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MODIFIED EXPONENTIAL CHEBYSHEV OPERATIONAL MATRICES OF DERIVATIVES FOR SOLVING HIGH-ORDER PARTIAL DIFFERENTIAL EQUATIONS IN UNBOUNDED DOMAINS

M. A. RAMADAN, K. R. MOHAMED, T. S. EL DANAF, M. A. ABD EL SALAM

Abstract. In this paper, a modified type of exponential Chebyshev operational matrices of derivatives is presented. The introduced operational matrices were employed for solving high-order linear partial differential equations (PDEs) with variable coefficients under general form of conditions by collocation method. The method is based on the approximation by the truncated double exponential Chebyshev (EC) series. The PDEs and conditions are transformed into block matrix equations, which correspond to a system of linear algebraic equations with the unknown EC coefficients, by using EC collocation points. Combining these matrix equations and then solving the system yields the EC coefficients of the solution function. Numerical examples are included to demonstrate the validity and applicability of the method.

1. Introduction

It is well known that the numerical methods have played an important role in solving PDEs. Some of the most known numerical methods that widely applied to solving PDEs are finite differences and finit element methods [22], [6]. Recently, various approximate methods are discussed, such as differential transform method, Adomian decomposition method and Homotopy analysis method see [25, 3, 23, 11, 13, 8, 20].

In addition, spectral methods are one of the principal methods for solving differential equations. The main idea of spectral methods is to approximate the solutions of differential equations by means of truncated series of orthogonal polynomials. The most used versions of spectral methods are tau, collocation, and Galerkin methods [9, 5, 1, 24, 14]. One of the most important orthogonal polynomials is Chebyshev polynomials. Mehamet Sezer [21] and Akyuz-Dascioglu [7] used the Chebyshev matrix method which is based on the Chebyshev coefficients for high order partial differential equations with complicated conditions, and most of them were on bounded intervals.

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Koc and Kurnaz [10] have proposed modified type of Chebyshev polynomials as an alternative to the solutions of PDEs given in all real domain. In their studies, the basis functions called exponential Chebyshev (EC) functions $E_n(x)$ that are orthogonal in $(-\infty, \infty)$ and applied to solve PDEs. This kind of extension tackles the problems over the whole real domain. In our previous reports [15] and [16] we introduced a modified form of the operational matrix of the derivatives by processing the truncation made by Koc [10] and applied it to ordinary and systems differential equations defined in whole rang. Recently, we reported a new operational matrix of derivatives of EC functions for solving ODEs in unbounded domains [17]. Also a new operational matrix of derivatives based on exponential Chebyshev of the second kind (ESC) functions introduced by us and employed to solve ordinary and partial differential equations with variable coefficients in unbounded domains using the collocation method [18] and [19]. In this paper, we introduce a modification of the operational matrices of the partial derivaties given in [10], that based on the relations between EC functions and their derivatives. It is very effective method for direct solution of PDEs with complicated conditions and it is also useful to obtain the approximate solution in whole domain.

The rest of the paper is organized as follows; in section 2, the definition, properties and the operational matrices of EC functions are listed, in section 3, the form of high-order linear non-homogeneous partial differential equations is presented, in section 4, we formulated the fundamental matrix relation based on collocation points, in section 5, method of solution is presented and finally, section 6 contains numerical illustrations and results that are compared with the exact solutions to demonstrate the applicability and accuracy of the present method.

2. Properties of double EC functions

Basu [4], has given the product $T_{r,s}(x,y) = T_r(x) \cdot T_s(x)$,which is a private form of Chebyshev polynomials. Mason [12] also has used a Chebyshev polynomial expression for an infinitely differentiable function $u(x,\ y)$ defined on the square S $(-\infty < x, y < \infty)$, where $T_r(x)$ and $T_s(y)$ are Chebyshev polynomials of the first kind.

2.1. **Definition.** The double functions are in the following form

$$E_{r,s}(x,y) = E_r(x) \cdot E_s(y), \tag{1}$$

where $E_r(x)$ and $E_s(y)$ are EC functions of the form

$$E_r(x) = T_r\left(\frac{e^x-1}{e^x+1}\right), \qquad E_s(y) = T_s\left(\frac{e^y-1}{e^y+1}\right).$$

The recurrence relation takes the form

$$\begin{aligned} E_{r+1,s}(x,y) &= \left\{2\left(\tfrac{e^x-1}{e^x+1}\right)E_r(x) - E_{r-1}\left(x\right)\right\} \cdot E_s(y), \qquad r \geq 1 \\ E_{r,s+1}(x,y) &= E_r(x) \cdot \left\{2\left(\tfrac{e^y-1}{e^y+1}\right)E_s(y) - E_{s-1}\left(y\right)\right\}. \qquad s \geq 1 \end{aligned} \tag{2}$$

2.2. **Orthogonality of double EC functions.** If the function $f(x,y)$ is continuous in S, then $E_{r,s}(x,y)$ are orthogonal with respect to the weight function

$$w(x,y) = \sqrt{e^{x+y}} / \left(e^x+1\right)\left(e^y+1\right),$$

with the orthogonality condition [10, 15, 16]

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_{i,j}(x,y)E_{k,l}(x,y)w(x,y)dxdy = \begin{cases} \pi^2 & i=j=k=l=0 \\ \frac{\pi^2}{4} & i=k\neq 0,\ j=l\neq 0 \\ \frac{\pi^2}{2} & i=k=0,\ j=l\neq 0 \\ & or\ \ i=k\neq 0,\ j=l=0 \\ 0 & other\,wise \end{cases} . \tag{3}$$

Also the product relation of double EC functions used in the partial derivatives relations is given by

$$E_{m,n}(x,y)\cdot E_{i,j}(x,y) = \frac{1}{4}[E_{m+i,n+j}(x,y)+E_{m+i,|n-j|}(x,y)+E_{|m-i|,n+j}(x,y)+E_{|m-i|,|n-j|}(x,y)].$$

2.3. **Function expansion in terms of double EC functions.** A function $u(x,y)$ well defined over the square S, can be expanded as [12]

$$u(x,y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_{r,s}E_{r,s}(x,y), \tag{4}$$

where

$$a_{r,s} = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y)E_{r,s}(x,y)w(x,y)dxdy}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_{r,s}^2(x,y)w(x,y)dxdy}.$$

If $u(x,y)$ in expression (4) is truncated to $n,\ m<\infty$ in terms of the double EC functions, it will take the form

$$U(x,y) \cong \sum_{r=0}^{m}\sum_{s=0}^{n} a_{r,s}E_{r,s}(x,y) = \mathbf{E}(x,y)\cdot\mathbf{A}, \tag{5}$$

where $\mathbf{E}(x,y)$ is $1\times(m+1)(n+1)$ vector with elements $E_{r,s}(x,y)$ and $\boldsymbol{A}$ is an unknown coefficient column vector are of the form

$$\begin{aligned}\mathbf{E}(x,y) = [&E_{0,0}(x,y)\quad E_{0,1}(x,y)\quad ....\quad E_{0,n}(x,y)\ \ E_{1,0}(x,y)\quad E_{1,1}(x,y)\quad ....\quad E_{1,n}(x,y) \\ &.....\ \ E_{m,0}(x,y)\quad E_{m,1}(x,y)\quad ....\quad E_{m,n}(x,y)],\end{aligned} \tag{6}$$

$$\begin{aligned}\mathbf{A} = [&a_{0,0}(x,y)\quad a_{0,1}(x,y)\quad ....\quad a_{0,n}(x,y)\ \ a_{1,0}(x,y)\quad a_{1,1}(x,y)\quad ....\quad a_{1,n}(x,y) \\ &.....\ \ a_{m,0}(x,y)\quad a_{m,1}(x,y)\quad ....\quad a_{m,n}(x,y)]^T.\end{aligned} \tag{7}$$

2.4. **The derivatives of double EC functions.**

Proposition 1. *The relation between the row vector $\boldsymbol{E}(x,y)$ and its (k)th-order derivative is given as*

$$\boldsymbol{E}^{(i,j)}(x,y) \cong \boldsymbol{E}(x,y)(\boldsymbol{D}_x)^i(\boldsymbol{D}_y)^j, \tag{8}$$

where, $\boldsymbol{D}_x$ and $\boldsymbol{D}_y$ are the $(m+1)(n+1)\times(m+1)(n+1)$ operational matrices for the derivatives, and the general form of them is

$$\boldsymbol{D}_x = diag\left(\frac{\alpha}{4}\boldsymbol{I},\ \ \boldsymbol{0},\ \ \frac{-\alpha}{4}\boldsymbol{I}\right)^T, \qquad \alpha\ = 0,\ 1,\ ...,\ m \tag{9}$$

and

$$\boldsymbol{D}_y = \begin{bmatrix} \mu & 0 & \cdots & 0 \\ 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu \end{bmatrix}^T, \quad \mu = diag\left(\frac{\beta}{4},\ 0,\ \frac{-\beta}{4}\right), \qquad \beta\ = 0,\ 1,\ ...,\ n. \quad (10)$$

We note that $\boldsymbol{I}$ *and* $\boldsymbol{0}$ *are* $(n+1)\times(n+1)$ *identity and zero matrices in the block matrix* $\boldsymbol{D}_x$ *which is* $(m+1)\times(m+1)$*. Also* μ *is matrix is* $(n+1)\times(n+1)$ *in the block matrix* $\boldsymbol{D}_y$ *which is* $(m+1)\times(m+1)$*.*

For more details for definition 2.1 and the properties mentioned in subsections 2.2, 2.3, 2.4, proposition.1 and its proof we refer the reader to [10], [15] and [16]. Now we have noted that, in [17] Koc and Kurnaz considered that $E_{r,s}^{(1,0)}(x,y) = E_{r,s}^{(0,0)}(x,y) = 0$ for $r > m$ and $E_{r,s}^{(0,1)}(x,y) = E_{r,s}^{(0,0)}(x,y) = 0$ for $s > n$ in their work. This consideration based on truncation in the operational matrices $\mathbf{D}_x$ and $\mathbf{D}_y$ to be square matrices and the matrix multiplication become possible. Now we introduce a modification to the operational matrices of EC functions to include the neglected terms that processes this truncation in $\mathbf{D}_x$ and $\mathbf{D}_y$ in the next proposition.

Proposition 2. *The (i, j)-th partial derivatives of the row vector* $\boldsymbol{E}(x,y)$ *is given as*

$$\begin{aligned}\boldsymbol{E}^{(i,j)}(x,y) = \boldsymbol{E}(x,y)(\boldsymbol{D}_x)^i(\boldsymbol{D}_y)^j + \textstyle\sum_{l=0}^{j-1} \boldsymbol{B}_2^{(0,-l+j-1)}(x,y)(\boldsymbol{D}_y)^l(\boldsymbol{D}_x)^i \\ + \textstyle\sum_{k=0}^{i-1} \boldsymbol{B}_1^{(-k+i-1,0)}(x,y)\ (\boldsymbol{D}_x)^k(\boldsymbol{D}_y)^j.\end{aligned} \quad (11)$$

where, $\boldsymbol{D}_x$ *and* $\boldsymbol{D}_y$ *are given as before in* (9), (10)*, where* $\boldsymbol{B}_1(x,y)$*,* $\boldsymbol{B}_2(x,y)$ *are* $1\times(m+1)(n+1)$ *row vectors:*

$$\boldsymbol{B}_1(x,y) = [0 \quad 0 \quad ... \quad 0 \quad \frac{-m}{4}E_{m+1,0}(x,y) \quad \frac{-m}{4}E_{m+1,1}(x,y) \quad ... \quad \frac{-m}{4}E_{m+1,n}(x,y)], \quad (12)$$

$$\boldsymbol{B}_2(x,y)[0 \quad 0\ ...\ \frac{-n}{4}E_{0,n+1}(x,y) \quad 0 \quad 0\ ...\ \frac{-n}{4}E_{2,n+1}(x,y)\ ...\ 0 \quad 0\ ...\ \frac{-n}{4}E_{m,n+1}(x,y)]. \quad (13)$$

Before we prove our proposition we note that the two summations in (11) are actual terms to get the equality sign that was truncated in (8). These added terms will improve the obtained approximate solutions as will be shown in the numerical examples in section 6.

Proof. The first partial derivatives of the $\mathbf{E}(x,y)$ can be expressed with equality sign by

$$\mathbf{E}^{(1,0)}(x,y) = \mathbf{E}(x,y)\mathbf{D}_x + \mathbf{B}_1(x,y), \quad \mathbf{E}^{(0,1)}(x,y) = \mathbf{E}(x,y)\mathbf{D}_y + \mathbf{B}_2(x,y), \quad (14)$$

consequently, to obtain the matrix $\mathbf{E}^{(i,j)}(x,y)$, we can use the relation (14) as

$$\begin{aligned}&\mathbf{E}^{(1,0)}(x,y) = \mathbf{E}(x,y)\mathbf{D}_x + \mathbf{B}_1(x,y),\\ &\mathbf{E}^{(2,0)}(x,y) = \mathbf{E}^{(1,0)}(x,y)\mathbf{D}_x + \mathbf{B}_1^{(1,0)}(x,y) = \ (\mathbf{E}(x,y)\mathbf{D}_x + \mathbf{B}_1(x,y))\,\mathbf{D}_x + \mathbf{B}_1^{(1,0)}(x,y),\end{aligned}$$

then, by induction we get *i-th* partial derivative with respect to x as

$$\mathbf{E}^{(i,0)}(x,y) = \mathbf{E}(x,y)(\mathbf{D}_x)^i + \sum_{k=0}^{i-1} \mathbf{B}_1^{(-k+i-1,0)}(x,y)\ (\mathbf{D}_x)^k\ , \quad i \geq 1 \qquad (15)$$

where $\mathbf{B}_1^{(i,0)}(x,y) = [0 \quad 0 \,........\, 0 \quad \frac{-m}{4} E_{m+1,0}^{(i,0)}(x,y) \quad \frac{-m}{4} E_{m+1,1}^{(i,0)}(x,y) \,....\, \frac{-m}{4} E_{m+1,n}^{(i,0)}(x,y)]$.

Now, we will find the *j-th* partial derivative of the relation (15) with respect to the variable y as

$$\begin{aligned}\mathbf{E}^{(i,1)}(x,y) &= \mathbf{E}^{(0,1)}(x,y)(\mathbf{D}_x)^i + \textstyle\sum_{k=0}^{i-1} \mathbf{B}_1^{(-k+i-1,1)}(x,y)\ (\mathbf{D}_x)^k \\ &= \mathbf{E}(x,y)\mathbf{D}_y(\mathbf{D}_x)^i + \mathbf{B}_2(x,y)(\mathbf{D}_x)^i + \textstyle\sum_{k=0}^{i-1} \mathbf{B}_1^{(-k+i-1,1)}(x,y)\ (\mathbf{D}_x)^k,\end{aligned}$$

and

$$\mathbf{E}^{(i,2)}(x,y) = \mathbf{E}(x,y)(\mathbf{D}_y)^2(\mathbf{D}_x)^i + \mathbf{B}_2^{(0,1)}(x,y)(\mathbf{D}_x)^i + \sum_{k=0}^{i-1} \mathbf{B}_1^{(-k+i-1,2)}(x,y)\ (\mathbf{D}_x)^k,$$

finally, by induction we get $(i,\ j)$-*th* partial derivatives as

$$\begin{aligned}\mathbf{E}^{(i,j)}(x,y) = \mathbf{E}(x,y)(\mathbf{D}_y)^j(\mathbf{D}_x)^i &+ \textstyle\sum_{l=0}^{j-1} \mathbf{B}_2^{(0,-l+j-1)}(x,y)(\mathbf{D}_y)^l\ (\mathbf{D}_x)^i \\ &+ \textstyle\sum_{k=0}^{i-1} \mathbf{B}_1^{(-k+i-1,j)}(x,y)\ (\mathbf{D}_x)^k.\end{aligned} \tag{16}$$

Similarly, if we begin with the partial derivative of the variable y then we find the $(i,\ j)$-*th* partial derivatives as

$$\begin{aligned}\mathbf{E}^{(i,j)}(x,y) = \mathbf{E}(x,y)(\mathbf{D}_x)^i(\mathbf{D}_y)^j &+ \textstyle\sum_{l=0}^{j-1} \mathbf{B}_2^{(i,l+j-1)}(x,y)(\mathbf{D}_y)^l \\ &+ \textstyle\sum_{k=0}^{i-1} \mathbf{B}_1^{(-k+i-1,0)}(x,y)\ (\mathbf{D}_x)^k(\mathbf{D}_y)^j.\end{aligned} \tag{17}$$

Then from (16) and (17) we find that

$$\begin{aligned}\mathbf{E}^{(i,j)}(x,y) = \mathbf{E}(x,y)(\mathbf{D}_x)^i(\mathbf{D}_y)^j &+ \textstyle\sum_{l=0}^{j-1} \mathbf{B}_2^{(0,-l+j-1)}(x,y)(\mathbf{D}_y)^l(\mathbf{D}_x)^i \\ &+ \textstyle\sum_{k=0}^{i-1} \mathbf{B}_1^{(-k+i-1,0)}(x,y)\ (\mathbf{D}_x)^k(\mathbf{D}_y)^j.\end{aligned} \tag{18}$$

which end the proof. □

3. Application of the introduced modified version of derivatives for high-order PDEs

The form of high-order linear non-homogeneous partial differential equations with variable coefficients in unbounded domains is

$$\sum_{i=0}^{p}\sum_{j=0}^{r} q_{i,j}(x,y)\, u^{(i,j)}(x,y) = f(x,y), -\infty < x,\ y < \infty, \tag{19}$$

with the complicated conditions [21], [7], [10]

$$\sum_{t=1}^{\rho}\sum_{k=0}^{p}\sum_{j=0}^{r} b_{i,j}^{t} u^{(i,j)}\,(\omega_t,\eta_t) = \lambda\,,$$

and / or

$$\sum_{t=1}^{\nu}\sum_{k=0}^{p}\sum_{j=0}^{r} c_{i,j}^{t}(x) u^{(i,j)}\,(x,\gamma_t) = g(x)\,, \tag{20}$$

and / or

$$\sum_{t=1}^{\theta}\sum_{k=0}^{p}\sum_{j=0}^{r} d_{i,j}^{t}(y)\, u^{(i,j)}\,(\varepsilon_t, y) = h(y)\,,$$

where the $u^{(0,0)}(x,y) = u(x,y)$, $u^{(i,j)}(x,y) = \frac{\partial^{i+j}}{\partial x^i \partial y^j} u(x,y)$ and $q_{i,j}(x,y)$, $f(x,y)$, $c_{i,j}^{t}(x)$, $g(x)$, d_{ij}^{k}, $h(y)$ are known functions on the square S $(-\infty < x,\ y < \infty,)$, and $\omega_t, \eta_t,\ \gamma_t,\ \varepsilon_t$ are constant $\in\ (\infty, -\infty)$ and may be one or more of them tends to infinity. Now, we consider that the approximate solution $U(x,y)$ to the exact solution $u(x,y)$ of Eq. (19) defined by expression (5) and its *(i, j)-th* partial derivatives defined by Eq. (18) as

$$U(x,y)=\sum_{r=0}^{m}\sum_{s=0}^{n}a_{r,s}E_{r,s}(x,y),=\mathbf{E}(x,y)\cdot\mathbf{A}, \tag{21}$$

and

$$\begin{aligned}U^{(i,j)}(x,y)=[&\mathbf{E}(x,y)(\mathbf{D}_x)^i(\mathbf{D}_y)^j+\textstyle\sum_{l=0}^{j-1}\mathbf{B}_2^{(0,-l+j-1)}(x,y)(\mathbf{D}_y)^l(\mathbf{D}_x)^i\\&+\textstyle\sum_{k=0}^{i-1}\mathbf{B}_1^{(-k+i-1,0)}(x,y)\,(\mathbf{D}_x)^k(\mathbf{D}_y)^j]\mathbf{A}.\end{aligned}\quad . \tag{22}$$

4. Fundamental Matrix Relations

Let us define the collocation points [10], [15] and [16], so that $-\infty < x_i,\ y_i < \infty$, as

$$\begin{aligned}x_k=Ln\left(\frac{1+\cos\left(\frac{k\pi}{m}\right)}{1-\cos\left(\frac{k\pi}{m}\right)}\right),\quad y_l=Ln\left(\frac{1+\cos\left(\frac{l\pi}{n}\right)}{1-\cos\left(\frac{l\pi}{n}\right)}\right),\\(k=1,\ ...,\ m-1,\ l=1,\ ...,\ n-1)\end{aligned} \tag{23}$$

and at the boundaries

$$(k=0,\ \ k=m)\ \ x_0\to\infty,\ x_m\to-\infty,(l=0,\ \ l=n)\ \ y_0\to\infty,\ y_n\to-\infty,$$

since the double EC functions are convergent at both boundaries $\pm\infty$, namely their values are ± 1. The appearance of infinity in the collocation points does not cause a loss or divergence in the method. Then, we substitute the collocation points (23) into Eq. (19) to obtain

$$\sum_{i=0}^{p}\sum_{j=0}^{r}q_{i,j}(x_k,y_l)\,u^{(i,j)}(x_k,y_l)=f(x_k,y_l), \tag{24}$$

The system (24) can be written in the matrix form

$$\sum_{i=0}^{p}\sum_{j=0}^{r}\mathbf{Q}_{i,j}\ \mathbf{U}^{(i,j)}=\mathbf{F}\qquad p\le m,\quad r\le n, \tag{25}$$

where $\mathbf{Q}_{i,j}$ denotes the diagonal matrix with inner elements are $q_{i,j}(x_k,y_l)$ and $\boldsymbol{F}$ denotes the column matrix with the elements $f(x_k,y_l)$ where $k=0,\ 1,\ 2,\ ...,\ m$; $l=0,\ 1,\ 2,\ ...,\ n$, by substituting the collocation points (23) into derivatives of the unknown function as in Eq. (22) yields

$$\mathbf{U}^{(i,j)}=\begin{bmatrix}U^{(i,j)}(x_0,\ y_0)\\ \vdots\\ U^{(i,j)}(x_0,\ y_n)\\ U^{(i,j)}(x_1,\ y_0)\\ \vdots\\ U^{(i,j)}(x_1,\ y_n)\\ \vdots\\ U^{(i,j)}(x_n,\ y_m)\end{bmatrix}=\begin{matrix}[\mathbf{E}(\mathbf{D}_x)^i(\mathbf{D}_y)^j+\\ \sum_{l=0}^{j-1}\mathbf{B}_2^{(0,-l+j-1)}(\mathbf{D}_y)^l(\mathbf{D}_x)^i+\\ \sum_{k=0}^{i-1}\mathbf{B}_1^{(-k+i-1,0)}\ (\mathbf{D}_x)^k(\mathbf{D}_y)^j]\mathbf{A}\end{matrix}\quad , \tag{26}$$

where

$$\begin{aligned}\mathbf{E}=[&\mathbf{E}(x_0,y_0)\ \ \mathbf{E}(x_0,y_1)\ ...\ \mathbf{E}(x_0,y_n)\ \ \mathbf{E}(x_1,y_0)\ \ \mathbf{E}(x_1,y_1)\ ...\ \mathbf{E}(x_1,y_n)\\&...\ \ \mathbf{E}(x_m,y_0)\ \ \mathbf{E}(x_m,y_1)\ \ ...\ \ \mathbf{E}(x_m,y_n)]^T,\end{aligned}$$

and

$$\mathbf{B}_1 = [\mathbf{B}_1(x_0,y_0) \quad \mathbf{B}_1(x_0,y_1) \ldots \mathbf{B}_1(x_0,y_n) \quad \mathbf{B}_1(x_1,y_0) \quad \mathbf{B}_1(x_1,y_1) \ldots \mathbf{B}_1(x_1,y_n) \\ \ldots \ \mathbf{B}_1(x_m,y_0) \quad \mathbf{B}_1(x_m,y_1) \quad \ldots \quad \mathbf{B}_1(x_m,y_n)]^T,$$

$$\mathbf{B}_2 = [\mathbf{B}_2(x_0,y_0) \quad \mathbf{B}_2(x_0,y_1) \ldots \mathbf{B}_2(x_0,y_n) \quad \mathbf{B}_2(x_1,y_0) \quad \mathbf{B}_2(x_1,y_1) \ldots \mathbf{B}_2(x_1,y_n) \\ \ldots \ \mathbf{B}_2(x_m,y_0) \quad \mathbf{B}_2(x_m,y_1) \quad \ldots \quad \mathbf{B}_2(x_m,y_n)]^T.$$

Therefore, from Eq. (25), we get a system of equations "fundamental matrix" for the PDE will be in the form

$$\left(\sum_{i=0}^{p}\sum_{j=0}^{r} \mathbf{Q}_{i,j} \left\{ \begin{array}{l} \mathbf{E}(x,y)(\mathbf{D}_x)^i(\mathbf{D}_y)^j + \\ \sum_{l=0}^{j-1} \mathbf{B}_2^{(0,-l+j-1)}(x,y)(\mathbf{D}_y)^l(\mathbf{D}_x)^i + \\ \sum_{k=0}^{i-1} \mathbf{B}_1^{(-k+i-1,0)}(x,y)\ (\mathbf{D}_x)^k(\mathbf{D}_y)^j \end{array} \right\}\right) \mathbf{A} = \mathbf{F}, \tag{27}$$

which corresponds to a system of $(m+1)(n+1)$ linear algebraic equations with $(m+1)(n+1)$ double EC coefficients $a_{r,s}$ unknowns. By substituting the collocation points (23) in the conditions (20) by same procedure before we get the fundamental matrices for conditions as

$$\begin{array}{l}
\sum_{l=1}^{\rho}\sum_{k=0}^{p}\sum_{j=0}^{r} b_{i,j}^{t}(x_k) \left\{ \begin{array}{l} \mathbf{E}\,(\omega_t,\eta_t)\,(\mathbf{D}_x)^i(\mathbf{D}_y)^j + \\ \sum_{l=0}^{j-1} \mathbf{B}_2^{(0,-l+j-1)}(\omega_t,\eta_t)(\mathbf{D}_y)^l(\mathbf{D}_x)^i + \\ \sum_{k=0}^{i-1} \mathbf{B}_1^{(-k+i-1,0)}\,(\omega_t,\eta_t)\ (\mathbf{D}_x)^k(\mathbf{D}_y)^j \end{array} \right\} \mathbf{A} = \lambda, \\
\sum_{t=1}^{\nu}\sum_{k=0}^{p}\sum_{j=0}^{r} c_{i,j}^{t}(x_k) \left\{ \begin{array}{l} \mathbf{E}\,(x_k,\gamma_t)\,(\mathbf{D}_x)^i(\mathbf{D}_y)^j + \\ \sum_{l=0}^{j-1} \mathbf{B}_2^{(0,-l+j-1)}(x_k,\gamma_t)(\mathbf{D}_y)^l(\mathbf{D}_x)^i + \\ \sum_{k=0}^{i-1} \mathbf{B}_1^{(-k+i-1,0)}\,(x_k,\gamma_t)\ (\mathbf{D}_x)^k(\mathbf{D}_y)^j \end{array} \right\} \mathbf{A} = g(x_k) \\
\sum_{t=1}^{\theta}\sum_{k=0}^{p}\sum_{j=0}^{r} d_{i,j}^{t}(y_l) \left\{ \begin{array}{l} \mathbf{E}\,(\varepsilon_t,y)\,(\mathbf{D}_x)^i(\mathbf{D}_y)^j + \\ \sum_{l=0}^{j-1} \mathbf{B}_2^{(0,-l+j-1)}\,(\varepsilon_t,y)\,(\mathbf{D}_y)^l(\mathbf{D}_x)^i + \\ \sum_{k=0}^{i-1} \mathbf{B}_1^{(-k+i-1,0)}\,(\varepsilon_t,y)\ (\mathbf{D}_x)^k(\mathbf{D}_y)^j \end{array} \right\} \mathbf{A} = h(y_l)\,.
\end{array} \tag{28}$$

It is also noted that the structure of matrices $\mathbf{Q}_{i,j}$and $\mathbf{F}$ vary according to the number of collocation points and the structure of the problem. However,$\mathbf{E}$, $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{D}_x$ and $\mathbf{D}_y$ do not change their nature for fixed values of m and n which are truncation limits of the EC series. In other words, the changes in $\mathbf{E}$, $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{D}_x$ and $\mathbf{D}_y$ are just dependent on the number of collocation points.

5. Method of Solution

The fundamental matrix (27) for Eq. (19) corresponding to a system of $(m+1)(n+1)$ algebraic equations for the $(m+1)(n+1)$ unknown coefficients $[a_{0,0},\ a_{0,1},\ \ldots\ a_{0,n},\ a_{1,0},\ a_{1,1},;\ldots\ a_{1,n},\ \ldots.\ ,a_{m,0},\ a_{m,1},\ \ldots\ a_{m,n}]$. We can write the matrix (27) as

$$\mathbf{W}\ \mathbf{A}{=}\mathbf{F} \quad or \quad [\mathbf{W};\ \mathbf{F}], \tag{29}$$

and we can obtain the matrix form for the conditions by means of (28) in a compact form as

$$\mathbf{V}\ \mathbf{A} = \mathbf{R} \qquad or \quad [\mathbf{V};\mathbf{R}], \tag{30}$$

where $\boldsymbol{V}$ is a $h \times (m+1)(n+1)$ matrix and $\boldsymbol{R}$ is a $h \times 1$ matrix, so that h is the rank of the all row matrices as in (28) belong to the given conditions.

Then (29) together with (30) can be written following compact form:

$$\mathbf{W}^*\mathbf{A} = \mathbf{F}^*, \quad or \quad [\mathbf{W}^*; \mathbf{F}^*]. \tag{31}$$

Furthermore, the system (31) can be formed by appending the rows (30) on conditions to the system (29). Then the size of the system of algebraic equations increases and therefore $\mathbf{W}^*$ becomes a rectangular matrix. To solve this new system, the generalized inverse of $\mathbf{W}^*$ can be used [7], and so the double EC coefficients can be found as

$$\mathbf{A} = \text{geninv}(\mathbf{W}^*)\cdot\mathbf{F}^*.$$

The method procedure can be summarized by the following algorithm:
1. Calculating the matrix $\mathbf{W}$
2. Forming the matrix $\mathbf{W}^*$ by adding $\mathbf{V}$
3. Solving the system of algebraic equations and gitting the unnkwon coefficients

6. Test examples

We consider some numerical examples that are numerically treated by the above mentioned method. The numerical computations are carried out by the Mathematica. 7.0, with usual PC (Intel processor CORE i3 2.53 GHz, 2.00 GB RAM).

Example: 6.1

Consider the following differential equation

$$u^{(2,1)} + \frac{1}{1+e^x}u^{(1,0)} = f(x,y), \quad x, y \in (-\infty,\ \infty), \tag{32}$$

to be the test problem, with exact solution

$$u(x,y) = \left(1 + \frac{4}{1 - Coshx}\right)\left(Tanh\frac{y}{2}\right),$$

where, the function $f(x,y)$takes the form

$$f(x,y) = \frac{1}{4}Sech^4\left(\frac{x}{2}\right)\ Sech^2\left(\frac{y}{2}\right)(4 + (1 + Sinhx)\ Sinhy - Coshx\ (2 + Sinhy)),$$

and the conditions for this test example are

$$\begin{array}{l} u(x,y) = \frac{-3+Coshx}{1+Coshx},\ \ at\ \ y \to \infty, \\ u(x,y) = -1 \quad at\ \ x \to \infty\ \ and\ \ at\ \ y \to -\infty, \\ u(0,0) = 0,\ \ and\ \ u(x,0)\ = 0 \quad at\ \ x \to -\infty\ . \end{array}$$

The fundamental matrix takes the form

$$\left\{\mathbf{Q}_{1,0}\left[\mathbf{E}(\mathbf{D}_x)^1 + \mathbf{B}_1\right] + \mathbf{Q}_{2,1}\left[\mathbf{E}(\mathbf{D}_x)^2(\mathbf{D}_y)^1 + \mathbf{B_2}(\mathbf{D}_x)^2 + \mathbf{B}_1\mathbf{D}_y\mathbf{D}_x + \mathbf{B}_1^{(1,0)}\mathbf{D}_y\right]\right\}\mathbf{A} = \mathbf{F},$$

We take m=n=8, where, the approximate solution given by

$$U(x,y) = a_{0,0}E_{,00}(x,y) + a_{0,1}E_{0,1}(x,y) + \cdots + a_{8,8}E_{8,8}(x,y),$$

then, by using the algorithm of the method we get the matrix of coefficinets as,

$$\begin{array}{l} a_{0,0} =\ a_{0,1} =\ ...\ = a_{0,8} = 0, \\ a_{1,0} = 0,\ a_{1,1} = ...\ = a_{1,8} = 0, \\ a_{2,0} = 0,\ \ a_{2,1} = 1,\ \ a_{2,2} =\ ...\ = a_{2,8} = 0, \\ \qquad \vdots \\ a_{8,0} =\ a_{8,1} =\ ... =\ a_{8,8} = 0, \end{array}$$

then, $U(x,y) = E_{2,1}(x,y)$, that is close to

$$U(x,y) = \left(-1+2\left(\frac{e^x-1}{e^x+1}\right)^2\right)\left(\frac{e^y-1}{e^y+1}\right) = \left(1+\frac{4}{1-Coshx}\right)\left(Tanh\frac{y}{2}\right),$$

which represent the exact solution of the problem, the CPU time used by the program is 55.068 seconds.

Example: 6.2

Consider the following differential equation [10] and [19]

$$u_{xy} - \frac{2}{1+e^x}u_y = \frac{4e^y}{(1+e^x)^2(1+e^y)^2}, \quad x,y \in (-\infty,\ \infty), \tag{33}$$

with conditions

$$u_y(0,y) = 0, \qquad u(x,0) = 0.$$

The fundamental matrix takes the form

$$\left\{\mathbf{Q}_{0,1}\left[\mathbf{E}(\mathbf{D}_y)^1 + \mathbf{B_2}\right] + \mathbf{Q}_{1,1}\left[\mathbf{E}(\mathbf{D}_x)^1(\mathbf{D}_y)^1 + \mathbf{B_2}\mathbf{D_x} + \mathbf{B}_1\mathbf{D}_y\right]\right\}\mathbf{A} = \mathbf{F},$$

We take *m=n=8*, where, the approximate solution given by

$$U(x,y) = a_{0,0}E_{,00}(x,y) + a_{0,1}E_{0,1}(x,y) + \cdots + a_{8,8}E_{8,8}(x,y),$$

then, after the augmented matrix of the system and conditions are computed, we obtain the solution as,

$$\begin{aligned}
&a_{0,0} = \ a_{0,1} = \ ... \ = a_{0,8} = 0,\\
&a_{1,0} = 0,\ a_{1,1} = 1,\ a_{1,2} = a_{1,3} = ... \ = a_{1,8} = 0,\\
&a_{2,0} = \ a_{2,1} = \ ... \ = a_{2,8} = 0,\\
&\qquad \vdots\\
&a_{8,0} = \ a_{8,1} = \ ... = \ a_{8,8} = 0,
\end{aligned}$$

then, $U(x,y) = E_{1,1}(x,y)$, that is close to

$$U(x,y) = \left(\frac{e^x-1}{e^x+1}\right)\left(\frac{e^y-1}{e^y+1}\right) = \left(\frac{e^{x+y}-e^x-e^y+1}{(e^x+1)(e^y+1)}\right),$$

which represent the exact solution of the problem. On the other hand, solution given in [10] at $n = m = 15$ the approximate solution doesn't give the exact solution, also the time used is 44.898 seconds.

Example: 6.3

The Cauchy problem [21], for the one-dimensional homogeneous wave equation is given by

$$\begin{aligned}
&u_{yy} - c^2u_{xx} = 0, \quad -\infty < x < \infty, \quad y \in [0,\ \infty),\\
&u(x,0) = f(x), \quad u_y(x,0) = g(x), \quad -\infty < x < \infty.
\end{aligned} \tag{34}$$

The solution of this problem can be interpreted as the amplitude of a sound wave propagating in very long and narrow pipe, which in practice can be considered as one-dimensional infinite medium. The initial conditions f, g are given functions that represent the amplitude u and the velocity u_y of the string at time $y = 0$. The exact solution of (34) is given by D'Alembert's formula

$$u(x,y) = \frac{1}{2}\left[f(x+cy) + f(x-cy)\right] + \frac{1}{2c}\int_{x-cy}^{x+cy} g(s)ds$$

Thus, if we take $f(x) = Sech(x)$ and $g(x) = 0$, we use our present method to solve (34), at $n = m = 8, 10$ by using double EC collocation points, we obtain the approximate solution $U(x, y)$.

In Table.1, the exact and approximate solutions are listed according to different values of *x, y*. The calculation of L_2 norm $\left(L_2 = \sqrt{h \sum_{i=0}^{I} (u^i - U^i)^2}\right)$ presented in Table.2, shows that the grater $n,\ m$ give good accuracy at step size $h = 0.1, x \in [-2,\ 2], y \in [0,\ 1]$ and, that our proposed method is accurate more than method presented in [10].

Table.3 compares between the CPU time of our method and the method given in [10], and shows that our method takes more time because of the truncation in the other algorithim. "The time is mentioned by seconds".

In Figure.1 we seek the contour plots of the exact, approximate solutions (n=m=8, 10) and the approximate solution by the method given in [10], such that $x \in [-2,\ 2], y \in [0,\ 1]$. In Figure.2 the error function of exact and approximate solutions for example 6.3 given where $x \in [0, 1], y \in [0, 1]$.

Table. 1 comparing the approximate and exact solution

x	y	Exact solution	Our method n=m=8	Abs error	Our method n=m=10	Abs error
0	1	0.648054	0.64752	5.34×10^{-4}	0.647913	1.41×10^{-4}
0.1	0.9	0.697877	0.697648	2.28×10^{-4}	0.697785	9.19×10^{-5}
0.2	0.5	0.876667	0.876798	1.131×10^{-4}	0.876665	1.76×10^{-6}
0.3	0.7	0.786531	0.78637	1.161×10^{-4}	0.786684	1.52×10^{-4}
0.4	0.6	0.814191	0.8137	4.91×10^{-4}	0.814417	2.25×10^{-4}
0.5	0.5	0.824027	0.823361	6.66×10^{-4}	0.824273	2.46×10^{-4}
0.6	0.3	0.827211	0.826981	2.30×10^{-4}	0.827267	5.62×10^{-5}
0.7	0.4	0.777981	0.777705	2.76×10^{-4}	0.778422	4.41×10^{-4}
0.8	0.7	0.710058	0.711906	1.84×10^{-3}	0.712759	2.7×10^{-3}
0.9	0.2	0.69802	0.697918	1.01×10^{-4}	0.698057	3.74×10^{-5}
1	1	0.632901	0.641166	8.26×10^{-3}	0.635261	2.36×10^{-3}

Table. 2 comparing the L_2norm

	our method L_2	Method [10] L_2
n=m=8	1.57006×10^{-3}	3.58864×10^{-2}
n=m=10	1.116309×10^{-3}	1.4611×10^{-2}

Table. 3 comparing the CPU time (seconds)

	our method	Method [10]
$n = m = 8$	69.403	7.785
$n = m = 10$	102.821	14.946

contour plot exact solution contour plot [10] $n = m = 8$
contour plot present method $n = m = 10$ contour plot present method $n = m = 10$

Figure .1 contour plots for example 6.3 $x \in [-2, 2], y \in [0, 1]$

Error functions for $n = m = 10$ Error functions for $n = m = 8$

Figure .2 error function of exact and approximate solutions for example 6.3 $x \in [0, 1], y \in [0, 1]$

Example: 6.4

Let us consider the Poisson equation [21], [19] and [2]

$$\bigtriangledown^2 u = f(x,y), \quad 0 \leq x,\ y \leq 1 \tag{35}$$

Poisson equation arises in steady state heat problems with time independent heat sources, where the Dirichlet boundary conditions in general form is

$$\begin{array}{ll} u(0,y) = f_1(y), & u(x,0) = g_1(x), \\ u(1,y) = f_2(y), & u(x,1) = g_2(x). \end{array}$$

If we chose the exact solution to be as

$$u(x,y) = (1+e^x)^{-1}(1+e^y)^{-1},$$

then, we find

$$\begin{array}{rl} f_1(y) = \frac{1}{2}(1+e^y)^{-1}, & g_1(x) = \frac{1}{2}(1+e^x)^{-1}, \\ f_2(y) = (1+e^y)^{-1}(1+e)^{-1}, & g_2(x) = (1+e)^{-1}(1+e^x)^{-1}. \end{array}$$

Appling our present method to solve (35), at $n = m = 8$ by using double EC collocation points, we obtain the approximate solution

$$U(x,y) = 0.25E_{0,0}(x,y) - 0.25E_{0,1}(x,y) - 0.25E_{1,0}(x,y) + 0.25E_{1,1}(x,y).$$

By simplifying the previous relation we reach to

$$U(x,y) = (1+e^x)^{-1}(1+e^y)^{-1},$$

which represent the exact solution of Poisson equation (35) with the connected conditions. In figure 3 we seek the contour plots of the exact and approximate solutions where, $x,\ y \in [0,1]$.

contour plot exact solution contour plot approximate solution

Figure .3 contour plots for example 6.4 $x \in [0,1], y \in [0,1]$

7. Conclusion

In this paper, a modified type of collocation method for solving high-order linear partial differential equations with variable coefficients under most general form of conditions is investigated. The method based on the approximation by the truncated double exponential Chebyshev (EC) series, and modified definition of the partial derivatives are presented. All principles and properties of this modification type are derived and introduced by us as a new definition. The PDEs and conditions are transformed into block matrix equations, which correspond to a system of linear algebraic equations with the unknown EC coefficients, by using EC collocation points. The generalized inverse is used to solve this linear system and finding the EC coefficients. Illustrative examples are used to demonstrate the applicability, effectiveness and the accuracy of the proposed technique. In addition, an interesting feature of this method is to find the analytical exact solution if the equation has an exact solution of rational exponential form. The method can also be extended to high-order nonlinear partial differential equation with variable coefficients, but some modifications are required.

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Mohamed A. Ramadan
Mathematics Department, Faculty of Science, Menoufia University, Shebein El-Koom, EGYPT
E-mail address: ramadanmohamed13@yahoo.com

Kamal R. Mohamed
Mathematics Department, Faculty of Science, Al-Azhar University, Nasr-City,11884, Cairo, EGYPT
E-mail address: kamal_raslan@yahoo.com

Talaat S. El Danaf
Mathematics Department, Faculty of Science, Menoufia University, Shebein El-Koom, EGYPT
E-mail address: talaat11@yahoo.com

Mohamed A. Abd El Salam
Mathematics Department, Faculty of Science, Al-Azhar University, Nasr-City,11884, Cairo, EGYPT
E-mail address: mohamed_salam1985@yahoo.com