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DYNAMICS OF A SYSTEM OF $k-$DIFFERENCE EQUATIONS

N. HADDAD, J. F. T. RABAGO

ABSTRACT. This note deals with the the boundedness and global stability of the non-negative solutions of the system of difference equations

$$y_{n+1}^{(1)} = \frac{a_1 y_n^{(1)}}{b_1 + c_1 (y_n^{(2)})^{p_1}},\ y_{n+1}^{(2)} = \frac{a_2 y_n^{(2)}}{b_2 + c_2 (y_n^{(3)})^{p_2}}, \ldots,$$
$$y_{n+1}^{(k-1)} = \frac{a_{k-1} y_n^{(k-1)}}{b_{k-1} + c_{k-1} (y_n^{(k)})^{p_{k-1}}}, y_{n+1}^{(k)} = \frac{a_k y_n^{(k)}}{b_k + c_k (y_n^{(1)})^{p_k}},$$

where $n \in \mathbb{N}_0$, $p_1, p_2, \ldots, p_k \in \mathbb{N}$, $k = 2, 3, ...$, the parameters a_i, b_i, c_i, $i = 1, \ldots, k$, and the initial conditions $y_0^{(1)}, y_0^{(2)}, \ldots, y_0^{(k)}$ are non-negative real numbers.

1. Introduction and preliminaries

Our aim in this paper is to investigate the qualitative behavior of solutions of the system of nonlinear difference equations

$$y_{n+1}^{(1)} = \frac{a_1 y_n^{(1)}}{b_1 + c_1 (y_n^{(2)})^{p_1}},\ y_{n+1}^{(2)} = \frac{a_2 y_n^{(2)}}{b_2 + c_2 (y_n^{(3)})^{p_2}},\ \ldots, y_{n+1}^{(k)} = \frac{a_k y_n^{(k)}}{b_k + c_k (y_n^{(1)})^{p_k}}, \quad \text{(S)}$$

where $n \in \mathbb{N}_0$, $p_1, p_2, \ldots, p_k \in \mathbb{N}$, $k = 2, 3, ...$, the parameters a_i, b_i, c_i, $i = 1, \ldots, k$ and the initial conditions $y_0^{(1)}, y_0^{(2)}, \ldots, y_0^{(k)}$ are non-negative real numbers. This investigation is motivated by a work of Yang and Yang [24] on the case when $k = 2$ and $p_1 = p_2$.

Recently, the study of difference equations have gained the interest of many mathematicians and became one of the hot topic in analysis (see, e.g., [1]–[27]). Some of these results dealt with the qualitative behavior of solutions of various systems of nonlinear difference equations, while others focused on finding the solutions in closed forms. In this paper, as in [24], we are interested on the boundedness and stability of solutions of system (S). We emphasize that our work generalizes the results exhibited in [24].

The change of variables

$$x_n^{(i)} = \left(\frac{c_{i-1}}{b_{i-1}}\right)^{\frac{1}{p_{i-1}}} y_n^{(i)}, \quad i = 1, \ldots, k,$$

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with $c_0 = c_k$, $b_0 = b_k$, $p_0 = p_k$ reduces system (S) to the system

$$x_{n+1}^{(i)} = \frac{\alpha_i x_n^{(i)}}{1 + (x_n^{(i+1)})^{p_i}}, \quad i = 1, \dots, k, \ n \in \mathbb{N}_0, \tag{M}$$

with the convention $x_n^{(k+1)} = x_n^{(1)}$. Let $\mathbf{X}_n = (x_n^{(1)}, \dots, x_n^{(k)})^T$, then the system (M) can be written in the vectorial form:

$$\mathbf{X}_{n+1} = \mathbf{F}(\mathbf{X}_n) = (f_1(\mathbf{X}_n), \dots, f_k(\mathbf{X}_n))^T, \quad n \in \mathbb{N}_0,$$

where

$$f_i(\mathbf{X}_n) = f_i(x_n^{(1)}, \dots, x_n^{(k)}) = \frac{\alpha_i x_n^{(i)}}{1 + (x_n^{(i+1)})^{p_i}}, \quad n \in \mathbb{N}_0.$$

Definition 1 A point $\bar{\mathbf{X}} = (\bar{x}_1, \dots, \bar{x}_k) \in [0, \infty[^k$ is called an *equilibrium point* of the system (M) if

$$\bar{\mathbf{X}} = \mathbf{F}(\bar{\mathbf{X}})$$

that is

$$\bar{x}_i = f_i(\bar{x}_1, \dots, \bar{x}_k) \tag{P.1}$$

for all $i = 1, \dots, k$.

The linearized system associated of the system (M) about an equilibrium point $\bar{\mathbf{X}}$ has the form

$$\mathbf{Y}_{n+1} = \mathbf{J_F}(\bar{\mathbf{X}})\ \mathbf{Y}_n, \quad n \in \mathbb{N}_0 \tag{P.2}$$

where $\mathbf{Y}_n := (y_n^{(1)}, \dots, y_n^{(k)})^T$ and $\mathbf{J_F}(\bar{\mathbf{X}}) := \mathrm{D}\mathbf{F}(\bar{\mathbf{X}})$ denotes the Jacobian matrix of $\mathbf{F}$ evaluated at point $\bar{\mathbf{X}}$. The characteristic equation of system (P.2) is given by

$$\det(\mathbf{J_F} - \lambda \mathbf{I}_k) = 0, \tag{P.3}$$

where $\mathbf{I}_k$ denotes the $k \times k$ identity matrix.

Theorem 1 [Linearized Stability Theorem]

i) If all roots of the characteristic equation (P.3) have absolute value less than one, then the equilibrium $\bar{\mathbf{X}}$ of system (M) is *locally asymptotically stable.*
ii) If at least one root of the characteristic equation (P.3) have absolute value greater than one, then the equilibrium $\bar{\mathbf{X}}$ of system (M) is *unstable.*

Now, in our work, the equilibrium point of system (M) are the non-negative solutions of the system

$$\bar{x}_i = \frac{\alpha_i \bar{x}_i}{1 + (\bar{x}_{i+1})^{p_i}}, \quad i = 1, \dots, k.$$

Clearly, $\bar{\mathbf{X}} := (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_k) = (0, 0, \dots, 0) =: \mathbf{0}$ is always an equilibrium point of system (M). However, when $\alpha_i > 1$ for all $i = 1, \dots, k$, system (M) also has the unique positive equilibrium point

$$\bar{\mathbf{X}}^+ := ((\alpha_k - 1)^{\frac{1}{p_k}}, (\alpha_1 - 1)^{\frac{1}{p_1}}, \dots, (\alpha_{k-1} - 1)^{\frac{1}{p_{k-1}}}).$$

2. Local and Global Stability of the Equilibrium Points

In theorems follows we examine the local and global stability of the trivial equilibrium $\mathbf{0}$ of system (M).

Theorem 2

i) If $\alpha_i < 1$, for every $i = 1, \ldots, k$, then the trivial equilibrium is locally asymptotically stable.

ii) If there exist an $i_0 \in \{1, \ldots, k\}$ such that $\alpha_{i_0} > 1$, then the trivial equilibrium is unstable.

Proof. The linearized system associated to system (M) about the trivial equilibrium is given by

$$\mathbf{X}_{n+1} = \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_k)\mathbf{X}_n$$

where $\mathbf{X}_n := (x_n^{(1)}, \ldots, x_n^{(k)})^T$ and its characteristic equation is

$$(\lambda - \alpha_1)(\lambda - \alpha_2) \cdots (\lambda - \alpha_k) = 0.$$

Using Theorem 1, the desired result follows.

Theorem 3 Assume that $\alpha_i < 1$ for all $i = 1, \ldots, k$, then the trivial equilibrium is globally asymptotically stable.

Proof. Let $\{\mathbf{X}_n\}_{n=0}^{\infty}$ be a solution of system (M). It suffices to show that the trivial equilibrium is a global attractor of all solutions of system (M), i.e. it suffices to prove that $x_n^{(i)} \to 0$ as $n \to \infty$ for all $i = 1, \ldots, k$. Note that

$$x_{n+1}^{(i)} = \frac{\alpha_i x_n^{(i)}}{1 + (x_n^{(i+1)})^{p_i}} \leq \alpha_i x_n^{(i)}$$

for all $i = 1, \ldots, k$. Then, by discrete analogue of Grönwall's inequality, we get $x_n^{(i)} \leq \alpha_i^n x_0^{(i)}$, for all $i = 1, \ldots, k$. Letting n goes to infinity, we get $\lim_{n\to\infty} x_n^{(i)} = 0$ as desired.

In theorem follow we examine the stability of the positive equilibrium $\bar{\mathbf{X}}^+$ of (M).

Theorem 4 Let $\alpha_i > 1$ for all $i = 1, \ldots, k$. Then, regardless of the parity of k, the positive equilibrium point $\bar{\mathbf{X}}^+$ of system (M) is always unstable.

Proof. The linearized system about the positive equilibrium point $\bar{\mathbf{X}}^+$ is

$$\mathbf{Y}_{n+1} = \mathbf{B}\,\mathbf{Y}_n,$$

where $\mathbf{Y}_n := (y_n^{(1)}, \ldots, y_n^{(k)})^T$ and

$$\mathbf{B} := [b_{i\,j}] = \begin{cases} 1, & \text{if } j = i, \\ \dfrac{-p_i \bar{x}_i (\bar{x}_{i+1})^{p_i - 1}}{\alpha_i}, & \text{if } j = i+1,\ i = 1, \ldots, k-1, \\ \dfrac{-p_k \bar{x}_k (\bar{x}_1)^{p_k - 1}}{\alpha_k}, & \text{if } i = k \text{ and } j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the characteristic equation is given by

$$(1 - \lambda)^k + (-1)^{k-1} b_{1\,2} b_{2\,3} \cdots b_{(k-1)\,k} b_{k\,1} = 0$$

and so

$$(1-\lambda)^k - \prod_{i=1}^{k} \frac{p_i(\alpha_i - 1)}{\alpha_i} = 0.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous differentiable function such that

$$f(x) = (1-x)^k - \prod_{i=1}^{k} \frac{p_i(\alpha_i - 1)}{\alpha_i},$$

then

$$f'(x) = -k(1-x)^{k-1}.$$

We consider two cases for k.

<u>CASE 1</u>. Suppose k is even, i.e. $k = 2l$ for some non-negative integer l. Since f is continuous on $(1, \infty)$, and $f(1) = -\prod_{i=1}^{2l} \frac{p_i(\alpha_i - 1)}{\alpha_i} < 0$ and $\lim_{x\to\infty} f(x) = +\infty$, then there exists $\lambda_1 > 1$ such that $f(\lambda_1) = 0$. Hence, it follows that the positive equilibrium $\bar{\mathbf{X}}^+$ is unstable.

<u>CASE 2</u>. Meanwhile, if k is odd, i.e. $k = 2l + 1$ for some non-negative integer l, then we have $\lim_{x\to-\infty} f(x) = +\infty$ while $\lim_{x\to+\infty} f(x) = -\infty$. Since f is strict decreasing, then by the Mean Value Theorem, there exists λ_1, real unique, such that $f(\lambda_1) = 0$.

Here on, we examine whether λ_1 is inside or outside the unit disk. First, we note that $f(1) = -\prod_{i=1}^{2l+1} \frac{p_i(\alpha_i - 1)}{\alpha_i} < 0$. Then, for all $x \in (1, \infty)$ we have $f(x) < -\prod_{i=1}^{2l+1} \frac{p_i(\alpha_i - 1)}{\alpha_i}$, i.e. $f(x) \neq 0$. Thus, λ_1 must be outside the interval $(1, \infty)$. Furthermore, we have $f(-1) = 2^{2l+1} - \prod_{i=1}^{2l+1} \frac{p_i(\alpha_i - 1)}{\alpha_i}$, from which we distinguish two possibilities.

<u>POSSIBILITY 1</u>. Suppose $2^{2l+1} < \prod_{i=1}^{2l+1} \frac{p_i(\alpha_i - 1)}{\alpha_i}$. Then, $f(-1) < 0$ which, in turn, implies that $\lambda_1 \in (-\infty, -1)$. By Theorem 1, it follows that the positive equilibrium $\bar{\mathbf{X}}^+$ is unstable.

<u>POSSIBILITY 2</u>. Meanwhile, if $2^{2l+1} \geq \prod_{i=1}^{2l+1} \frac{p_i(\alpha_i - 1)}{\alpha_i}$, then when $f(-1) > 0$, we get $\lambda_1 \in (-1, 1)$ and when $f(-1) = 0$, λ_1 takes exactly the value of -1. In addition, the remaining roots of f, which we denote by $\{\lambda_j\}_2^{2l+1}$, must be complex conjugate pairs.

Let us consider the function g defined as

$$g(y) = y^{2l+1} - \prod_{i=1}^{2l+1} \frac{p_i(\alpha_i - 1)}{\alpha_i}.$$

Clearly, g admits $2l+1$ roots. Denote these roots by $\{\beta_j\}_1^{2l+1}$. Then, by Viéte's formula, we have $\sum_{j=1}^{2l+1} \beta_j = 0$. Evidently, $\lambda_j = 1 - \beta_j$ $(j = 1, \dots, 2l+1)$, are roots of f. Hence, $\sum_{j=1}^{2l+1} \lambda_j = 2l+1$. Thus, $\sum_{j=2}^{2l+1} \lambda_j > 2l$, since $\lambda_1 < 1$ (the real root of the function f). Now, since $\{\lambda_j\}_2^{2l+1}$ are complex conjugate roots (which come in pairs), then we have $\lambda_{2j} = a_j + ib_j$ and $\lambda_{2j+1} = a_j - ib_j$ for $j = 1, \dots, l$, where $a_j, b_j \in \mathbb{R}$. Therefore,

$$\sum_{j=2}^{2l+1} \lambda_j = 2\sum_{i=1}^{l} a_i > 2l.$$

Thus, there exists $r \in \{1, \dots, l\}$ such that $a_r > 1$. So, it follows that

$$|\lambda_{2r}| = |\lambda_{2r+1}| = |a_r + ib_r| = |a_r - ib_r| > 1$$

and by Theorem 1, it follows that the positive equilibrium $\bar{\mathbf{X}}^+$ is again unstable.

3. Existence of Unbounded Solutions

Here we investigate the existence of unbounded solutions of system (M).

Theorem 5 Assume that $k = 2l$, $\alpha_i > 1$ for all $i = 1, \dots, 2l$ and let $\{\mathbf{X}_n\}_0^\infty$ be a solution of system (M) such that one of the following hypothesis holds

(H.1) $x_0^{(2j)} \geq \bar{x}_{2j}$ and $x_0^{(2j-1)} < \bar{x}_{2j-1}$ with $j = 1, \dots, l$, or

(H.2) $x_0^{(2j)} < \bar{x}_{2j}$ and $x_0^{(2j-1)} \geq \bar{x}_{2j-1}$ with $j = 1, \dots, l$.

Then, $\{\mathbf{X}_n\}_0^\infty$ non-oscillates about the equilibrium point $\bar{\mathbf{X}} = (\bar{x}_1, \dots, \bar{x}_{2l})$.

Proof. Let $\{\mathbf{X}_n\}_0^\infty$ be a solution of system (M), such that hypothesis (H.1) holds, then.

$$x_1^{(2j)} = \frac{\alpha_{2j} x_0^{(2j)}}{1 + (x_0^{(2j+1)})^{p_{2j+1}}} \geq \frac{\alpha_{2j}\bar{x}_{2j}}{1 + (\bar{x}_{2j+1})^{p_{2j+1}}} = \bar{x}_{2j},$$

and

$$x_1^{(2j-1)} = \frac{\alpha_{2j-1} x_0^{(2j-1)}}{1 + (x_0^{(2j)})^{p_{2j}}} < \frac{\alpha_{2j-1}\bar{x}_{2j-1}}{1 + (\bar{x}_{2j})^{p_{2j}}} = \bar{x}_{2j-1},$$

for all $j = 1, \dots, l$. By induction, one easily finds that

$$x_n^{(2j)} \geq \bar{x}_{2j} \qquad \text{and} \qquad x_n^{(2j-1)} < \bar{x}_{2j-1}$$

for all $n \in \mathbb{N}_0$ and $j = 1, \dots, l$. Similarly, if hypothesis (H.2) holds, we get

$$x_n^{(2j)} < \bar{x}_{2j} \qquad \text{and} \qquad x_n^{(2j-1)} \geq \bar{x}_{2j-1}$$

for all $n \in \mathbb{N}_0$ and $j = 1, \dots, l$. This proves the theorem.

The following theorem can be proven in similar fashion as in the previous result.

Theorem 6 Assume that $k = 2l+1$, $\alpha_i > 1$ for all $i = 1, \dots, 2l+1$ and let $\{\mathbf{X}_n\}_0^\infty$ be a solution of system (M), if one of the following hypothesis holds.

(H.1) $x_0^{(2j)} \geq \bar{x}_{2j}$, $x_0^{(2j-1)} < \bar{x}_{2j-1}$ with $j = 1, \ldots, l$, and $x_0^{(2l+1)} < \bar{x}_{2l+1}$, or

(H.2) $x_0^{(2j)} < \bar{x}_{2j}$, $x_0^{(2j-1)} \geq \bar{x}_{2j-1}$ with $j = 1, \ldots, l$, and $x_0^{(2l+1)} \geq \bar{x}_{2l+1}$.

Then, $\{\mathbf{X}_n\}_0^\infty$ non-oscillates about the equilibrium point $\bar{\mathbf{X}} = (\bar{x}_1, \ldots, \bar{x}_{2l+1})$.

Theorem 7 Assume that $k = 2l$, $\alpha_i > 1$ for all $i = 1, \ldots, 2l$ and let $\{\mathbf{X}_n\}_0^\infty$ be a solution of system (M), then the following statements are true.

(i) If $(x_0^{(2j)}, x_0^{(2j-1)}) \in (0, \bar{x}_{2j}) \times (\bar{x}_{2j-1}, \infty)$ $(j = 1, \ldots, l)$, then

$$\lim_{n\to\infty} x_n^{(2j)} = 0 \quad \text{and} \quad \lim_{n\to\infty} x_n^{(2j-1)} = \infty$$

for all $j = 1, \ldots, l$.

(ii) If $(x_0^{(2j)}, x_0^{(2j-1)}) \in (\bar{x}_{2j}, \infty) \times (0, \bar{x}_{2j-1})$ $(j = 1, \ldots, l)$, then

$$\lim_{n\to\infty} x_n^{(2j)} = \infty \quad \text{and} \quad \lim_{n\to\infty} x_n^{(2j-1)} = 0$$

for all $j = 1, \ldots, l$.

Proof. The proof of the above results are similar, so we only prove (i). Let $\{\mathbf{X}_n\}_0^\infty$ be a solution of system (M). If $(x_0^{(2j)}, x_0^{(2j-1)}) \in (0, \bar{x}_{2j}) \times (\bar{x}_{2j-1}, \infty)$ $(j = 1, \ldots, l)$, then from Theorem 5, we have

$$x_n^{(2j)} < \bar{x}_{2j} \quad \text{and} \quad x_n^{(2j-1)} > \bar{x}_{2j-1}$$

for all $j = 1, \ldots, l$ and $n \in \mathbb{N}_0$. Hence,

$$x_{n+1}^{(2j)} = \frac{\alpha_{2j} x_n^{(2j)}}{1 + (x_n^{(2j+1)})^{p_{2j+1}}} < \frac{\alpha_{2j} x_n^{(2j)}}{1 + (\bar{x}_{2j+1})^{p_{2j+1}}} = x_n^{(2j)},$$

and

$$x_{n+1}^{(2j-1)} = \frac{\alpha_{2j-1} x_n^{(2j-1)}}{1 + (x_n^{(2j)})^{p_{2j}}} > \frac{\alpha_{2j-1} x_n^{(2j-1)}}{1 + (\bar{x}_{2j})^{p_{2j}}} = x_n^{(2j-1)},$$

for all $j = 1, \ldots, l$ and so, the results follow.

The following theorem can be proven in similar fashion as in the previous result.

Theorem 8 Assume that $k = 2l+1$, $\alpha_i > 1$ for all $i = 1, \ldots, 2l+1$ and let $\{\mathbf{X}_n\}_0^\infty$ be a solution of system (M), then the following statements are true.

(i) If $(x_0^{(2j)}, x_0^{(2j-1)}) \in (0, \bar{x}_{2j}) \times (\bar{x}_{2j-1}, \infty)$ $(j = 1, \ldots, l)$ and $x_0^{(2l+1)} > \bar{x}_{2l+1}$, then

$$\lim_{n\to\infty} x_n^{(2j)} = 0 \quad \text{and} \quad \lim_{n\to\infty} x_n^{(2j-1)} = \infty$$

for all $j = 1, \ldots, l$, and

$$\lim_{n\to\infty} x_n^{(2l+1)} = \infty.$$

(ii) If $(x_0^{(2j)}, x_0^{(2j-1)}) \in (\bar{x}_{2j}, \infty) \times (0, \bar{x}_{2j-1})$ $(j = 1, \ldots, l)$ and $x_0^{(2l+1)} < \bar{x}_{2l+1}$, then

$$\lim_{n\to\infty} x_n^{(2j)} = \infty \quad \text{and} \quad \lim_{n\to\infty} x_n^{(2j-1)} = 0$$

for all $j = 1, \ldots, l$, and

$$\lim_{n\to\infty} x_n^{(2l+1)} = \infty.$$

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Nabila Haddad
LMAM Laboratory, Mathematics Department, Jijel University, Jijel 18000, Algeria
E-mail address: nabilahaddadt@yahoo.com

Julius Fergy T. Rabago
Department of Mathematics and Computer Science, College of Science, University of the Philippines Baguio, Baguio City 2600, Philippines
E-mail address: jfrabago@gmail.com