Electronic Journal of Mathematical Analysis and Applications
Vol. 5(2) July 2017, pp. 266-270.
ISSN: 2090-729X(online)
http://fcag-egypt.com/Journals/EJMAA/

NEW INTEGRAL INEQUALITIES VIA STRONGLY CONVEXITY

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ABSTRACT. In this note, we establish the estimate of $\int_a^b (x-a)^p (b-x)^q f(x)\,dx$ in the cases where f also $|f|^\lambda$ are strongly convex functions.

1. Introduction

It is well known that convexity plays an important and central role in many areas, such as economic, finance, optimization, and game theory. Due to its diverse applications this concept has been extended and generalized in several directions.

We recall that a function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex, if for all $x, y \in I$ and $t \in [0,1]$, the following inequality $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$ holds. In [8] Polyak gave the concept of strongly convexity as follows a function $f : [a,b] \to \mathbb{R}$ is said to be strongly convex with modulus $c > 0$, if the following inequality $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2$ holds for all $x, y \in [a,b]$ and $t \in [0,1]$.

We also recall that the generalized quadrature formula of Gauss-Jacobi type has the following form

$$\int_a^b (x-a)^p (b-x)^q f(x)\,dx = \sum_{k=0}^{m} B_{m,k} f(\gamma_k) + \Re_m[f] \tag{1}$$

for certain $B_{m,k}, \gamma_k$ and the remainder term $\Re_m[f]$, see[9].

In [7] Özdemir et al. gave the estimate of the left hand side of equality (1) when the function f is quasi-convex on $[a,b] \subset \mathbb{R}^+$ with $0 \leq a < b < \infty$, as follows

$$\begin{aligned}\int_a^b (x-a)^p (b-x)^q f(x)\,dx \quad &\leq \quad (b-a)^{p+q+1} \beta(p+1, q+1) \\ &\quad \times \max\{f(a), f(b)\}.\end{aligned}$$

In [3, 4] Liu discussed the left hand side of (1) in the cases where $|f|^{\frac{k}{k-1}}$ and $|f|^l$ are quasi-convex and (α, m)-convex, and P-convex functions. Iscan et al. [2] treated

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2010 *Mathematics Subject Classification.* 26D15, 26D20, 26A51.
Key words and phrases. Integral inequality, strongly convex function, Hölder inequality.
Submitted Dec. 13, 2016.

the equality (1) in the cases where f and $|f|^{\lambda}$ are harmonically convex functions. In [5] Muddassar et al., discussed the left hand side of equality (1) in the cases where $|f|, |f|^{\frac{k}{k-1}}$ and $|f|^{l}$ are s-(α, m)-convex functions. In [1] Ahmad gave the estimate of the left had side of (1) when $|f|, |f|^{\frac{k}{k-1}}$ and $|f|^{l}$ are P-preinvex and prequasiinvex functions. In [6] Noor et al., established the estimate of the left hand side of (1) for strongly generalized harmonic convex functions with modulus $c > 0$, and derived several other cases.

In the present note we establish a new estimate of the left hand side of equality (1) in the case where f and $|f|^{\lambda}$for $\lambda > 1$ are strongly convex functions with modulus $c > 0$.

2. Main results

In order to prove the results we need the following Lemma

Lemma 1.[3] Let $f : [a, b] \subset [0, \infty) \to \mathbb{R}$ be continuous on $[a, b]$ such that $f \in L([a, b])$, $a < b$. Then the equality

$$\int_a^b (x-a)^p (b-x)^q f(x)\, dx = (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q f(ta + (1-t)b)\, dt$$

holds for some fixed $p, q > 0$.

Theorem 1. Let $f : [a, b] \subset [0, \infty) \to \mathbb{R}$ be integrable function on $[a, b]$. If f is strongly convex with modulus $c > 0$ and $p, q > 0$, then we have

$$\begin{aligned} \int_a^b (x-a)^p (b-x)^q f(x)\, dx \quad &\leq \quad (b-a)^{p+q+1} \ \left[f(a)\beta(q+2, p+1) \right. \\ &\quad \left. + f(b)\beta(q+1, p+2) - \frac{c}{6}(b-a)^2 \right]. \end{aligned} \tag{2}$$

Proof. From Lemma 1, and strongly convexity of f, we have

$$\begin{aligned} \int_a^b (x-a)^p (b-x)^q f(x)\, dx \quad &\leq \quad (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q \left(tf(a) + (1-t)f(b)\right. \\ &\quad \left. - ct(1-t)(b-a)^2\right) dt \\ &= \quad (b-a)^{p+q+1} \ f(a) \int_0^1 (1-t)^p t^{q+1} dt \\ &\quad + (b-a)^{p+q+1} \ f(b) \int_0^1 (1-t)^{p+1} t^q dt \\ &\quad -c(b-a)^{p+q+3} \int_0^1 t(1-t) dt \\ &= \quad (b-a)^{p+q+1} \ f(a)\beta(q+2, p+1) \\ &\quad + (b-a)^{p+q+1} \ f(b)\beta(q+1, p+2) \\ &\quad -\frac{c}{6}(b-a)^{p+q+3}, \end{aligned}$$

which is the desired result. $\square$

Theorem 2. Let $f:[a,b]\subset[0,\infty)\to\mathbb{R}$ be integrable function on $[a,b]$, and let $\lambda>1$. If $|f|^{\lambda}$is strongly convex with modulus $c>0$ for $p,q>0$, then we have

$$\int_a^b (x-a)^p (b-x)^q f(x)\,dx \le (b-a)^{p+q+1}\left(\beta(q+1,p+1)\right)^{1-\frac{1}{\lambda}}$$
$$\times\left(|f(a)|^{\lambda}\ \beta(q+2,p+1)+|f(b)|^{\lambda}\beta(q+1,p+2)-\frac{c}{6}(b-a)^2\right)^{\frac{1}{\lambda}}. \tag{3}$$

Proof. From Lemma 1, properties of modulus, and power mean inequality, we have

$$\begin{aligned}\int_a^b (x-a)^p (b-x)^q f(x)\,dx &\le (b-a)^{p+q+1}\left(\int_0^1 (1-t)^p t^q dt\right)^{1-\frac{1}{\lambda}}\\ &\quad\times\left(\int_0^1 (1-t)^p t^q \left|f(ta+(1-t)b)\right|^{\lambda} dt\right)^{\frac{1}{\lambda}}\\ &= (b-a)^{p+q+1}\left(\beta(q+1,p+1)\right)^{1-\frac{1}{\lambda}}\\ &\quad\times\left(\int_0^1 (1-t)^p t^q \left|f(ta+(1-t)b)\right|^{\lambda} dt\right)^{\frac{1}{\lambda}}.\end{aligned} \tag{4}$$

Since $|f|^{\lambda}$is strongly convex, we deduce

$$\begin{aligned}\int_a^b (x-a)^p (b-x)^q f(x)\,dx &\le (b-a)^{p+q+1}\left(\beta(q+1,p+1)\right)^{1-\frac{1}{\lambda}}\Bigg(|f(a)|^{\lambda}\int_0^1 (1-t)^p t^{q+1}dt\\ &\quad+|f(b)|^{\lambda}\int_0^1 (1-t)^{p+1} t^q dt-c(b-a)^2\int_0^1 t(1-t)dt\Bigg)^{\frac{1}{\lambda}}\\ &= (b-a)^{p+q+1}\left(\beta(q+1,p+1)\right)^{1-\frac{1}{\lambda}}\Big(|f(a)|^{\lambda}\ \beta(q+2,p+1)\\ &\quad+|f(b)|^{\lambda}\beta(q+1,p+2)-\frac{c}{6}(b-a)^2\Big)^{\frac{1}{\lambda}},\end{aligned}$$

which is the desired result. $\square$

Theorem 3. If all the assumptions of Theorem 2 are satisfied, then we have

$$\int_a^b (x-a)^p (b-x)^q f(x)\,dx \le \frac{(b-a)^{p+q+1}}{2^{\frac{1}{\lambda}}}\left(\beta\left(\tfrac{q\lambda}{\lambda-1}+1,\tfrac{p\lambda}{\lambda-1}+1\right)\right)^{1-\frac{1}{\lambda}}$$
$$\times\left(|f(a)|^{\lambda}\ +|f(b)|^{\lambda}-\frac{c}{3}(b-a)^2\right)^{\frac{1}{\lambda}}. \tag{5}$$

Proof. From Lemma 1, properties of modulus, and Hölder inequality, we have

$$
\begin{aligned}
\int_a^b (x-a)^p (b-x)^q f(x)\,dx &\le (b-a)^{p+q+1}\left(\int_0^1 (1-t)^{\frac{p\lambda}{\lambda-1}} t^{\frac{q\lambda}{\lambda-1}}\,dt\right)^{1-\frac{1}{\lambda}} \\
&\quad\times\left(\int_0^1 |f(ta+(1-t)b)|^{\lambda}\,dt\right)^{\frac{1}{\lambda}} \\
&= (b-a)^{p+q+1}\left(\beta\left(\tfrac{q\lambda}{\lambda-1}+1,\tfrac{p\lambda}{\lambda-1}+1\right)\right)^{1-\frac{1}{\lambda}} \\
&\quad\times\left(\int_0^1 |f(ta+(1-t)b)|^{\lambda}\,dt\right)^{\frac{1}{\lambda}}. \qquad (6)
\end{aligned}
$$

Since $|f|^{\lambda}$is strongly convex, we get

$$
\begin{aligned}
\int_a^b (x-a)^p (b-x)^q f(x)\,dx &\le (b-a)^{p+q+1}\left(\beta\left(\tfrac{q\lambda}{\lambda-1}+1,\tfrac{p\lambda}{\lambda-1}+1\right)\right)^{1-\frac{1}{\lambda}} \\
&\quad\times\left(\int_0^1 \left(t|f(a)|^{\lambda}+(1-t)|f(b)|^{\lambda}-ct(1-t)(b-a)^2\right)dt\right)^{\frac{1}{\lambda}} \\
&= \frac{(b-a)^{p+q+1}}{2^{\frac{1}{\lambda}}}\left(\beta\left(\tfrac{q\lambda}{\lambda-1}+1,\tfrac{p\lambda}{\lambda-1}+1\right)\right)^{1-\frac{1}{\lambda}} \\
&\quad\times\left(|f(a)|^{\lambda}+|f(b)|^{\lambda}-\frac{c}{3}(b-a)^2\right)^{\frac{1}{\lambda}},
\end{aligned}
$$

which is the desired result. □

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