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RIGHT AND LEFT DISLOCATED b-METRIC SPACES AND FIXED POINT THEOREMS

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Abstract. Using the concept of generalized contraction, some fixed point theorems are investigated in the context of right and left dislocated b-metric spaces. We have proved φ-contraction and Reich type contraction in right and left dislocated b-metric spaces.

1. Introduction

One branch of generalizations of celebrated Banach contraction principle is based on the replacement of contraction condition imposed on $T : X \to X$, where (X, d) is a complete metric space. The weaker condition described by Browder [1] as, $d(Tx, Ty) \leq \varphi d(x, y)$ for all $x, y \in X$, where φ is a comparison function introduced by Berinde [2]. Reich [3] generalized the Banach contraction principle by introducing a new type of contraction condition which were given the name of Reich type contraction. In similar direction Istratescu [4] introduced the convex type contraction and generalized Banach contraction principle for such a type of contraction condition.

The notion of b-metric space was introduced by Czerwik [5] in connection with some problems concerning with the convergence of non-measurable functions with respect to measure. Fixed point theorems regarding b-metric spaces was obtained in [6] and [7]. In 2013, Shukla [8] generalized the notion of b-metric spaces and introduced the concept of partial b-metric spaces. Rahman and Sarwar [9] further generalized the concept of b-metric space and initiated the notion of dislocated quasi-b-metric space. Fixed point theorems in dislocated quasi-b-metric spaces are established by the researchers in [10] and [11].

Recently in 2017, Mujeeb and Sarwar [12] investigated right and left dislocated b-metric spaces and proved some fixed point results in such type of spaces.

In this work, we have proved φ-contraction and Reich type of contraction in the setting of right and left dislocated b-metric space which generalize and extend some existing fixed point results of the literature in these newly discovered spaces.

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2. Preliminaries

Definition 2.1.[9]. Let X be a non-empty set and $k \geq 1$ be a real number then a mapping $d : X \times X \to [0, \infty)$ is called dislocated quasi-b-metric if $\forall\ x, y, z \in X$

(d_1) $d(x, y) = d(y, x) = 0$ implies that $x = y$;

(d_2) $d(x, y) \leq k[d(x, z) + d(z, y)]$.

The pair (X, d) is called dislocated quasi-b-metric space or shortly (dq b-metric) space.

Definition 2.2.[12]. Let X be a non empty set. Let $k \geq 1$ be a real number then a mapping $d : X \times X \to [0, \infty)$ is called right dislocated b-metric if $\forall\ x, y, z \in X$ satisfying

$rd_1)$ $d(x, y) = d(y, x) = 0$ implies that $x = y$;

$rd_2)$ $d(x, y) \leq k[d(x, z) + d(y, z)]$.

And the pair (X, d) is called right dislocated b-metric (rd b-metric) space.

Definition 2.3.[12]. Let X be a non empty set. Let $k \geq 1$ be a real number then a mapping $d : X \times X \to [0, \infty)$ is called left dislocated b-metric if $\forall\ x, y, z \in X$ satisfying

$ld_1)$ $d(x, y) = d(y, x) = 0$ implies that $x = y$;

$ld_2)$ $d(x, y) \leq k[d(z, x) + d(z, y)]$.

And the pair (X, d) is called left dislocated b-metric (ld b-metric) space.

Remarks. For some interesting properties and examples of right and left dislocated b-metric space see [12].

Definition 2.4.[12]. A sequence $\{x_n\}$ in X is called rd b-convergent in X if there exists $x \in X$ such that $\lim_{n\to\infty} d(x, x_n) = 0$. In this case x is called the rd b-limit of the sequence $\{x_n\}$.

Unlike b-metric space rd b-metric space need not be left and right convergent. But in case of rd b-metric space it is rd b-convergent only.

Definition 2.5.[12]. A sequence $\{x_n\}$ in X is called ld b-convergent in X if there exists $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$. In this case x is called the ld b-limit of the sequence $\{x_n\}$.

In case of ld b-metric space a convergent sequence need only to be ld b-convergent.

Remarks. Since the notion of ld b-metric space is look like a dual notion of rd b-metric space. Therefore, we state the following definitions and some basic properties for right dislocated b-metric spaces only which may be easily carried out for left dislocated b-metric spaces.

The following definitions can be found in [12].

Definition 2.6. A sequence $\{x_n\}$ in rd or ld b-metric space is called Cauchy sequence if for $\epsilon > 0$ there exist $n_0 \in N$, such that for $m > n \geq n_0$, we have $d(x_n, x_m) < \epsilon$.

Definition 2.7. A rd or ld $b-$metric space (X, d) is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition 2.8. Let (X, d) be a rd or ld b-metric space. A mapping $T : X \to X$ is called contraction if $k \geq 1$ there exists a constant $\alpha \in [0, 1)$ with $k\alpha < 1$ and for all $x, y \in X$ satisfying

$$d(Tx, Ty) \leq \alpha d(x, y).$$

The following result may be seen in [12].

Lemma 1. Every subsequence of rd or ld b-convergent sequence to x_0 is rd b-convergent to x_0.

Lemma 2. Limit of convergent sequence in rd or ld b-metric space is unique.
Lemma 3. Let (X, d) be a rd or ld b-metric space and $\{x_n\}$ be a sequence in rd b-metric space such that

$$d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n) \tag{1}$$

for $n = 1, 2, 3, ...$ and $0 \leq \alpha k < 1$ where $\alpha \in [0, 1)$ and k is defined in rd b-metric space. Then $\{x_n\}$ is a Cauchy sequence in X.
Lemma 4. Let (X, d) be a rd or ld b-metric space. If $T : X \to X$ is a contraction. Then T is rd b-continuous.
Theorem 1. Let (X, d) be a complete rd or (ld) b-metric space. If $T : X \to X$ is a contraction. Then T has a unique fixed point.
Theorem 2.[13]. Every φ-contraction $T : X \to X$ where (X, d) is a complete metric space, is a Picard's operator.
Definition 2.9.[2]. A map $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called comparison function if it satisfies:

(1) φ is monotonic increasing;
(2) The sequence $\{\varphi^n(t)\}_{n=0}^{\infty}$ converge to zero for all $t \in \mathbb{R}_+$ where φ^n stand for nth iterate of φ.
If φ satisfies:
(3) $\sum\limits_{k=0}^{\infty} \varphi^k(t)$ converge for all $t \in \mathbb{R}_+$.

Then φ is called (c)-comparison function.
Thus every comparison function is c-comparison function. A prototype example for comparison function is

$$\varphi(t) = \alpha t \quad t \in \mathbb{R}_+ \;\; 0 \leq \alpha < 1.$$

Some more examples and properties of comparison and c-comparison function can be found in [2].

3. Main Results

Theorem 1. Let (X, d) be a complete right (left) dislocated b-metric space and $T : X \to X$ be a continuous function for $k \geq 1$ satisfying

$$d(Tx, Ty) \leq \varphi d(x, y) \tag{2}$$

for all $x, y \in X$ where φ is a comparison function. Then T has a unique fixed point in X.
Proof. Let x_0 be arbitrary in X we define a sequence $\{x_n\}$ in X as following

$$x_0, x_1 = Tx_0, x_2 = Tx_1, ..., x_{n+1} = Tx_n \text{ for all } \quad n \in \mathrm{N}.$$

Now to show that $\{x_n\}$ is a Cauchy sequence in X consider

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n).$$

Using (2) we have

$$d(x_n, x_{n+1}) \leq \varphi d(x_{n-1}, x_n). \tag{3}$$

Similarly one can show that

$$d(x_{n-1}, x_n) \leq \varphi d(x_{n-2}, x_{n-1}). \tag{4}$$

Putting (3) in (4) we have

$$d(x_n, x_{n+1}) \leq \varphi^2 d(x_{n-2}, x_{n-1}).$$

Proceeding in similar manner we get

$$d(x_{n-1}, x_n) \leq \varphi^n d(x_0, x_1). \tag{5}$$

To show that $\{x_n\}$ is a Cauchy sequence consider $m > n$ and using (rd_2) or (ld_2) we have

$$d(x_n, x_m) \leq k \cdot d(x_n, x_{n+1}) + k^2 \cdot d(x_{n+1}, x_{n+2}) + k^3 \cdot d(x_{n+2}, x_{n+3}) + ...$$

Using (5) the above equation become

$$d(x_n, x_m) \leq k \cdot \varphi^n d(x_0, x_1) + k^2 \cdot \varphi^{n+1} d(x_0, x_1) + k^3 \cdot \varphi^{n+2} d(x_0, x_1) + ...$$

Since φ is a comparison function so taking $n, m \to \infty$ we get

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

Which show that $\{x_n\}$ is a Cauchy sequence in complete right (left) dislocated b-metric space X. So there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Now to show that z is the fixed point of T. Since $x_n \to z$ as $n \to \infty$ using the continuity of T we have

$$\lim_{n\to\infty} Tx_n = Tz$$

which implies that

$$\lim_{n\to\infty} x_{n+1} = Tz.$$

Thus $Tz = z$. So z is the fixed point of T.

Uniqueness: Suppose that T has two fixed points z and w for $z \neq w$. Consider

$$d(z, w) = d(Tz, Tw).$$

Using (2) we have

$$d(z, w) \leq \varphi d(z, w).$$

Since φ is a comparison function so the above inequality is possible only if $d(z, w) = 0$ similarly one can show that $d(w, z) = 0$. So by (d_1) $z = w$. Hence T has a unique fixed point in X.

Remark. Theorem 1 generalize Banach contraction principle and the result established by Matkowski [13] in right (left) dislocated b-metric spaces.

Theorem 2. Let (X, d) be a complete right or (left) dislocated b-metric space and $T : X \to X$ is a continuous self-mapping satisfying

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) + \beta \cdot d(x, Tx) + \gamma \cdot d(y, Ty) \tag{6}$$

for all $x, y \in X$ and $\alpha, \beta, \gamma \geq 0$ with $k\alpha + k\beta + \gamma < 1$ where $k \geq 1$. Then T has a unique fixed point in X.

Proof. Let x_0 be arbitrary in X we define a sequence $\{x_n\}$ in X as following

$$x_0, x_1 = Tx_0, x_2 = Tx_1, ..., x_{n+1} = Tx_n.$$

Now to show that $\{x_n\}$ is a Cauchy sequence consider

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n).$$

Using (6) we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \alpha \cdot d(x_{n-1} + x_n) + \beta \cdot d(x_{n-1}, Tx_{n-1}) + \gamma \cdot d(x_n, Tx_n).$$

By the definition of the sequence we get

$$d(x_n, x_{n+1}) \leq \alpha \cdot d(x_{n-1}, x_n) + \beta \cdot d(x_{n-1}, x_n) + \gamma \cdot d(x_n, x_{n+1}).$$

Simplification yields

$$d(x_n, x_{n+1}) \leq \frac{\alpha + \beta}{1 - \gamma} \cdot d(x_{n-1}, x_n).$$

Let

$$h = \frac{\alpha + \beta}{1 - \gamma} < \frac{1}{k}.$$

So the above inequality become

$$d(x_n, x_{n+1}) \leq h \cdot d(x_{n-1}, x_n).$$

Also

$$d(x_{n-1}, x_n) \leq h \cdot d(x_{n-2}, x_{n-1}).$$

Thus

$$d(x_n, x_{n+1}) \leq h^2 \cdot d(x_{n-2}, x_{n-1}).$$

Similarly proceeding we get

$$d(x_n, x_{n+1}) \leq h^n \cdot d(x_0, x_1).$$

Since $h < \frac{1}{k}$. Taking limit $n \to \infty$, so $h^n \to 0$ and

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

So by Lemma 3 $\{x_n\}$ is a Cauchy sequence in complete right or (left) dislocated b-metric space so there must exist $u \in X$ such that

$$\lim_{n\to\infty} (x_n, u) = 0.$$

Now to show that u is the fixed point of T. Since $x_n \to u$ as $n \to \infty$ using the continuity of T we have

$$\lim_{n\to\infty} Tx_n = Tu$$

which implies that

$$\lim_{n\to\infty} x_{n+1} = Tu.$$

Thus $Tu = u$. So u is the fixed point of T.

Uniqueness: Let T have two fixed points i.e u, v with $u \neq v$ then we have

$$d(u, v) = d(Tu, Tv) \leq \alpha \cdot d(u, v) + \beta \cdot d(u, Tu) + \gamma \cdot d(v, Tv)$$

$$d(u, v) = d(Tu, Tv) \leq \alpha \cdot d(u, v) + \beta \cdot d(u, u) + \gamma \cdot d(v, v).$$

Putting $u = v$ in (6) one can easily show that $d(u, u) = d(v, v) = 0$. Thus the above equation become

$$d(u, v) \leq \alpha \cdot d(u, v).$$

The above inequality is possible only if $d(u, v) = 0$ similarly one can show that $d(v, u) = 0$. So by (d_1) we get that $u = v$. Thus fixed point of T is unique.

Corollary. Let (X, d) be a complete right or (left) dislocated b-metric space and $T : X \to X$ is a continuous self-mapping satisfying

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) + \beta \cdot d(x, Tx)$$

for all $x, y \in X$ and $\alpha, \beta \geq 0$ with $k\alpha + k\beta < 1$ where $k \geq 1$. Then T has a unique fixed point in X.

Corollary. Let (X, d) be a complete right or (left) dislocated b-metric space and $T : X \to X$ is a continuous self-mapping satisfying

$$d(Tx, Ty) \leq \alpha \cdot d(x, y)$$

for all $x, y \in X$ and $\alpha \geq 0$ with $0 \leq k\alpha < 1$ where $k \geq 1$. Then T has a unique fixed point in X.

Remarks. Theorem 2 generalize Reich type contraction and extend Banach contraction principle and convex type contraction in complete right or (left) dislocated b-metric spaces.

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