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APPROXIMATE CONTROLLABILITY OF DAMPED SECOND-ORDER IMPULSIVE NEUTRAL STOCHASTIC INTEGRO-DIFFERENTIAL SYSTEM WITH STATE-DEPENDENT DELAY

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Abstract. The objective of this paper is to study the approximate controllability for a class of damped second-order impulsive neutral stochastic integro-differential system with state-dependent delay in Hilbert spaces under the assumptions that the corresponding linear system is approximately controllable. By employing a fixed point theorem for condensing maps combined with theories of a strongly continuous cosine families of bounded linear operators, a set of sufficient conditions are derived for achieving the required result. As an application, an example is provided to illustrate our results.

1. Introduction

In this paper, we shall consider the approximate controllability for a class of damped second-order impulsive neutral stochastic integro-differential system with state-dependent delay of the form:

$$\begin{cases} d\big[x'(t) - G\big(t, x_t, \int_0^t g(t,s,x_s)ds\big)\big] = \big[Ax(t) + \mathcal{D}x'(t) + Bu(t)\big]dt + F\big(t, x_{\rho(t,x_t)}\big)dw(t), \\ \qquad\qquad t \neq t_k, \quad k = \{1, \cdots, m\} := \overline{1,m}, \quad t \in J := [0,T], \\ \Delta x(t_k) = I_k^1(x_{t_k}), \quad \Delta x'(t_k) = I_k^2(x_{t_k}), \quad k = \overline{1,m}, \\ x_0 = \varphi \in \mathcal{B}, \quad x'(0) = x_1 \in \mathbb{H}, \end{cases} \tag{1.1}$$

where $x(\cdot)$ is a stochastic process taking values in a real separable Hilbert space $\mathbb{H}$; $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of a strongly continuous cosine family on $\mathbb{H}$. The function control $u(\cdot) \in L_2^{\mathcal{F}}(J,U)$ of admissible control functions for a separable Hilbert space U, $B : U \to \mathbb{H}$ is a bounded linear operator, and $\mathcal{D}$ is a bounded linear operator on a Hilbert space $\mathbb{H}$ with $D(\mathcal{D}) \subset D(A)$. The history $x_t : (-\infty, 0] \to \mathbb{H}$, $x_t(\theta) = x(t+\theta)$ for $t \geq 0$, belong to the phase space $\mathcal{B}$, which will be described in Section 2. Assume that the mappings $G : J \times \mathcal{B} \times \mathbb{H} \to \mathbb{H}$, $F : J \times \mathcal{B} \to \mathcal{L}_2^0$, $g : J \times J \times \mathcal{B} \to \mathbb{H}$, $I_k^1, I_k^2 : \mathcal{B} \to \mathbb{H}$, $k = \overline{1,m}$, $\rho : J \times \mathcal{B} \to (-\infty, T]$ are

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appropriate functions to be specified later. Furthermore, let $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$ be prefixed points, and $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, represents the jump of the function x at time t_k with I_k determining the size of the jump, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. Similarly $x'(t_k^+)$ and $x'(t_k^-)$ denote, respectively, the right and left limits of $x'(t)$ at t_k. Let $\varphi(t) \in L_2(\Omega, \mathcal{B})$ and $x_1(t)$ be $\mathbb{H}$-valued $\mathcal{F}_t$-measurable random variables independent of the Wiener process $\{w(t)\}$ with a finite second moment.

Approximate controllability is one of the fundamental concept in mathematical control theory and plays an important role in both deterministic and stochastic control systems. It is well known that controllability of deterministic systems are widely used in many fields of science and technology (for instance, see [5, 49]). Stochastic control theory is stochastic generalization of classic control theory. The theory of controllability of differential equations in infinite dimensional spaces has been extensively studied in the literature, and the details can be found in various papers and monographs [4, 7, 25, 50] and the references therein. Besides white noise or stochastic perturbation, many systems like predator-prey systems arising from realistic models depend heavily on the histories or impulsive effect [19, 26]. Therefore, there is a real need to discuss stochastic impulsive functional differential systems with infinite delay. On the existence and the controllability for these equations we refer the reader to (for example, see [9, 11, 20, 22, 21, 24, 25, 34, 47, 48] and the references therein).

On the other hand, in recent years, second-order differential equations have been gained much attentions since it not only exists widely but also can be used to study many phenomena in the real lives. In many cases it is advantageous to treat the second-order abstract differential equations directly than to convert them to first-order systems (for instance, see [14]). Second-order equations have been examined in [44]. The deterministic version for the existence and the controllability of second-order differential equations have been thoroughly studied by several authors (see [1, 2, 3, 6, 10, 13, 16, 42, 43, 44] and the references therein) while the controllability for stochastic version are not yet sufficiently investigated, and there are only few works on it [8, 23, 30, 31, 32, 33, 36, 35, 38].

Furthermore, functional differential equations with state-dependence is a special type of functional differential equations and it have become more important in various mathematical models in the study of population dynamics, biology, ecology and epidemic, etc. For this reason, in recent years, control problem for differential equations with state-dependence has attracted much attention of researchers. To be more precise, in [1], Arthi and Balachandran discussed the controllability of second-order impulsive functional differential equations with state-dependent delay by means of the Sadovskii fixed point theorem. By using Schauder's fixed point theorem, Sakthivel and Anandhi [39] investigated the approximate controllability of impulsive differential equations with state-dependent delay. Yan [46] proved sufficient conditions for the approximate controllability of partial neutral functional differential systems of fractional order with state-dependent delay by using the Krasnoselskii-Schaefer type fixed point theorem with the fractional power of operators. More recently, also by using Schauder's fixed point theorem, Sakthivel and Ren [40] established the approximate controllability of fractional differential equations with state-dependent delay. Besides, in dynamical systems damping is another important issue, it may be mathematically modelled as a force synchronous with the velocity of the object but opposite in direction to it. Hence, in this manuscript, we will also study damped second-order stochastic differential equations. On the damped second-order differential equations, we refer the reader to (for example, see [2, 3, 18, 27, 45] and the references therein). However, to the best of our knowledge, it seems that little is known about approximate controllability for a class of damped second-order impulsive neutral stochastic integro-differential system with state-dependent delay and the aim of this paper is to fill this gap. The results presented in the current manuscript constitute a continuation and generalization of the controllability results from [1, 2, 3, 6, 8, 24, 31, 32, 38, 39, 46] to the damped second-order impulsive neutral stochastic integro-differential system with state-dependent delay in Hilbert spaces settings.

The main techniques used in this paper include the Sadovskii fixed point theorem combined with theories of a strongly continuous cosine families of bounded linear operators.

The structure of this paper is as follows: In Section 2, we briefly present some basic notations, preliminaries and assumptions. The main results in Section 3 are devoted to study the approximate controllability for the system (1.1) with their proofs. At last, an example is presented to illustrate the main results.

2. Preliminaries

In what follows we recall some basic definitions, notations, lemmas and results for stochastic equations in infinite dimensions and cosine families of operators. For more details on this section, we refer the reader to [12, 13, 43].

Let $(\mathbb{H}, \|\cdot\|_{\mathbb{H}}, \langle\cdot,\cdot\rangle)$ and $(\mathbb{K}, \|\cdot\|_{\mathbb{K}}, \langle\cdot,\cdot\rangle)$ denote two real separable Hilbert spaces, with their vectors norms and their inner products, respectively. We denote by $\mathcal{L}(\mathbb{K};\mathbb{H})$ be the set of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$, which is equipped with the usual operator norm $\|\cdot\|$. In this paper, we use the symbol $\|\cdot\|$ to denote norms of operators regardless of the spaces potentially involved when no confusion possibly arises. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ be a complete filtered probability space satisfying the usual condition (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets). Let $w = (w(t))_{t\geq 0}$ be a Q-Wiener process defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ with the covariance operator Q such that $Tr(Q) < \infty$. We assume that $\mathcal{F}_t = \sigma(\{w(s) : 0 \leq s \leq t\})$ is the σ-algebra generated by w and $\mathcal{F}_T = \mathcal{F}$. We also assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in $\mathbb{K}$, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k$, $k = 1, 2, ...$, and a sequence of independent Brownian motions $\{\beta_k\}_{k\geq 1}$ such that

$$\langle w(t), e\rangle_{\mathbb{K}} = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\langle e_k, e\rangle_{\mathbb{K}}\beta_k(t), \quad e \in \mathbb{K}, t \geq 0.$$

Let $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}}\mathbb{K};\mathbb{H})$ be the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}\mathbb{K}$ into $\mathbb{H}$ with the inner product $\langle\Psi, \phi\rangle_{\mathcal{L}_2^0} = Tr[\Psi Q\phi^*]$, where ϕ^* is the adjoint of the operator ϕ.

Next, to be able to access approximate controllability for the system (1.1), we need to introduce theory of cosine functions of operators and the second order abstract Cauchy problem.

Definition 2.1. *[(1)]*

(1) *The one-parameter family $\{C(t)\}_{t\in\mathbb{R}} \subset \mathcal{L}(\mathbb{H})$ is said to be a strongly continuous cosine family if the following hold:*

[(i)]$C(0) = I$, I is the identity operators in $\mathbb{H}$; $C(t)x$ is continuous in t on $\mathbb{R}$ for any $x \in \mathbb{H}$; $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$.

(2) *The corresponding strongly continuous sine family* $\{S(t)\}_{t\in\mathbb{R}} \subset \mathcal{L}(\mathbb{H})$, *associated to the given strongly continuous cosine family* $\{C(t)\}_{t\in\mathbb{R}} \subset \mathcal{L}(\mathbb{H})$ *is defined by*

$$S(t)x = \int_0^t C(s)xds, \quad t \in \mathbb{R}, x \in \mathbb{H}.$$

(3) *The infinitesimal generator* $A : \mathbb{H} \to \mathbb{H}$ *of* $\{C(t)\}_{t\in\mathbb{R}} \subset \mathcal{L}(\mathbb{H})$ *is given by*

$$Ax = \frac{d^2}{dt^2}C(t)x\Big|_{t=0},$$

for all $x \in D(A) = \{x \in \mathbb{H} : C(\cdot) \in \mathsf{C}^2(\mathbb{R}, \mathbb{H})\}$.

It is well known that the infinitesimal generator A is a closed, densely defined operator on $\mathbb{H}$, and the following properties hold, see Travis and Webb [43].

Proposition 2.1. *Suppose that* A *is the infinitesimal generator of a cosine family of operators* $\{C(t)\}_{t\in\mathbb{R}}$. *Then, the following hold:*
[(i)]There exist a pair of constants $M_A \geq 1$ *and* $\alpha \geq 0$ *such that* $\|C(t)\| \leq M_A e^{\alpha|t|}$ *and hence,* $\|S(t)\| \leq M_A e^{\alpha|t|}$; $A\int_s^r S(u)xdu = [C(r) - C(s)]x$, *for all* $0 \leq s \leq r < \infty$; *There exist* $N \geq 1$ *such that* $\|S(s) - S(r)\| \leq N\big|\int_s^r e^{\alpha|s|}ds\big|$, $0 \leq s \leq r < \infty$.

Thanks to the Proposition 2.1 and the uniform boundedness principle, as a direct consequence we see that both $\{C(t)\}_{t\in J}$ and $\{S(t)\}_{t\in J}$ are uniformly bounded by $\widetilde{M} = M_A e^{\alpha|T|}$.

The existence of solutions for the second order linear abstract Cauchy problem

$$\begin{cases} x''(t) &= Ax(t) + h(t), \quad t \in J, \\ x(0) &= z, \quad x'(0) = w, \end{cases} \tag{2.1}$$

where $h : J \to \mathbb{H}$ is an integrable function has been discussed in [41]. Similarly, the existence of solutions of the semilinear second order abstract Cauchy problem it has been treated in [43].

Definition 2.2. *The function* $x(\cdot)$ *given by*

$$x(t) = C(t)z + S(t)w + \int_0^t S(t-s)h(s)ds, \quad t \in J,$$

is called a mild solution of (2.1), and that when $z \in \mathbb{H}$, $x(\cdot)$ *is continuously differentiable and*

$$x'(t) = AS(t)z + C(t)w + \int_0^t C(t-s)h(s)ds, \quad t \in J.$$

For additional details about cosine function theory, we refer to the reader to [41, 43].

Definition 2.3. *Denote the space* $\mathcal{M}^2, \overline{\mathcal{M}}^2$*-formed by all* $\mathcal{F}_t$*-adapted measurable,* $\mathbb{H}$*-valued stochastic process* $x = x(t), t \in J$ *such that*
[(i)]$\mathcal{M}^2 := \mathcal{M}^2(J, \mathbb{H})$
$= \{x : J \to \mathbb{H},\ x|_{(t_k,t_{k+1}]} \in \mathsf{C}((t_k, t_{k+1}], \mathbb{H})$ *and* $x(t_k^+)$ *there exists,* $\forall k = \overline{1,m}\}$. $\overline{\mathcal{M}}^2 := \overline{\mathcal{M}}^2(J, \mathbb{H})$
$= \{x \in \mathcal{M}^2,\ x|_{(t_k,t_{k+1}]} \in \mathsf{C}^1((t_k, t_{k+1}], \mathbb{H})$ *and* $x'(t_k^+)$ *there exists,* $\forall k = \overline{1,m}\}$. *For all*

$x \in \mathcal{M}^2, \overline{\mathcal{M}}^2$,

$$\|x\|_{\mathcal{M}^2} := \Big(\mathbf{E}\sup_{t\in J}\|x(t)\|^2\Big)^{\frac{1}{2}}, \quad \|x\|_{\overline{\mathcal{M}}^2} = \|x\|_{\mathcal{M}^2} + \|x'\|_{\mathcal{M}^2}.$$

Then, it is obvious that $\mathcal{M}^2$, $\overline{\mathcal{M}}^2$ with the above norms are Banach spaces.

The collection of all strongly-measurable, square-integrable $\mathbb{H}$-valued random variables, denoted by $L_2(\Omega, \mathcal{F}, \mathbf{P}; \mathbb{H}) := L_2(\Omega, \mathbb{H})$, is a Banach space equipped with norm $\|x\|_{L_2} = \big(\mathbf{E}\|x\|^2\big)^{\frac{1}{2}}$. Let $\mathsf{C}(J, L_2(\Omega, \mathbb{H}))$ be the Banach space of all continuous map from J to $L_2(\Omega, \mathbb{H})$ satisfying the condition $\mathbf{E}\sup_{t\in J}\|x(t)\|^2 < \infty$. An important subspace is given by $L_2^0(\Omega, \mathbb{H}) = \{f \in L_2(\Omega, \mathbb{H}) :$ f is $\mathcal{F}_0$-measurable$\}$.

To simplify the notations, we put $t_0 = 0$, $t_{m+1} = T$ and for $v \in \mathcal{M}^2$ we denote by $\widetilde{v}_k \in C([t_k, t_{k+1}], L_2(\Omega, \mathbb{H}))$, $k = 0, 1, \cdots, m$, the function given by

$$\widetilde{v}_k(t) = \begin{cases} v(t), & for \quad t \in (t_k, t_{k+1}], \\ v(t_k^+), & for \quad t = t_k. \end{cases}$$

Moreover, for $B \subseteq \mathcal{M}^2$ we denote by $\widetilde{B}_k = \{\widetilde{v}_k : v \in B\}$, $k = 0, 1, \cdots, m$.

To prove our results, we need the following lemma introduced in Yan and Zhang [48].

Lemma 2.1. *([48], Lemma 2.7) A set $B \subseteq \mathcal{M}^2$ is relatively compact in $\mathcal{M}^2$, if and only if, the set $\widetilde{B}_k$ is relatively compact in $\mathsf{C}([t_k, t_{k+1}], L_2(\Omega, \mathbb{H}))$, for every $k = 0, 1, \cdots, m$.*

In the whole of this work, we suppose that the phase space $\mathcal{B}$ is axiomatically defined, we use the approach proposed in [15]. More precisely, we have the following definition.

Definition 2.4. *The phase space $\mathcal{B}((-\infty, 0], \mathbb{H})$ (denoted by $\mathcal{B}$ for brevity) is the space of $\mathcal{F}_0$-measurable functions from $(-\infty, 0]$ to $\mathbb{H}$ endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$, which satisfies the following axioms:*

($\mathbf{A}_1$) If $x : (-\infty, T] \to \mathbb{H}$, $T > 0$, is such that $x_0 \in \mathcal{B}$, then for every $t \in [0, T]$, the following properties hold:

[(i)]$x_t \in \mathcal{B}$; $\|x(t)\|_{\mathbb{H}} \leq L\|x_t\|_{\mathcal{B}}$, which is equivalent to $\|\varphi(0)\|_{\mathbb{H}} \leq L\|\varphi\|_{\mathcal{B}}$ for every $\varphi \in \mathcal{B}$; $\|x_t\|_{\mathcal{B}} \leq M(t)\sup_{0\leq s\leq t}\|x(s)\|_{\mathbb{H}} + N(t)\|x_0\|_{\mathcal{B}}$,

where $L > 0$ is a constant; $M, N : [0, +\infty) \to [1, +\infty)$, $M(\cdot)$ is continuous, $N(\cdot)$ is locally bounded, and M, N are independent of $x(\cdot)$.

($\mathbf{A}_2$) The space $\mathcal{B}$ is complete.

Remark 2.1. *In retarded functional differential equations without impulses, the axioms of the phase space $\mathcal{B}$ include the continuity of the function $t \to x_t$, see [19] for details. Due to the impulses, this property is not satisfied in (1.1) and, for this reason, has been unconsidered in our description of $\mathcal{B}$.*

Next, we give an example to illustrate the above definition.

Example 2.1. *Let $\alpha < 0$, define the phase space*

$$\mathcal{B} := \Big\{\phi \in \mathsf{C}\big((-\infty, 0]; L^2([0, \pi])\big) : \lim_{\sigma\to-\infty} e^{\sigma\alpha}\phi(\sigma) \quad \textit{exists in } L^2([0, \pi])\Big\}$$

and let $\|\phi\|_{\mathcal{B}} = \sup_{\sigma\in(-\infty,0]} e^{\sigma\alpha}\|\phi(\sigma)\|_{L^2([0,\pi])}$. *Then,* $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ *is a Banach space and satisfies the axioms* $(\mathbf{A}_1)$ *and* $(\mathbf{A}_2)$ *with* $L = 1$, $M(t) = \max\{1, e^{-\alpha t}\}$, $N(t) = e^{-\alpha t}$.

Remark 2.2. *As a consequence of the phase space axioms, for convenience, the property (iii) in Definition 2.4 can be replaced by the following condition:*

$$\|x_t\|_{\mathcal{B}} \leq \widetilde{M}_T \sup_{s\in J}\mathbf{E}\|x(s)\|_{\mathbb{H}} + M_T\mathbf{E}\|\varphi\|_{\mathcal{B}},$$

where $\widetilde{M}_T := \sup_{s\in J} M(s)$, $M_T := \sup_{s\in J} N(s)$, *(see* [48], *Lemma 2.8).*

We will take the help of fixed point theorem due to Sadovskii, which is an extension of Schauder's principle and the contraction principle.

Lemma 2.2. *([37]) Let* Θ *be a condensing operator on a Banach space* $\mathbb{H}$, *that is,* Θ *is continuous and takes bounded sets into bounded sets, and* $\mu(\Theta(A)) \leq \mu(A)$ *for every bounded set* A *of* $\mathbb{H}$ *with* $\mu(A) > 0$. *If* $\Theta(B) \subset B$ *for a convex, close and bounded set* B *of* $\mathbb{H}$, *then* Θ *has a fixed point in* $\mathbb{H}$ *(where* $\mu(\cdot)$ *denotes Kuratowski's measure of noncompactness).*

Now, motivated by Definition 2.2, we give the following definition of mild solution for (1.1).

Definition 2.5. *An* $\mathcal{F}_t$*-adapted stochastic process* $x : J_T \to \mathbb{H}$ *is called a mild solution of (1.1) on* $J_T = (-\infty, T]$ *if* $x_0 = \varphi \in \mathcal{B}$ *and* $x'(0) = x_1 \in \mathbb{H}$ *satisfying* $x_0, x_1 \in L_2^0(\Omega, \mathbb{H})$ *such that the following conditions hold:*

[(i)]$\{x_t : t \in J\}$ *is a* $\mathcal{B}$*-valued stochastic process;* $x|_J \in \mathcal{M}^2$ *and* $x(t)$ *satisfies the following integral equation:*

$$\begin{aligned} x(t) =& C(t)x_0 + S(t)[x_1 - G(0, x_0, 0)] \\ &+ \sum_{k=0}^{j-1}\big[S(t-t_{k+1})\mathcal{D}x(t_{k+1}^-) - S(t-t_k)\mathcal{D}x(t_k^+)\big] - S(t-t_j)\mathcal{D}x(t_j^+) \\ &+ \int_0^t C(t-s)\mathcal{D}x(s)ds + \int_0^t S(t-s)Bu(s)ds \\ &+ \int_0^t C(t-s)G(s, x_s, \int_0^s g(s,\tau,x_\tau)d\tau)ds + \sum_{0<t_k<t} C(t-t_k)I_k^1(x_{t_k}) \\ &+ \int_0^t S(t-s)F\big(s, x_{\rho(s,x_s)}\big)dw(s) + \sum_{0<t_k<t} S(t-t_k)I_k^2(x_{t_k}), \forall t \in [t_j, t_{j+1}], j = \overline{0,m}, \end{aligned} \tag{2.2}$$

$\Delta x(t_k) = I_k^1(x_{t_k})$, $\Delta x'(t_k) = I_k^2(x_{t_k})$, $k = \overline{1,m}$.

Remark 2.3. *The equation (2.2) can also be written as*

$$\begin{aligned} x(t) =& C(t)x_0 + S(t)[x_1 - G(0, x_0, 0)] \\ &+ \int_0^t S(t-s)\mathcal{D}x'(s)ds + \int_0^t S(t-s)Bu(s)ds \end{aligned}$$

$$+\int_0^t C(t-s)G\big(s, x_s, \int_0^s g(s,\tau,x_\tau)d\tau\big)ds + \sum_{0<t_k<t} C(t-t_k)I_k^1(x_{t_k})$$
$$+\int_0^t S(t-s)F\big(s, x_{\rho(s,x_s)}\big)dw(s) + \sum_{0<t_k<t} S(t-t_k)I_k^2(x_{t_k}), \quad t \in J.$$

It is convenient to introduce the relevant operators and the basic controllability condition.

(i) The operator $L_0^T \in \mathcal{L}(L_2^{\mathcal{F}}(J,\mathbb{H}), L_2(\Omega, \mathcal{F}_T, \mathbb{H}))$ is defined by

$$L_0^T u = \int_0^T S(T-s)Bu(s)ds,$$

where $L_2^{\mathcal{F}}(J,\mathbb{H})$ is the space of all $\mathcal{F}_t$-adapted, H-valued measurable square integrable processes on $J\times\Omega$. Clearly the adjoint $(L_0^T)^* : L_2(\Omega,\mathcal{F}_T,\mathbb{H}) \to L_2^{\mathcal{F}}(J,\mathbb{H})$ is defined by

$$[(L_0^T)^* z](t) = B^* S^*(T-t)\mathbf{E}\{z \mid \mathcal{F}_t\}.$$

(ii) The controllability operator Π_0^T associated with the linear stochastic system of (1.1) is defined by

$$\Pi_0^T\{\cdot\} = L_0^T(L_0^T)^*\{\cdot\} = \int_0^T S(T-t)BB^*S^*(T-t)\mathbf{E}\{\cdot \mid \mathcal{F}_t\}dt.$$

which belongs to $\mathcal{L}(L_2(\Omega,\mathcal{F}_T,\mathbb{H}), L_2(\Omega,\mathcal{F}_T,\mathbb{H}))$ and the controllability operator $\Gamma_s^T \in \mathcal{L}(\mathbb{H},\mathbb{H})$ is

$$\Gamma_s^T = \int_s^T S(T-t)BB^*S^*(T-t)dt, \quad 0 \le s < t.$$

Let $x(t;u)$ denotes state value of the system (1.1) at time t corresponding to the control $u \in L_2^{\mathcal{F}}(J,U)$. In particular, the state of system (1.1) at $t=T$, $x(T;u)$ is called the terminal state with control u. $\mathcal{R}_T := \mathcal{R}(T;u) = \{x(T;u) : u(\cdot) \in L_2^{\mathcal{F}}(J,U)\}$ is called the reachable set of the system (1.1).

Definition 2.6. *The stochastic system (1.1) is said to be approximately controllable on the interval J if for every $x_0, x_1 \in L_2^0(\Omega,\mathbb{H})$, there is some control $u(\cdot) \in L_2^{\mathcal{F}}(J,U)$,*

$$\overline{\mathcal{R}}_T = L_2(\Omega,\mathcal{F}_T,H),$$

where $\overline{\mathcal{R}}_T$ is the closure of the reachable set.

Lemma 2.3. *([28]) For any $h \in L_2(\Omega,\mathcal{F}_T,H)$, there exists $z \in L_2^{\mathcal{F}}(J,\mathcal{L}_2^0)$ such that $h = \mathbf{E}h + \int_J z(s)dw(s)$.*

In order to establish the results, we assume the following hypotheses:

[($\mathbf{H}0$)]The function $t \to \varphi_t$ is continuous from $\Sigma(\rho^-) = \{\rho(s,\varphi) \le 0, (s,\varphi) \in J\times\mathcal{B}\}$ into $\mathcal{B}$ and there exists a continuous and bounded function $l^\varphi : \Sigma(\rho^-) \to (0,\infty)$ such that $\|\varphi_t\|_{\mathcal{B}} \le l^\varphi(t)\|\varphi\|_{\mathcal{B}}$ for each $t \in \Sigma(\rho^-)$.

[($\mathbf{H}1$)]The cosine family of operators $\{C(t)\}_{t\in J}$ on $\mathbb{H}$ and the corresponding sine family $\{S(t)\}_{t\in J}$ are compact for $t>0$, and there exists positive constants M_B, M_C, $M_{\mathcal{D}}$, M_S such that for all $t\in J$

$$\|B\|^2\le M_B,\quad \|C(t)\|^2\le M_C,\quad \|\mathcal{D}\|^2\le M_{\mathcal{D}},\quad \|S(t)\|^2\le M_S.$$

[($\mathbf{H}2$)]There exists positive constants $M_g,\widetilde{M}_g$ such that for all $t,s\in J$, $x,y\in\mathcal{B}$

$$\mathbf{E}\Big\|\int_0^t[g(t,s,x)-g(t,s,y)]ds\Big\|^2\le M_g\|x-y\|^2_{\mathcal{B}}$$

and $\widetilde{M}_g=\sup_{(t,s)\in J\times J}\Big(\|\int_0^t g(t,s,0)ds\|^2\Big)$.

[($\mathbf{H}3$)]The function $G:J\times\mathcal{B}\times\mathbb{H}\to\mathbb{H}$ is continuous and there exists positive constants $M_G,\widetilde{M}_G$ such that for all $t\in J$, $x_1,x_2\in\mathcal{B}$, $y_1,y_2\in\mathbb{H}$

$$\mathbf{E}\|G(t,x_1,y_1)-G(t,x_2,y_2)\|^2\le M_G(\|x_1-x_2|^2_{\mathcal{B}}+\mathbf{E}\|y_1-y_2\|^2)$$

and $\widetilde{M}_G=\sup_{t\in J}\|G(t,0,0)\|^2$.

[($\mathbf{H}4$)]The function $F:J\times\mathcal{B}\to\mathcal{L}_2^0$ satisfies the following conditions:[(i)]

(1) (a) The function $F(\cdot,x):J\to\mathcal{L}_2^0$ is strongly measurable for each $x\in\mathcal{B}$.
(b) The function $F(t,\cdot):\mathcal{B}\to\mathcal{L}_2^0$ is continuous for almost all $t\in J$.
(c) There exists an integrable function $\zeta_F:J\to[0,\infty)$ and a continuous nondecreasing function $\Psi_F:[0,\infty)\to(0,\infty)$ such that for every $(t,x)\in J\times\mathcal{B}$

$$\mathbf{E}\|F(t,x)\|^2_{\mathcal{L}_2^0}\le\zeta_F(t)\Psi_F(\|x\|^2_{\mathcal{B}}).$$

[($\mathbf{H}5$)]There exists positive constants $M_{I_k^1}$, $M_{I_k^2}$ such that for all $x,y\in\mathcal{B}$

$$\mathbf{E}\|I_k^1(x)-I_k^1(y)\|^2\le M_{I_k^1}\|x-y\|^2_{\mathcal{B}},\quad \mathbf{E}\|I_k^2(x)-I_k^2(y)\|^2\le M_{I_k^2}\|x-y\|^2_{\mathcal{B}}.$$

[($\mathbf{H}6$)]The functions $I_k^1,I_k^2:\mathcal{B}\to\mathbb{H}$, $k=\overline{1,m}$ are completely continuous, and there are continuous nondecreasing functions $\Omega_k,\Theta_k:[0,\infty)\to(0,\infty)$, $k=\overline{1,m}$ and any $x\in\mathcal{B}$ such that

$$\mathbf{E}\|I_k^1(x)\|^2\le\Omega_k(\|x\|^2_{\mathcal{B}}),\quad \mathbf{E}\|I_k^2(x)\|^2\le\Theta_k(\|x\|^2_{\mathcal{B}}),\quad k=\overline{1,m}.$$

[($\mathbf{H}7$)]For $0\le t<T$, the operator $\alpha\mathcal{R}(\alpha,\Gamma_t^T):=\alpha(\alpha I+\Gamma_t^T)^{-1}\to 0$ as $\alpha\to 0^+$ in the strong operator topology.

[($\mathbf{H}8$)]The function $G:J\times\mathcal{B}\times\mathbb{H}\to\mathbb{H}$ and $F:J\times\mathcal{B}\to\mathcal{L}_2^0$ are uniformly bounded.

Remark 2.4. *In view of [29], the assumption ($\mathbf{H}7$) is equivalent to the linear system of (1.1) is approximately controllable.*

Lemma 2.4. *Let $x:(-\infty,T]\to\mathbb{H}$ such that $x_0=\varphi$ and $x|_J\in\mathcal{M}^2$. If ($\mathbf{H}0$) be hold, then*

$$\|x_s\|_{\mathcal{B}}\le(M_T+l_0^{\varphi})\|\varphi\|_{\mathcal{B}}+\widetilde{M}_T\sup\big\{\|x(\theta)\|;\theta\in[0,\max\{0,s\}]\big\},s\in\Sigma(\rho^-)\cup J,$$

where $l_0^{\varphi}=\sup_{t\in\Sigma(\rho^-)}l^{\varphi}(t)$ (see [48], Lemma 3.3).

3. Main results

In this section sufficient conditions are established for the approximate controllability of the stochastic control system (1.1) under the assumption that the associated linear system is approximately controllable.

Theorem 3.1. *Assume that the assumptions* (**H0**) − (**H5**) *hold. If*

$$10\Big(1+\frac{10}{\alpha^2}M_B^2M_S^2T^2\Big)\bigg[\big((4m+1)M_S+TM_C\big)M_{\mathcal{D}}+4M_C\widetilde{M}_T^2m\sum_{k=1}^{m}M_{I_k^1}+4M_S\widetilde{M}_T^2m\sum_{k=1}^{m}M_{I_k^2} + 4M_CM_GT^2\widetilde{M}_T^2(1+M_g)+2\widetilde{M}_T^2M_STr(Q)\lim_{\varepsilon\to\infty}\inf\frac{\Psi_F(\varepsilon)}{\varepsilon}\int_J\zeta_F(s)ds\bigg]\le 1, \tag{3.1}$$

then the system (1.1) has at least one mild solution on J.

(1) *Proof.* For all $\alpha>0$, we define the control for the system (1.1) as

$$\begin{aligned}
u_\alpha(t,x)=B^*S^*(T-t)\bigg\{&R(\alpha,\Pi_0^T)\Big[\mathbf{E}h-C(T)\varphi(0)-S(T)[x_1-G(0,\varphi,0)]\\
&-\sum_{k=0}^{j-1}\big[S(T-t_{k+1})\mathcal{D}x(t_{k+1}^-)-S(T-t_k)\mathcal{D}x(t_k^+)\big]+S(T-t_j)\mathcal{D}x(t_j^+)\\
&-\sum_{0<t_k<T}C(T-t_k)I_k^1(x_{t_k})-\sum_{0<t_k<T}S(T-t_k)I_k^2(x_{t_k})\Big]\\
&-\int_0^T R(\alpha,\Pi_s^T)C(T-s)\mathcal{D}x(s)ds+\int_0^T R(\alpha,\Pi_s^T)z(s)dw(s)\\
&-\int_0^T R(\alpha,\Pi_s^T)C(T-s)G(s,x_s,\int_0^s g(s,\tau,x_\tau)d\tau)ds\\
&-\int_0^T R(\alpha,\Pi_s^T)S(T-s)F\big(s,x_{\rho(s,x_s)}\big)dw(s)\bigg\}.
\end{aligned}\tag{3.2}$$

We consider the space $\Upsilon=\{x\in\mathcal{M}^2 : x(0)=\varphi(0)=0\}$ endowed with the uniform convergence topology and define the operator $\mathcal{P}_\alpha:\Upsilon\to\Upsilon$ by $(\mathcal{P}_\alpha x)_0=0$ and for all $t\in[t_j,t_{j+1}]$, every $j=\overline{0,m}$,

$$\begin{aligned}
(\mathcal{P}_\alpha x)(t)=&C(t)\varphi(0)+S(t)[x_1-G(0,\varphi,0)]\\
&+\sum_{k=0}^{j-1}\big[S(t-t_{k+1})\mathcal{D}\overline{x}(t_{k+1}^-)-S(t-t_k)\mathcal{D}\overline{x}(t_k^+)\big]-S(t-t_j)\mathcal{D}\overline{x}(t_j^+)\\
&+\int_0^t C(t-s)\mathcal{D}\overline{x}(s)ds+\int_0^t S(t-s)Bu_\alpha(s,\overline{x})ds\\
&+\int_0^t C(t-s)G(s,\overline{x}_s,\int_0^s g(s,\tau,\overline{x}_\tau)d\tau)ds+\sum_{0<t_k<t}C(t-t_k)I_k^1(\overline{x}_{t_k})\\
&+\int_0^t S(t-s)F\big(s,\overline{x}_{\rho(s,\overline{x}_s)}\big)dw(s)+\sum_{0<t_k<t}S(t-t_k)I_k^2(\overline{x}_{t_k}),
\end{aligned}$$

where $\overline{x} : (-\infty, T] \to \mathbb{H}$ is the extension of x to $(-\infty, T]$ such that $\overline{x}_0 = \varphi$ and $\overline{x} = x|_J$. From Remark 2.2 and our assumptions, we infer that $\mathcal{P}_\alpha x \in \mathcal{M}^2$.

Let $\overline{\varphi} : (-\infty, T] \to \mathbb{H}$ is the extension of φ to $(-\infty, T]$ such that $\overline{\varphi}(\theta) = \varphi(0) = 0$ on J and $l_0^\varphi = \sup_{t\in\Sigma(\rho^-)} l^\varphi(t)$. For $r > 0$, let

$$B_r(0, \Upsilon) := \{y \in \Upsilon : \|y\|^2 \le r\}$$

then, for each r, $B_r(0, \Upsilon)$ is a bounded closed convex set in Υ.

Lemma 3.1. *Under the assumptions of Theorem 3.1, then there exists $r > 0$ such that $\mathcal{P}_\alpha\big(B_r(0, \Upsilon)\big) \subseteq B_r(0, \Upsilon)$.*

Proof. If this property is not true, then for each $r > 0$ and $t^r \in J$ there exists a function $x^r(t^r) \in B_r(0, \Upsilon)$ such that $\mathbf{E}\|(\mathcal{P}_\alpha x^r)(t^r)\|^2 > r$. Then, by Lemma 2.4, assumptions $(\mathbf{H0})-(\mathbf{H5})$, Hölder's inequality and Burkholder-Davis-Gundy's inequality, we have

$$\begin{aligned}
&\mathbf{E}\|u_\alpha(t^r, \overline{x}^r)\|^2 \\
&\le \frac{10}{\alpha^2} M_B M_S \Bigg\{ \|\mathbf{E}h\|^2 + M_C L^2 \mathbf{E}\|\varphi\|^2_{\mathcal{B}} + 2M_S(\mathbf{E}\|x_1\|^2 + \mathbf{E}\|G(0,\varphi,0)\|^2) \\
&\quad + \big((4m+1)M_S + TM_C\big) M_{\mathcal{D}} r + 2M_C m \sum_{k=1}^{m} \Big(2M_{I_k^1}\big[(M_T + l_0^\varphi)^2\|\varphi\|^2_{\mathcal{B}} + \widetilde{M}_T^2 r\big] + \|I_k^1(0)\|^2\Big) \\
&\quad + 2M_S m \sum_{k=1}^{m} \Big(2M_{I_k^2}\big[(M_T + l_0^\varphi)^2\|\varphi\|^2_{\mathcal{B}} + \widetilde{M}_T^2 r\big] + \|I_k^2(0)\|^2\Big) + Tr(Q)\int_J \|z(s)\|^2_{\mathcal{L}_2^0} ds \\
&\quad + M_C M_G T^2 \Big[\Big(4(M_T + l_0^\varphi)^2\|\varphi\|^2_{\mathcal{B}} + 4\widetilde{M}_T^2 r + 2\widetilde{M}_G\Big) + M_g\Big(4(M_T + l_0^\varphi)^2\|\varphi\|^2_{\mathcal{B}} + 4\widetilde{M}_T^2 r \\
&\quad + 2\widetilde{M}_g\Big)\Big] + M_S Tr(Q)\Psi_F\Big(2(M_T + l_0^\varphi)^2\|\varphi\|^2_{\mathcal{B}} + 2\widetilde{M}_T^2 r\Big)\int_J \zeta_F(s)ds \Bigg\} := \Delta. \qquad (3.3)
\end{aligned}$$

Thanks to (3.3) we get

$$\begin{aligned}
&r < \mathbf{E}\|(\mathcal{P}_\alpha x^r)(t^r)\|^2 \\
&\le 10\Bigg\{\frac{10}{\alpha^2} M_B^2 M_S^2 T^2 \Big(\|\mathbf{E}h\|^2 + Tr(Q)\int_J \|z(s)\|^2_{\mathcal{L}_2^0} ds\Big) \\
&\quad + \Big(1 + \frac{10}{\alpha^2} M_B^2 M_S^2 T^2\Big)\Bigg(\Big[M_C L^2 \mathbf{E}\|\varphi\|^2_{\mathcal{B}} + 2M_S(\mathbf{E}\|x_1\|^2 + \mathbf{E}\|G(0,\varphi,0)\|^2)\Big] \\
&\quad + 2M_C m \sum_{k=1}^{m} \Big(2M_{I_k^1}\big[(M_T + l_0^\varphi)^2\|\varphi\|^2_{\mathcal{B}}\big] + \|I_k^1(0)\|^2\Big) \\
&\quad + 2M_S m \sum_{k=1}^{m} \Big(2M_{I_k^2}\big[(M_T + l_0^\varphi)^2\|\varphi\|^2_{\mathcal{B}}\big] + \|I_k^2(0)\|^2\Big) + 2M_C M_G T^2 \Big[\Big(2(M_T + l_0^\varphi)^2 \\
&\quad \times \|\varphi\|^2_{\mathcal{B}} + \widetilde{M}_G\Big) + M_g\Big(2(M_T + l_0^\varphi)^2\|\varphi\|^2_{\mathcal{B}} + \widetilde{M}_g\Big)\Big]\Bigg) + \Big(1 + \frac{10}{\alpha^2} M_B^2 M_S^2 T^2\Big)
\end{aligned}$$

$$\times\Bigg[\Big(\big((4m+1)M_S+TM_C\big)M_{\mathcal{D}}+4M_C\widetilde{M}_T^2m\sum_{k=1}^{m}M_{I_k^1}+4M_S\widetilde{M}_T^2m\sum_{k=1}^{m}M_{I_k^2}+4M_CM_GT^2$$
$$\times\widetilde{M}_T^2(1+M_g)\Big)r+M_STr(Q)\Psi_F\Big(2(M_T+l_0^{\varphi})^2\|\varphi\|_{\mathcal{B}}^2+2\widetilde{M}_T^2r\Big)\int_J\zeta_F(s)ds\Bigg]\Bigg\}. \tag{3.4}$$

Dividing both sides of (3.4) by r and taking the limit as $r\to\infty$, we infer that

$$10\Big(1+\frac{10}{\alpha^2}M_B^2M_S^2T^2\Big)\Bigg[\big((4m+1)M_S+TM_C\big)M_{\mathcal{D}}+4M_C\widetilde{M}_T^2m\sum_{k=1}^{m}M_{I_k^1}+4M_S\widetilde{M}_T^2m\sum_{k=1}^{m}M_{I_k^2}$$
$$+4M_CM_GT^2\widetilde{M}_T^2(1+M_g)+2\widetilde{M}_T^2M_STr(Q)\lim_{\varepsilon\to\infty}\inf\frac{\Psi_F(\varepsilon)}{\varepsilon}\int_J\zeta_F(s)ds\Bigg]\geq 1,$$

which is contradictory with our assumption (3.1). Thus, for some $r>0$, $\mathcal{P}_\alpha\big(B_r(0,\Upsilon)\big)\subseteq B_r(0,\Upsilon)$. □

To prove that $\mathcal{P}_\alpha$ is a condensing operator, we decompose $\mathcal{P}_\alpha=\mathcal{P}_\alpha^1+\mathcal{P}_\alpha^2$, where $\mathcal{P}_\alpha^1,\mathcal{P}_\alpha^2$ are defined on $B_r(0,\Upsilon)$, respectively, by

$$\begin{aligned}(\mathcal{P}_\alpha^1x)(t)=&C(t)\varphi(0)+S(t)[x_1-G(0,\varphi,0)]\\&+\sum_{k=0}^{j-1}\big[S(t-t_{k+1})\mathcal{D}\overline{x}(t_{k+1}^-)-S(t-t_k)\mathcal{D}\overline{x}(t_k^+)\big]-S(t-t_j)\mathcal{D}\overline{x}(t_j^+)\\&+\int_0^tC(t-s)\mathcal{D}\overline{x}(s)ds+\sum_{0<t_k<t}C(t-t_k)I_k^1(\overline{x}_{t_k})\\&+\int_0^tC(t-s)G(s,\overline{x}_s,\int_0^sg(s,\tau,\overline{x}_\tau)d\tau)ds+\sum_{0<t_k<t}S(t-t_k)I_k^2(\overline{x}_{t_k}),\\(\mathcal{P}_\alpha^2x)(t)=&\int_0^tS(t-s)F\big(s,\overline{x}_{\rho(s,\overline{x}_s)}\big)dw(s)+\int_0^tS(t-s)Bu_\alpha(s,\overline{x})ds.\end{aligned}$$

Lemma 3.2. *Under the assumptions of Theorem 3.1, then $\mathcal{P}_\alpha^1$ is a contractive mapping.*

Proof. In view of (**H1**) $-$ (**H3**), (**H5**) and note that $\|x_s-y_s\|_{\mathcal{B}}\leq\widetilde{M}_T\sup_{s\in J}\mathbf{E}\|x(s)-y(s)\|$, for any $x,y\in B_r(0,\Upsilon)$, we see that

$$\mathbf{E}\|(\mathcal{P}_\alpha^1x)(t)-(\mathcal{P}_\alpha^1y)(t)\|^2\leq 6\Big[\big((4m+1)M_S+TM_C\big)M_{\mathcal{D}}+M_C\widetilde{M}_T^2m\sum_{k=1}^{m}M_{I_k^1}+M_S\widetilde{M}_T^2m\sum_{k=1}^{m}M_{I_k^2}$$
$$+M_CM_GT^2\widetilde{M}_T^2(1+M_g)\Big]\sup_{s\in J}\mathbf{E}\|\overline{x}(s)-\overline{y}(s)\|^2.$$

Taking the supremum over t, we obtain

$$\|(\mathcal{P}_\alpha^1x)-(\mathcal{P}_\alpha^1y)\|_{\mathcal{M}^2}^2\leq 6\Big[\big((4m+1)M_S+TM_C\big)M_{\mathcal{D}}+M_C\widetilde{M}_T^2m\sum_{k=1}^{m}M_{I_k^1}+M_S\widetilde{M}_T^2m\sum_{k=1}^{m}M_{I_k^2}$$
$$+M_CM_GT^2\widetilde{M}_T^2(1+M_g)\Big]\|x-y\|_{\mathcal{M}^2}^2.$$

By assumption (3.1), we infer that $\mathcal{P}_\alpha^1$ is a contractive mapping. □

Lemma 3.3. *Under the assumptions of Theorem 3.1, then $\mathcal{P}_\alpha^2$ is continuous and compact on $B_r(0,\Upsilon)$.*

Proof. The proof divided into the following three steps.

Step 1. We show that $\mathcal{P}_\alpha^2$ is continuous. Let $\{x^n\}_{n=0}^{\infty} \subseteq B_r(0,\Upsilon)$, with $x^n \to x$ in $\mathcal{M}^2$. From the Axiom ($\mathbf{A}_1$), it is easy to see that $(\overline{x^n})_t \to \overline{x}_t$ as $n \to \infty$ uniformly for $t \in (-\infty, T]$. By assumptions (**H0**), (**H4**), for each $t \in J$, as $n \to \infty$, we obtain

$$F\big(t, \overline{x^n}_{\rho(t,(\overline{x^n})_t)}\big) \to F\big(t, \overline{x}_{\rho(t,\overline{x}_t)}\big),$$

and since

$$\mathbf{E}\Big\|F\big(t, \overline{x^n}_{\rho(t,(\overline{x^n})_t)}\big) - F\big(t, \overline{x}_{\rho(t,\overline{x}_t)}\big)\Big\|^2 \le 2\zeta_F(t)\Psi_F\big(r^\star\big),$$

where $r^\star := 2(M_T + l_0^\varphi)^2\|\varphi\|_{\mathcal{B}}^2 + 2\widetilde{M}_T^2 r$.

Then by the Lebesgue majorant Theorem, we can conclude that

$$\|(\mathcal{P}_\alpha^2 x^n) - (\mathcal{P}_\alpha^2 x)\|_{\mathcal{M}^2}^2 \xrightarrow{n \to \infty} 0.$$

Therefore, $\mathcal{P}_\alpha^2$ is continuous.

Step 2. The set $\mathcal{P}_\alpha^2\big(B_r(0,\Upsilon)\big) = \big\{(\mathcal{P}_\alpha^2 x)(t) : x \in B_r(0,\Upsilon)\big\}$ is relatively compact in $\mathbb{H}$, for every $t \in J$. Subsequently, we show that $\big\{(\mathcal{P}_\alpha^2 x)(t) : x \in B_r(0,\Upsilon)\big\}$ is uniformly bounded. Indeed, we have

$$\mathbf{E}\|(\mathcal{P}_\alpha^2 x)(t)\|^2 \le 2M_S Tr(Q)\Psi_F\big(r^\star\big)\int_J \zeta_F(s)ds + 2T^2 M_B M_S \Delta < \infty.$$

Thus, the set $\big\{(\mathcal{P}_\alpha^2 x)(t) : x \in B_r(0,\Upsilon)\big\}$ is uniformly bounded.

Step 3. The set $\big\{(\mathcal{P}_\alpha^2 x)(t) : x \in B_r(0,\Upsilon)\big\}$ is an equicontinuous family of functions on J. The functions $\big\{(\mathcal{P}_\alpha^2 x) : x \in B_r(0,\Upsilon)\big\}$ are equicontinuous at $t = 0$. For each $x \in B_r(0,\Upsilon)$ and $0 < t_1 < t_2 \le T$, we have

$$\begin{aligned}
&\mathbf{E}\|(\mathcal{P}_\alpha^2 x)(t_2) - (\mathcal{P}_\alpha^2 x)(t_1)\|^2 \\
&\le 4\Big[Tr(Q)\Psi_F\big(r^\star\big)\int_0^{t_1}\|S(t_2-s) - S(t_1-s)\|^2\zeta_F(s)ds + Tr(Q)M_S\Psi_F\big(r^\star\big)\int_{t_1}^{t_2}\zeta_F(s)ds \\
&\quad + M_B T\Delta\int_0^{t_1}\|S(t_2-s) - S(t_1-s)\|^2 ds + M_B M_S(t_2-t_1)^2\Delta\Big]. \qquad (3.5)
\end{aligned}$$

The inequality (3.5) tends to 0 by the continuity of the function $t \to \|S(t)\|$ and when $t_2 - t_1 \to 0$. Therefore, the left hand side of the inequality (3.5) tends to 0 as $t_2 - t_1 \to 0$. This implies that $\big\{(\mathcal{P}_\alpha^2 x)(t) : x \in B_r(0,\Upsilon)\big\}$ is a family of equicontinuous functions on J. Hence, by Arzelá-Ascoli's theorem we conclude that $\mathcal{P}_\alpha^2$ is compact. □

Therefore, the Sadovskii fixed point theorem allows us to conclude that system (1.1) has at least one mild solution on J. This completes the proof of Theorem 3.1. □

The following corollary follows immediately from Theorem 3.1 and provides a generalization and extension of the main existence result in [47].

Corollary 3.1. *Assume that $B = 0$ and $\mathcal{D} = 0$ in the system (1.1). Then, the system (1.1) has at least one mild solution on J.*

The next main result in this section concerning the approximate controllability of mild solutions of (1.1) can now be stated as follows:

Theorem 3.2. *Assume that the assumptions of Theorem 3.1 hold and in addition, hypothesis* (**H7**), (**H8**) *are satisfied. Then, the system (1.1) is approximately controllable on* J.

Proof. By Theorem 3.1, $\mathcal{P}_\alpha$ has a fixed point x^*_α in $\mathbb{H}$. By the stochastic Fubini theorem [12], it is easy to see that

$$\begin{aligned}x^*_\alpha(T) =& h - \alpha R(\alpha, \Pi_0^T)\Big[\mathbf{E}h - C(T)\varphi(0) - S(T)[x_1 - G(0,\varphi,0)]\\&+ \sum_{k=0}^{j-1}\big[S(T-t_{k+1})\mathcal{D}x^*(t^-_{k+1}) - S(T-t_k)\mathcal{D}x^*(t^+_k)\big] - S(T-t_j)\mathcal{D}x^*(t^+_j)\\&+ \sum_{0<t_k<T} C(T-t_k)I^1_k(x^*_{t_k}) + \sum_{0<t_k<T} S(T-t_k)I^2_k(x^*_{t_k})\Big]\\&+ \int_0^T \alpha R(\alpha,\Pi_s^T)C(T-s)\mathcal{D}x^*(s)ds - \int_0^T \alpha R(\alpha,\Pi_s^T)z(s)dw(s)\\&+ \int_0^T \alpha R(\alpha,\Pi_s^T)C(T-s)G(s,x^*_s,\int_0^s g(s,\tau,x^*_\tau)d\tau)ds\\&+ \int_0^T \alpha R(\alpha,\Pi_s^T)S(T-s)F\big(s,x^*_{\rho(s,x^*_s)}\big)dw(s).\end{aligned}$$

By assumption (**H8**), there exists a sequence, still denoted by

$$\Big\{G\big(s,x^*_s,\int_0^s g(s,\tau,x^*_\tau)d\tau\big), F\big(s,x^*_{\rho(s,x^*_s)}\big)\Big\},$$

weakly converging to, say, $\big\{G(s,w,g(s,\tau,w)), F(s,w)\big\}$ in $\mathbb{H}\times\mathcal{L}^0_2$. On the other hand, by assumption (**H7**), for all $0 \le t < T$, $\alpha R(\alpha,\Pi_s^T) \xrightarrow{\alpha\to 0^+} 0$ strongly and moreover $\|\alpha R(\alpha,\Pi_s^T)\| \le 1$. Therefore, by the Lebesgue majorant Theorem and the compactness of $C(t)$, $S(t)$, $t > 0$, it follows that

$$\begin{aligned}&\mathbf{E}\|x^*_\alpha(T) - h\|^2\\ \le& \mathbf{E}\Big\|\alpha R(\alpha,\Pi_0^T)\Big[\mathbf{E}h - C(T)\varphi(0) - S(T)[x_1 - G(0,\varphi,0)]\\&+ \sum_{k=0}^{j-1}\big[S(T-t_{k+1})\mathcal{D}x^*(t^-_{k+1}) - S(T-t_k)\mathcal{D}x^*(t^+_k)\big] - S(T-t_j)\mathcal{D}x^*(t^+_j)\\&+ \sum_{0<t_k<T} C(T-t_k)I^1_k(x^*_{t_k}) + \sum_{0<t_k<T} S(T-t_k)I^2_k(x^*_{t_k})\Big]\\&- \int_0^T \alpha R(\alpha,\Pi_s^T)C(T-s)\mathcal{D}x^*(s)ds + \int_0^T \alpha R(\alpha,\Pi_s^T)z(s)dw(s)\\&- \int_0^T \alpha R(\alpha,\Pi_s^T)C(T-s)G(s,x^*_s,\int_0^s g(s,\tau,x^*_\tau)d\tau)ds\\&- \int_0^T \alpha R(\alpha,\Pi_s^T)S(T-s)F\big(s,x^*_{\rho(s,x^*_s)}\big)dw(s)\Big\|^2 \xrightarrow{\alpha\to 0^+} 0.\end{aligned}$$

Thus, $x^*_\alpha(T) \to h$ holds, which shows that the system (1.1) is approximately controllable. Theorem 3.2 is proved. □

Now, according to Theorem 3.1 and Theorem 3.2, if assumption (**H5**) replaced by assumption (**H6**), then we can also get corresponding results as Theorem 3.1 and Theorem 3.2. Indeed, we have the following theorem.

Theorem 3.3. *Assume that the assumptions* (**H0**) – (**H4**) *and* (**H6**) *hold. If*

$$10\Big(1+\frac{10}{\alpha^2}M_B^2M_S^2T^2\Big)\Bigg[\big((4m+1)M_S+TM_C\big)M_{\mathcal{D}}+4M_C\widetilde{M}_T^2m\sum_{k=1}^{m}\lim_{\varepsilon\to\infty}\inf\frac{\Omega_k(\varepsilon)}{\varepsilon}$$
$$+4M_S\widetilde{M}_T^2m\sum_{k=1}^{m}\lim_{\varepsilon\to\infty}\inf\frac{\Theta_k(\varepsilon)}{\varepsilon}+4M_CM_GT^2\widetilde{M}_T^2(1+M_g)$$
$$+2\widetilde{M}_T^2M_STr(Q)\lim_{\varepsilon\to\infty}\inf\frac{\Psi_F(\varepsilon)}{\varepsilon}\int_J\zeta_F(s)ds\Bigg]\le 1,$$

then the system (1.1) has at least one mild solution on J.

Proof. We sketch the proof only, as it resembles the arguments of Theorem 3.1. Similarly as before, we define the operator $\mathcal{P}_\alpha : \Upsilon \to \Upsilon$ as in the Theorem 3.1, then we deduce that $\mathcal{P}_\alpha\big(B_r(0,\Upsilon)\big) \subseteq B_r(0,\Upsilon)$. Further, note that to prove that $\mathcal{P}_\alpha$ is a condensing operator, we introduce the decomposition $\mathcal{P}_\alpha = \mathcal{P}^1_\alpha + \mathcal{P}^2_\alpha + \mathcal{P}^3_\alpha$, where $\mathcal{P}^1_\alpha, \mathcal{P}^2_\alpha, \mathcal{P}^3_\alpha$ are defined on $B_r(0,\Upsilon)$, respectively, by

$$(\mathcal{P}^1_\alpha x)(t) = C(t)\varphi(0)+S(t)[x_1-G(0,\varphi,0)]-S(t-t_j)\mathcal{D}\overline{x}(t_j^+)$$
$$+\sum_{k=0}^{j-1}\big[S(t-t_{k+1})\mathcal{D}\overline{x}(t_{k+1}^-)-S(t-t_k)\mathcal{D}\overline{x}(t_k^+)\big]$$
$$+\int_0^t C(t-s)\mathcal{D}\overline{x}(s)ds+\int_0^t C(t-s)G(s,\overline{x}_s,\int_0^s g(s,\tau,\overline{x}_\tau)d\tau)ds,$$
$$(\mathcal{P}^2_\alpha x)(t) = \int_0^t S(t-s)F\big(s,\overline{x}_{\rho(s,\overline{x}_s)}\big)dw(s)+\int_0^t S(t-s)Bu_\alpha(s,\overline{x})ds,$$
$$(\mathcal{P}^3_\alpha x)(t) = \sum_{0<t_k<t}C(t-t_k)I^1_k(\overline{x}_{t_k})+\sum_{0<t_k<t}S(t-t_k)I^2_k(\overline{x}_{t_k}).$$

then, by the same way as in the proof of Theorem 3.1, we can easily shown that the operator $\mathcal{P}^1_\alpha$ is a contraction while $\mathcal{P}^2_\alpha$ is compact. On the other hand, from Lemma 2.1, by using the same arguments as Theorem 3.2 in [17] we can show that $\mathcal{P}^3_\alpha$ is compact. As a consequence of Lemma 2.2, we infer that $\mathcal{P}_\alpha = \mathcal{P}^1_\alpha + \mathcal{P}^2_\alpha + \mathcal{P}^3_\alpha$ has a fixed point which is the mild solution for the system (1.1) on J. Thus we have completed the proof of Theorem 3.3. □

Theorem 3.4. *Assume that the assumptions of Theorem 3.3 hold and in addition, hypothesis* (**H7**), (**H8**) *are satisfied. Then, the system (1.1) is approximately controllable on J.*

Proof. By the same way as in the proof of Theorem 3.2, we infer that the system (1.1) is approximately controllable on J. We omit it here. □

4. Application

In this section, the established previous results are applied to study the approximate controllability of the stochastic nonlinear wave equation with state-dependent delay. Specifically, we consider the following approximate controllability for a class of damped second-order impulsive neutral stochastic integro-differential system with state-dependent delay of the form:

$$\begin{cases} d\Big[\dfrac{\partial}{\partial t}y(t,\xi) - \widetilde{G}\big(t, y(t-\tau,\xi), \displaystyle\int_0^s \widetilde{g}(t,s,y(s-\tau,\xi))ds\big)\Big] \\ \quad = \Big[\dfrac{\partial^2}{\partial \xi^2}y(t,\xi) + \gamma\dfrac{\partial}{\partial t}y(t,\xi) + \displaystyle\int_0^\pi \delta(s)\dfrac{\partial}{\partial t}y(t,s)ds + Bu(t,\xi)\Big]dt \\ \qquad + \widetilde{F}\big[t, y(s-\rho_1(\tau)\rho_2(\|y(t)\|,\xi)\big]d\beta(s), \quad t\neq t_k, \quad t\in J, \quad \tau>0, \quad \xi\in[0,\pi], \\ \Delta y(t_k)(\xi) = \int_{-\infty}^{t_k}\eta_k(t_k-s)y(s,\xi)ds, \quad k=\overline{1,m}, \quad \xi\in[0,\pi], \\ \Delta y'(t_k)(\xi) = \int_{-\infty}^{t_k}\mu_k(t_k-s)y(s,\xi)ds, \quad k=\overline{1,m}, \quad \xi\in[0,\pi], \\ y(t,0)=y(t,\pi)=0, \quad \frac{\partial}{\partial t}y(0,\xi)=x_1(\xi), \quad t\in J, \quad \xi\in[0,\pi], \\ y(t,\xi)=\varphi(t,\xi), \quad t\in(-\infty,0], \quad \xi\in[0,\pi], \end{cases} \tag{4.1}$$

where $\beta(t)$ is a standard one-dimensional Wiener process in $\mathbb{H}$ defined on a stochastic basis $(\Omega,\mathcal{F},\mathbf{P})$, $0=t_0<t_1<\cdots<t_m<T$ are prefixed numbers, γ is a prefixed real number, $\delta\in\mathbb{H}$, $\rho_i:[0,\infty)\to[0,\infty)$, $i=1,2$ are continuous, and $\varphi\in\mathcal{B}$, where the phase space $\mathcal{B}$ introduced in Example 2.1. We take $\mathbb{H}=L^2([0,\pi])$ with the norm $\|\cdot\|$. Define $A:\mathbb{H}\to\mathbb{H}$ by $Ax=x''$ with domain

$$D(A)=\{x(\cdot)\in\mathbb{H}: x, x' \text{ are absolutely continuous}, \ x''\in\mathbb{H},\ x(0)=x(\pi)=0\}.$$

The spectrum of A consists of the eigenvalues $-n^2$ for $n\in\mathbb{N}$, with associated eigenvectors $e_n(\xi):=\sqrt{\frac{2}{\pi}}\sin n\xi$, $n=1,2,3,....$ Furthermore, the set $\{e_n:n\in\mathbb{N}\}$ is an orthogonal basics in $\mathbb{H}$. Then

$$Ax=\sum_{n=1}^{\infty} n^2\langle x,e_n\rangle e_n, \quad x\in D(A). \tag{4.2}$$

Using (4.2), one can easily verify that the operators $C(t)$ defined by

$$C(t)x=\sum_{n=1}^{\infty} cos(nt)\langle x,e_n\rangle e_n, \quad t\in\mathbb{R},$$

form a cosine function on $\mathbb{H}$, with associated sine function

$$S(t)x=\sum_{n=1}^{\infty}\frac{sin(nt)}{n}\langle x,e_n\rangle e_n, \quad t\in\mathbb{R}.$$

It is clear that (see Ref. [41]), for all $x\in\mathbb{H}$, $t\in\mathbb{R}$, $C(\cdot)x$ and $S(\cdot)x$ are periodic functions with $\|C(t)\|\le 1$ and $\|S(t)\|\le 1$.

Now, we define the linear continuous mapping B from

$$U=\big\{u=\sum_{n=2}^{\infty}u_n e_n \mid \|u\|_U^2:=\sum_{n=2}^{\infty}u_n^2<\infty\big\}$$

to $\mathbb{H}$ as follows:

$$Bu = 2u_2e_1 + \sum_{n=2}^{\infty} u_ne_n.$$

On the other hand, it is easy to see that if $x = \sum_{n=1}^{\infty}\langle x, e_n\rangle e_n$, then

$$B^*v = (2v_1 + v_2)e_2 + \sum_{n=3}^{\infty} v_ne_n$$

and let $\|B^*S^*(t)x\|^2 = 0$, then we infer that $x = 0$, which means that deterministic linear system corresponding to (4.1) is approximately controllable on J.

Obviously, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach space. Hence for $(t,\psi) \in J\times\mathcal{B}$, where $\psi(\theta)x = \psi(\theta, x)$, $(\theta, x) \in (-\infty, 0]\times[0,\pi]$. Let $y(t)(\xi) = y(t,\xi)$ and define the operators $G : J\times\mathcal{B}\times\mathbb{H} \to \mathbb{H}$, $F : J\times\mathcal{B} \to \mathcal{L}_2^0$, $g : J\times J\times\mathcal{B} \to \mathbb{H}$, $\mathcal{D} : \mathbb{H} \to \mathbb{H}$, $\rho : J\times\mathcal{B} \to (-\infty, T]$, $I_k^1, I_k^2 : \mathcal{B} \to \mathbb{H}$, $k = \overline{1,m}$ by

$$\begin{aligned}
G\Big(t,\psi,\int_0^s \widetilde{G}(t,s,\psi)ds\Big)(\xi) &= \widetilde{g}\Big(t,\psi(\theta,\xi),\int_0^s \widetilde{g}(t,s,\psi)ds\Big),\\
g(t,s,\psi)(\xi) &= \widetilde{g}\big(t,s,\psi(\theta,\xi)\big),\\
\mathcal{D}\psi(\xi) &= \gamma\psi(t,\xi) + \int_0^\pi \delta(s)\psi(t,s)ds,\\
F(t,\psi)(\xi) &= \widetilde{F}\big(t,\psi(\theta,\xi)\big),\\
\rho(t,\psi) &= \rho_1(t)\rho_2(\|\psi(0)\|),\\
I_k^1(t,\psi)(\xi) &= \int_{-\infty}^0 \eta_k(-s)\psi(\theta)(\xi)ds, \quad k=\overline{1,m},\\
I_k^2(t,\psi)(\xi) &= \int_{-\infty}^0 \mu_k(-s)\psi(\theta)(\xi)ds, \quad k=\overline{1,m}.
\end{aligned}$$

Then, the system (4.1) can be written in the abstract form as the system (1.1). Further, we can impose some suitable conditions on the above defined functions as those in the assumptions $(\mathbf{H0})-(\mathbf{H8})$. Therefore, by Theorem 3.2 and Theorem 3.4, we can conclude that the system (4.1) is approximately controllable on J.

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