^{m} a_k k_1 + a\sum_{k=1}^{m} a_k k_2 + k_1 + k_2.
\end{aligned}$$

Let

$$a \| X_0 \|_2 + a\sum_{k=1}^{m} a_k k_1 + a\sum_{k=1}^{m} a_k k_2 + k_1 + k_2 = \beta.$$

β is clearly a positive real number, then ($\| HX \|_C \leq \beta$), so $HX \in Q$ and hence $HQ \subset Q$ and is uniformly bounded.
For $t_1, t_2 \in R^+$, $t_1 < t_2$, let $| t_2 - t_1 | < \delta$, then

$$\begin{aligned}
\| HX(t_2) - HX(t_1) \|_2 &\leq \int_{t_1}^{t_2} \| f(s, X(s)) \|_2 \, ds + \left\| \int_{t_1}^{t_2} g(s)dW(s) \right\|_2 \\
&\leq \int_{t_1}^{t_2} l(s)ds + \sqrt{\int_{t_1}^{t_2} g^2(s)ds} \leq k_1 + k_2 \leq 2k
\end{aligned}$$

where $k = \sup\{k_1, k_2\}$.
Then $\{HX\}$ is a class of equicontinuous functions. Therefore the operator H is equicontinuous and uniformly bounded.
Suppose that $\{X_n\} \in C$ such that $X_n \to X$ with probability 1

So,

$$\begin{aligned}
\underset{n\to\infty}{l.i.m}\; HX_n(t) &= \underset{n\to\infty}{l.i.m}\left[aX_0 - a\sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s,X_n(s))ds - a\sum_{k=1}^{m} a_k \int_0^{\tau_k} g(s)dW(s)\right] \\
&+ \underset{n\to\infty}{l.i.m}\left[\int_0^t f(s,X_n(s))ds + \int_0^t g(s)dW(s)\right] \\
&= aX_0 - a\sum_{k=1}^{m} a_k \underset{n\to\infty}{l.i.m}\left[\int_0^{\tau_k} f(s,X_n(s))ds\right] - a\sum_{k=1}^{m} a_k \int_0^{\tau_k} g(s)dW(s) \\
&+ \underset{n\to\infty}{l.i.m}\left[\int_0^t f(s,X_n(s))ds\right] + \int_0^t g(s)dW(s).
\end{aligned}$$

Then applying stochastic Lebesgue dominated convergence theorem, we get

$$\begin{aligned}
\underset{n\to\infty}{l.i.m}\; HX_n(t) &= aX_0 - a\sum_{k=1}^{m} a_k \int_0^{\tau_k} \underset{n\to\infty}{l.i.m}\,[f(s,X_n(s))]ds - a\sum_{k=1}^{m} a_k \int_0^{\tau_k} g(s)dW(s) \\
&+ \int_0^t \underset{n\to\infty}{l.i.m}\,[f(s,X_n(s))]ds + g(s)dW(s) \\
&= aX_0 - a\sum_{k=1}^{m} a_k \int_0^{\tau_k} [f(s,\underset{n\to\infty}{l.i.m}\,X_n(s))]ds - a\sum_{k=1}^{m} a_k \int_0^{\tau_k} g(s)dW(s) \\
&+ \int_0^t [f(s,\underset{n\to\infty}{l.i.m}\,X_n(s))]ds + \int_0^t g(s)dW(s) \\
&= aX_0 - a\sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s,X(s))ds - a\sum_{k=1}^{m} a_k \int_0^{\tau_k} g(s)dW(s) \\
&+ \int_0^t f(s,x(s))ds + \int_0^t g(s)dW(s) = HX(t).
\end{aligned}$$

This proves that H is continuous operator, then H is continuous and compact. Applying Schauder fixed point theorem, we deduce that there exists a fixed point $X \in C$ which proves that there exists at least one solution of the stochastic differential equation (1)-(2) given by (3).

4. Maximal and minimal solution

Definition 4 Let $q(t)$ be a solution of the problem (1)-(2), then $q(t)$ is said to be a maximal solution of (1)-(2) if every solution $X(t)$ of (1)-(2) satisfies the inequality

$$E(X^2(t)) < E(q^2(t)).$$

A minimal solution $s(t)$ can be defined by similar way by reversing the above inequality i.e.

$$E(X^2(t)) > E(s^2(t)).$$

In this section f assumed to satisfy the following definition.

Definition 5 The function $f : [0,T] \times L_2(\Omega) \to L_2(\Omega)$ is said to be stochastically increasing if for any $X, Y \in L_2(\Omega)$ satisfying $\| X(t) \|_2 < \| Y(t) \|_2$ implies that

$$\| f(t,X(t)) \|_2 < \| f(t,Y(t)) \|_2 .$$

Also The function $f : [0,T] \times L_2(\Omega) \to L_2(\Omega)$ is said to be stochastically decreasing if for any $X, Y \in L_2(\Omega)$ satisfying $\| X(t) \|_2 < \| Y(t) \|_2$ implies that

$$\| f(t,X(t)) \|_2 > \| f(t,Y(t)) \|_2$$

Now we have the following lemma.

Lemma 4 let the assumptions (i)-(iii) be satisfied and let $X, Y \in C$ satisfying

$$\begin{aligned} \| X(t) \|_2 &\leq a \left(\| X_0 \|_2 + \sum_{k=1}^{m} a_k \int_0^{\tau_k} \| f(s,X(s)) \|_2 \, ds + \sum_{k=1}^{m} a_k \left\| \int_0^{\tau_k} g(s) dW(s) \right\|_2 \right) \\ &+ \int_0^t \| f(s,X(s)) \|_2 \, ds + \left\| \int_0^t g(s) dW(s) \right\|_2 , \end{aligned}$$

and

$$\begin{aligned} \| Y(t) \|_2 &\geq a \left(\| X_0 \|_2 + \sum_{k=1}^{m} a_k \int_0^{\tau_k} \| f(s,Y(s)) \|_2 \, ds + \sum_{k=1}^{m} a_k \left\| \int_0^{\tau_k} g(s) dW(s) \right\|_2 \right) \\ &+ \int_0^t \| f(s,Y(s)) \|_2 \, ds + \left\| \int_0^t g(s) dW(s) \right\|_2 . \end{aligned}$$

If $f(t,X)$ is stochastically increasing function, then

$$\| X(t) \|_2 < \| Y(t) \|_2 . \tag{4}$$

Proof. Let the conclusion 4 be false, then there exists t_1 such that

$$\| X(t_1) \|_2 = \| Y(t_1) \|_2, \ t_1 > 0 \tag{5}$$

and

$$\| X(t) \|_2 < \| Y(t) \|_2, \ 0 < t < t_1 \tag{6}$$

Now from definition 4 and equation 6, we obtain

$$\begin{aligned} \| X(t_1) \|_2 &\leq a \left(\| X_0 \|_2 + \sum_{k=1}^{m} a_k \int_0^{\tau_k} \| f(s,X(s)) \|_2 \, ds + \sum_{k=1}^{m} a_k \left\| \int_0^{\tau_k} g(s) dW(s) \right\|_2 \right) \\ &+ \int_0^{t_1} \| f(s,X(s)) \|_2 \, ds + \left\| \int_0^{t_1} g(s) dW(s) \right\|_2 \end{aligned}$$

$$\begin{aligned} &< a\left(\| X_0 \|_2 + \sum_{k=1}^{m} a_k \int_0^{\tau_k} \| f(s,Y(s)) \|_2 \, ds + \sum_{k=1}^{m} a_k \left\| \int_0^{\tau_k} g(s)dW(s) \right\|_2 \right) \\ &+ \int_0^{t_1} \| f(s,Y(s)) \|_2 \, ds + \left\| \int_0^{t_1} g(s)dW(s) \right\|_2 \\ &< \| Y(t) \|_2, \ 0 < t < t_1, \end{aligned}$$

which contradicts (5), then $\| X(t) \|_2 < \| Y(t) \|_2$.

Now we have the following theorem.

Theorem 5 Let the assumptions (i)-(iii) be satisfied. If $f(t,X)$ is stochastically increasing, then there exists a maximal solution of problem (1)-(2).

Proof. Let $\epsilon > 0$, be given, then

$$\begin{aligned} X_\epsilon(t) &= a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_\epsilon(s,X_\epsilon(s))ds - \sum_{k=1}^{m} a_k \int_0^{\tau_k} g_\epsilon(s)dW(s)\right) \\ &+ \int_0^{t} f_\epsilon(s,X_\epsilon(s))ds + \int_0^{t} g_\epsilon(s)dW(s), \end{aligned} \tag{7}$$

where

$$f_\epsilon(t,X_\epsilon(t)) = f(t,X_\epsilon(t)) + \epsilon$$

and

$$g_\epsilon(t) = g(t) + \epsilon.$$

Clearly the functions $f_\epsilon(t,X_\epsilon(t))$ and $g_\epsilon(t)$ satisfy the conditions (i)-(iii) and

$$\| f_\epsilon(t,X_\epsilon(t)) \|_2 \leq l(t) + \epsilon = \grave{l}(t),$$

then equation (7) is a solution of the problem (1)-(2) according to Theorem 3
Now let ϵ_1 and ϵ_2 be such that $0 < \epsilon_2 < \epsilon_1 < \epsilon$, then

$$\begin{aligned} X_{\epsilon_1}(t) &= a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f_{\epsilon_1}(s,X_{\epsilon_1}(s))ds - \sum_{k=1}^{m} a_k \int_0^{\tau_k} g_{\epsilon_1}(s)dW(s)\right) \\ &+ \int_0^{t} f_{\epsilon_1}(s,X_{\epsilon_1}(s))ds + \int_0^{t} g_{\epsilon_1}(s)dW(s) \\ &= a\left(X_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \left(f(s,X_{\epsilon_1}(s)) + \epsilon_1\right) ds - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \left(g(s) + \epsilon_1\right) dW(s)\right) \\ &+ \int_0^{t} \left(f(s,X_{\epsilon_1}(s)) + \epsilon_1\right) ds + \int_0^{t} \left(g(s) + \epsilon_1\right) dW(s). \end{aligned}$$

Now

$$\begin{aligned}
\| X_{\epsilon_2}(t) \|_2 &= \left\| aX_0 + \int_0^t (f(s, X_{\epsilon_2}(s)) + \epsilon_2)\, ds + \int_0^t (g(s) + \epsilon_2)\, dW(s) \right. \\
&- \left. a\sum_{k=1}^m a_k \int_0^{\tau_k} (f(s, X_{\epsilon_2}(s)) + \epsilon_2)\, ds - a\sum_{k=1}^m a_k \int_0^{\tau_k} (g(s) + \epsilon_2)\, dW(s) \right\|_2 \\
&\leq \left\| aX_0 + \int_0^t (f(s, X_{\epsilon_1}(s)) + \epsilon_1)\, ds + \int_0^t (g(s) + \epsilon_1)\, dW(s) \right. \\
&- \left. a\sum_{k=1}^m a_k \int_0^{\tau_k} (f(s, X_{\epsilon_2}(s)) + \epsilon_2)\, ds - a\sum_{k=1}^m a_k \int_0^{\tau_k} (g(s) + \epsilon_2)\, dW(s) \right\|_2 \\
&\leq \left\| X_{\epsilon_1}(t) + a\sum_{k=1}^m a_k \int_0^{\tau_k} (f(s, X_{\epsilon_1}(s)) + \epsilon_1)\, ds + a\sum_{k=1}^m a_k \int_0^{\tau_k} (g(s) + \epsilon_1)\, dW(s) \right. \\
&- \left. a\sum_{k=1}^m a_k \int_0^{\tau_k} (f(s, X_{\epsilon_2}(s)) + \epsilon_2)\, ds - a\sum_{k=1}^m a_k \int_0^{\tau_k} (g(s) + \epsilon_2)\, dW(s) \right\|_2 \\
&\leq \| X_{\epsilon_1}(t) \|_2 + a\sum_{k=1}^m a_k \int_0^{\tau_k} ||f(s, X_{\epsilon_1}(s)) - f(s, X_{\epsilon_2}(s)||_2\, ds \\
&+ a\sum_{k=1}^m a_k \int_0^{\tau_k} | \epsilon_1 - \epsilon_2 |\, ds + a\sum_{k=1}^m a_k \sqrt{\int_0^{\tau_k} | \epsilon_1 - \epsilon_2 |^2\, ds}.
\end{aligned}$$

Since ϵ_i are very small and near real numbers, then $| \epsilon_1 - \epsilon_2 | \to 0$, also the function $f(t, X(t))$ is stochastically increasing, then

$$||f(s, X_{\epsilon_1}(s)) - f(s, X_{\epsilon_2}(s)||_2 \to 0.$$

Hence

$$\| X_{\epsilon_2}(t) \|_2 \leq \| X_{\epsilon_1}(t) \|_2 .$$

For $\epsilon_n \leq \epsilon_{n-1} \leq .... \leq \epsilon_2 \leq \epsilon_1 \leq \epsilon$, we can prove that

$$\| X_{\epsilon_n}(t) \|_2 \| \leq \| X_{\epsilon_{n-1}}(t) \|_2 \leq ... \leq \| X_{\epsilon_2}(t) \|_2 \leq \| X_{\epsilon_1}(t) \|_2 \leq \| X_{\epsilon}(t) \|_2 .$$

As shown before in the proof of Theorem 3 the family of functions $X_\epsilon(t)$ defined by equation (3) is uniformly bounded and equi-continuous functions. Hence by Arzela Theorem [9], there exists a decreasing sequence ϵ_n such that $\epsilon \to 0$ as $n \to \infty$ and $\overset{l.i.m}{n\to\infty} X_{\epsilon_n}(t)$ exists uniformly in C.

Denote this limit by $q(t)$, then from the continuity of the function f_{ϵ_n} in the second argument, we can apply Lebesgue dominated convergence theorem to get

$$q(t) = \overset{l.i.m}{n\to\infty} X_{\epsilon_n}(t).$$

This proves that $q(t)$ is a solution of the problem (1)-(2).
Finally, we shall show that $q(t)$ is the maximal solution of the problem (1)-(2).
To do this, let $X(t)$ be any solution of the problem (1)-(2). Then

$$\| X_\epsilon(t) - X(t) \|_2 = \epsilon.$$

So

$$\| X_\epsilon(t) \|_2 - \| X(t) \|_2 \geq \epsilon.$$

As $\epsilon \to 0$, we obtain

$$\| X_\epsilon(t) \|_2 \geq \| X(t) \|_2 .$$

From the uniqueness of the maximal solution (see [2]), it is clear that $X_\epsilon(t)$ tends to $q(t)$ uniformly as $\epsilon \to 0$. This completes the proof.
By a similar way, we can prove the following theorem.
Theorem 6 Let the assumptions (i)-(iii) be satisfied. If $f(t, X)$ is stochastically decreasing, then there exists a minimal solution of the problem (1)-(2).

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