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EXISTENCE AND UNIQUENESS OF WEAK SOLUTIONS FOR TRANSPORT EQUATIONS

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Abstract. For a transport equation with the velocity field that has a particular form, we prove the existence and uniqueness of weak solutions. Moreover, we obtain the continuity for the unique weak solution.

1. Introduction

In the last few years, some progress has been made on the well-posedness of the transport equation:

$$\begin{cases} \partial_t u(t,x) + b(t,x)\cdot\nabla u(t,x) + c(t,x)u(t,x) = f(t,x), \ (t,x)\in(0,T]\times\mathbb{R}^d, \\ u(0,x) = u_0(x), \ x\in\mathbb{R}^d, \end{cases} \tag{1}$$

where $T>0$ is a given real number, $b:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$, $c,f:[0,T]\times\mathbb{R}^d\to\mathbb{R}$, $u_0:\mathbb{R}^d\to\mathbb{R}$ are Borel functions. The first remarkable result in this direction is due to DiPerna and Lions [12], where the authors derived the well-posedness of (1) in $L^1\cap L^\infty$-setting, if b is of class $L^1([0,T];W^{1,1}_{loc}(\mathbb{R}^d;\mathbb{R}^d))$ and satisfies suitable global conditions including L^∞-bounds on spatial divergence. Later, Diperna and Lions' work was strengthen by Lions [14] to the piecewise $W^{1,1}_{loc}$ velocity field. Recently, Le Bris and Lions [13] used the same technique developed in [12] to establish the existence and uniqueness of solutions for a class of transport equations and then founded the differentiability of solutions for differential equations with $W^{1,1}_{loc}$ velocity. We also refer to [17] for high order differentiability of solutions.

Using a slightly different philosophy, Ambrosio [1] (also see [3]) studied the continuity equation, i.e. $c(t,x)=\mathrm{div}b(t,x)$ and established the uniqueness of L^∞-solutions by assuming $b\in BV_{loc}$, whose distributional spatial divergence belongs to L^∞. Then using the renormalized technique for BV_{loc} coefficient, Ambrosio also proved the well-posedness for a class of hyperbolic systems of conservation laws [2, 4, 5]. But for general b, only with BV_{loc} regularity, counterexamples of nonuniqueness of weak solutions for (1) have been constructed and studied by many authors in recent years, such as see [7, 8, 9, 10, 11]. Thus to overcome the obstacle of nonuniqueness, restrictions need to be imposed on b that will weed out undesirable solutions.

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In this paper, we study (1) in which b has the following form:

$$b(t,x)=(b_1(t,x_1),b_2(t,x)), \tag{2}$$

where $x=(x_1,x_2)\in\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}=\mathbb{R}^d$, $b_1:[0,T]\times\mathbb{R}^{d_1}\to\mathbb{R}^{d_1}$, $b_2:[0,T]\times\mathbb{R}^d\to\mathbb{R}^{d_2}$. Our source to study (2) stems from conservation laws directly, and now let us state it briefly. Consider the following inhomogeneous scalar conservation law equation

$$\left\{\begin{array}{l}\partial_t\rho(t,x)+\mathrm{div}(G(t,\rho(t,x)))=A(t,\rho),\ (t,x)\in(0,T]\times\mathbb{R}^d,\\ \rho(t=0,x)=\rho_0(x),\ x\in\mathbb{R}^d.\end{array}\right. \tag{3}$$

If G and A are smooth, the method of vanishing viscosity implies the existence of weak solutions for (3). But, it seems to be difficult to get the uniqueness for weak solutions even for smooth G and A. An alternative, instructive way of viewing the weak solution ρ is by rewriting (3) in its kinetic form (see [15, 16]) using the Maxwellian $u(t,x,v)=1_{(0,\rho(t,x))}(v)-1_{(\rho(t,x),0)}(v)$,

$$\left\{\begin{array}{l}\partial_tu(t,x,v)+A(t,v)\partial_vu(t,x,v)+\partial_vG(t,v)\cdot\nabla_xu(t,x,v)\\ \quad=\partial_vm(t,x,v),\quad (t,x,v)\in(0,T]\times\mathbb{R}^d\times\mathbb{R},\\ u(t=0,x,v)=u_0(x,v)=1_{(0,\rho_0(x))}(v)-1_{(\rho_0(x),0)}(v),\ (x,v)\in\mathbb{R}^d\times\mathbb{R}.\end{array}\right. \tag{4}$$

Thus, we transform the nonlinear equation (3) into a linear ones, at the price of increasing the number of independent variables from d to $d+1$. In particular, to prove the uniqueness of solutions for (3), one should prove the uniqueness of solutions for (4). Since (4) is linear, we may establish the uniqueness of weak solutions for general G and A. When to study (4), a special feature is that the initial value should be L^1-integrable in the variable x, so we should establish the well-posedness of the transport problem with $u_0\in L^p(\mathbb{R};L^1(\mathbb{R}^d))$. In general, we consider (1) with $u_0\in L^p(\mathbb{R}^{d_1};L^1(\mathbb{R}^{d_2}))$, and in Section 2, we found the well-posedness of (1), (2).

Notations. $\mathcal{D}(\mathbb{R}^d)$ and $\mathcal{D}((0,T)\times\mathbb{R}^d)$ stand for the set of all smooth functions on $\mathbb{R}^d$ and $(0,T)\times\mathbb{R}^d$ with compact supports, respectively. Given a measurable function ς, ς^+ is defined by $\max\{\varsigma,0\}$. sgn is the sign function defined by $\mathrm{sgn}(\tau)=1_{\tau>0}(\tau)-1_{\tau<0}(\tau)$. The letter C will denote a positive constant, whose values may change in different places.

2. Transport equations

Set $\nabla=(\nabla_{x_1},\nabla_{x_2})$ and $\mathrm{div}=\mathrm{div}_{x_1}+\mathrm{div}_{x_2}$, we make the following assumptions:
$(H_1):b_1\in L^1([0,T];L^q_{loc}(\mathbb{R}^{d_1};\mathbb{R}^{d_1})),\ \mathrm{div}_{x_1}b_1\in L^1([0,T];L^\infty(\mathbb{R}^{d_1}))$;
$(H_2):b_2\in L^1([0,T];L^q_{loc}(\mathbb{R}^{d_1};L^\infty_{loc}(\mathbb{R}^{d_2};\mathbb{R}^{d_2}))),\ \mathrm{div}_{x_2}b_2\in L^1([0,T];L^\infty(\mathbb{R}^d))$;
$(H_3):f\in L^p([0,T]\times\mathbb{R}^{d_1};L^1(\mathbb{R}^{d_2})),u_0\in L^p(\mathbb{R}^{d_1};L^1(\mathbb{R}^{d_2})),c\in L^1([0,T];L^\infty(\mathbb{R}^d))$,
where $p\in[1,\infty)$ and $1/p+1/q=1$.

Definition 2.1 Let $p\in[1,\infty)$. $u\in L^\infty([0,T];L^p(\mathbb{R}^{d_1};L^1(\mathbb{R}^{d_2})))$ is called a weak solution of (1), (2) if for every $\varphi\in\mathcal{D}(\mathbb{R}^d)$ and $t\in[0,T]$,

$$\begin{aligned}\int_{\mathbb{R}^d}u(t,x)\varphi(x)dx&=\int_{\mathbb{R}^d}u_0(x)\varphi(x)dx+\int_0^t\int_{\mathbb{R}^d}b(s,x)\cdot\nabla\varphi(x)u(s,x)dxds\\ &+\int_0^t\int_{\mathbb{R}^d}\mathrm{div}b(s,x)\varphi(x)u(s,x)dxds+\int_0^t\int_{\mathbb{R}^d}f(s,x)\varphi(x)dxds\\ &-\int_0^t\int_{\mathbb{R}^d}c(s,x)u(s,x)\varphi(x)dxds.\end{aligned} \tag{5}$$

Before stating and proving the existence, let us give an auxiliary lemma.

Lemma 2.1([6, Theorem 2.2]) Let $A \subset \mathbb{R}^d$ be an open set, and let $u \in W^{1,1}_{loc}(A)$. Then for any Lipschitz function $\beta : \mathbb{R} \to \mathbb{R}$,

$$\nabla[\beta(u)] = \beta'(u)\nabla u.$$

We are now in a position to state and prove the existence of weak solutions.

Theorem 2.1 (Existence). Let $p \in [1, \infty)$ and let b, c, u_0 and f satisfy hypotheses $(H_1) - (H_3)$. Then there exists a weak solution of (1), (2).

Proof. Suppose $r > 0$. Let $B_{1,r}(0)$ be the ball in $\mathbb{R}^{d_1}$ with radius r and central 0, and let $B_{2,r}(0)$ be the corresponding ball in $\mathbb{R}^{d_2}$. Let ρ_1 and ρ_2 be two regularization kernels in variables x_1 and x_2, respectively, i.e.

$$0 \le \rho_i \in \mathcal{D}(\mathbb{R}^{d_i}), \quad \int_{\mathbb{R}^{d_i}} \rho_i(x_i)dx_i = 1, \; i = 1, 2. \tag{6}$$

For any $\varepsilon > 0$, we set $\rho_{\varepsilon,i}(\cdot) = \frac{1}{\varepsilon^{d_i}}\rho_i(\frac{\cdot}{\varepsilon}), \; i = 1, 2$. Define

$$\begin{cases} b_{\varepsilon,r}(t, x) = ((b_1(t)1_{B_{1,r}(0)}) * \rho_{\varepsilon,1}, \; (b_2(t)1_{B_{1,r}(0)}1_{B_{2,r}(0)}) * \rho_{\varepsilon,1} * \rho_{\varepsilon,2})(x), \\ c_{\varepsilon,r}(t, x) = (c(t)1_{B_{1,r}(0)}1_{B_{2,r}(0)}) * \rho_{\varepsilon,1} * \rho_{\varepsilon,2}(x). \end{cases}$$

Since $f \in L^p([0, T] \times \mathbb{R}^{d_1}; L^1(\mathbb{R}^{d_2}))$, $u_0 \in L^p(\mathbb{R}^{d_1}; L^1(\mathbb{R}^{d_2}))$, we can choose two sequences $\{u^n_0\} \subset \mathcal{D}(\mathbb{R}^d)$ and $\{f_n\} \subset \mathcal{D}((0, T) \times \mathbb{R}^d)$ such that

$$u^n_0 \longrightarrow u_0 \;\text{ in }\; L^p(\mathbb{R}^{d_1}; L^1(\mathbb{R}^{d_2})), \quad f_n \longrightarrow f \;\text{ in }\; L^p([0, T] \times \mathbb{R}^{d_1}; L^1(\mathbb{R}^{d_2})) \tag{7}$$

and

$$\begin{cases} \|u^n_0\|_{L^p(\mathbb{R}^{d_1};L^1(\mathbb{R}^{d_2}))} \le C\|u_0\|_{L^p(\mathbb{R}^{d_1};L^1(\mathbb{R}^{d_2}))}, \\ \|f_n\|_{L^p([0,T]\times\mathbb{R}^{d_1};L^1(\mathbb{R}^{d_2}))} \le C\|f\|_{L^p([0,T]\times\mathbb{R}^{d_1};L^1(\mathbb{R}^{d_2}))}. \end{cases}$$

Consider the following approximation problem

$$\begin{cases} \partial_t u_{n,\varepsilon,r}(t, x) + b_{\varepsilon,r}(t, x) \cdot \nabla u_{n,\varepsilon,r}(t, x) + c_{\varepsilon,r}(t, x)u_{n,\varepsilon,r} \\ \quad = f_n(t, x), \; (t, x) \in (0, T] \times \mathbb{R}^d, \\ u_{n,\varepsilon,r}(t = 0, x) = u^n_0(x), \; x \in \mathbb{R}^d. \end{cases} \tag{8}$$

By the classical characteristic method, there exists a unique smooth (in x) solution $u_{n,\varepsilon,r}(t, x)$ of (8) and $u_{n,\varepsilon,r}(t, x)$ satisfies (5).

If we choose $\beta(\tau) = |\tau|$, then it is Lipschitz continuous, by virtue of Lemma 2.1, we obtain

$$\begin{aligned} & \partial_t\beta(u_{n,\varepsilon,r}) + b_{\varepsilon,r}(t, x) \cdot \nabla\beta(u_{n,\varepsilon,r}) + c_{\varepsilon,r}(t, x)\beta'(u_{n,\varepsilon,r})u_{n,\varepsilon,r}(t, x) \\ = \; & \beta'(u_{n,\varepsilon,r})f_n(t, x), \end{aligned} \tag{9}$$

associated with $\beta(u_{n,\varepsilon,r}(t = 0, x)) = \beta(u^n_0(x))$. By integrating the identity (9) in x_2 over $\mathbb{R}^{d_2}$ and using the integration by parts, it turns to

$$\begin{aligned} & \partial_t \int_{\mathbb{R}^{d_2}} |u_{n,\varepsilon,r}|(t, x)dx_2 + b_{1,\varepsilon,r}(t, x_1) \cdot \nabla_{x_1} \int_{\mathbb{R}^{d_2}} |u_{n,\varepsilon,r}|(t, x)dx_2 \\ & + \int_{\mathbb{R}^{d_2}} c_{\varepsilon,r}(t, x)|u_{n,\varepsilon,r}|(t, x)dx_2 \\ = \; & \int_{\mathbb{R}^{d_2}} \mathrm{div}_{x_2} b_{2,\varepsilon,r}(t, x_1, x_2)|u_{n,\varepsilon,r}|(t, x_1, x_2)dx_2 \\ & + \int_{\mathbb{R}^{d_2}} \mathrm{sgn}(u_{n,\varepsilon,r})f_n(t, x_1, x_2)dx_2, \end{aligned} \tag{10}$$

where we have used the fact $\beta'(\tau) = \mathrm{sgn}(\tau)$. Setting

$$v_{n,\varepsilon,r}(t,x_1) = \int_{\mathbb{R}^{d_2}} |u_{n,\varepsilon,r}|(t,x_1,x_2)dx_2,$$

it causes to

$$\begin{cases} \partial_t v_{n,\varepsilon,r}(t,x_1) + b_{1,\varepsilon,r}(t,x_1)\cdot\nabla_{x_1} v_{n,\varepsilon,r}(t,x_1) = g_{n,\varepsilon,r}(t,x_1), \\ v_{n,\varepsilon,r}(t=0,x_1) = \int_{\mathbb{R}^{d_2}} |u_0^n|(x)dx_2, \end{cases} \tag{11}$$

where

$$\begin{aligned} g_{n,\varepsilon,r}(t,x_1) &= \int_{\mathbb{R}^{d_2}} [\mathrm{div}_{x_2} b_{2,\varepsilon,r}(t,x) - c_{\varepsilon,r}(t,x)]|u_{n,\varepsilon,r}|(t,x)dx_2 \\ &\quad + \int_{\mathbb{R}^{d_2}} \mathrm{sgn}(u_{n,\varepsilon,r}) f_n(t,x)dx_2. \end{aligned}$$

The arguments employed above for $\beta(\tau) = |\tau|$ in (8) adapted to $\beta(\tau) = |\tau|^p$ in (11) now, yields that

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^{d_1}} v_{n,\varepsilon,r}^p(t,x_1)dx_1 \\ =\ & \int_{\mathbb{R}^{d_1}} \mathrm{div}_{x_1} b_{1,\varepsilon,r}(t,x_1) v_{n,\varepsilon,r}^p(t,x_1)dx_1 + \int_{\mathbb{R}^{d_1}} p v_{n,\varepsilon,r}^{p-1}(t,x_1) g_{n,\varepsilon,r}(t,x_1)dx_1 \\ \leq\ & C(t)\int_{\mathbb{R}^{d_1}} v_{n,\varepsilon,r}^p(t,x_1)dx_1 + \|f_n(t,\cdot)\|_{L^p(\mathbb{R}^{d_1};L^1(\mathbb{R}^{d_2}))}, \end{aligned}$$

where

$$\begin{aligned} C(t) &= \|\mathrm{div}_{x_1} b_{1,\varepsilon,r}(t,\cdot)\|_{L^\infty(\mathbb{R}^{d_1})} + p\|\mathrm{div}_{x_2} b_{2,\varepsilon,r}(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \\ &\quad + p\|c_{\varepsilon,r}(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} + (p-1). \end{aligned}$$

Using the Grönwall lemma, we conclude that

$$\begin{aligned} &\int_{\mathbb{R}^{d_1}} v_{n,\varepsilon,r}^p(t,x_1)dx_1 \\ \leq\ & C\int_{\mathbb{R}^{d_1}} \Big[\int_{\mathbb{R}^{d_2}} |u_0^n|(x)dx_2\Big]^p dx_1 + C\int_0^T\int_{\mathbb{R}^{d_1}} \Big[\int_{\mathbb{R}^{d_2}} |f_n(x)|dx_2\Big]^p dx_1dt. \end{aligned}$$

So

$$\begin{aligned} &\int_{\mathbb{R}^{d_2}} \Big[\int_{\mathbb{R}^{d_1}} |u_{n,\varepsilon,r}|dx_2\Big]^p dx_1 \\ \leq\ & C\int_{\mathbb{R}^{d_1}} \Big[\int_{\mathbb{R}^{d_2}} |u_0|dx_2\Big]^p dx_1 + C\int_0^T\int_{\mathbb{R}^{d_1}} \Big[\int_{\mathbb{R}^{d_2}} |f|dx_2\Big]^p dx_1dt, \end{aligned} \tag{12}$$

for all $t \in [0,T]$.

The discussion applied above, with a slight change, also gives the estimate

$$\int_{\mathbb{R}^d} |u_{n,\varepsilon,r}|^{p'}dx \leq C\Big[\int_{\mathbb{R}^d} |u_0^n|^{p'}dx + \int_0^T\int_{\mathbb{R}^d} |f_n|^{p'}dxdt\Big], \tag{13}$$

for any $p' > p$, since $u_0^n \in \mathcal{D}(\mathbb{R}^d)$ and $f_n \in \mathcal{D}((0,T)\times\mathbb{R}^d)$.

From (12) and (13), the sequence $\{u_{n,\varepsilon,r}\}$ is weakly relatively compact in the space $L^\infty([0,T];L^p_{loc}(\mathbb{R}^{d_1};L^1_{loc}(\mathbb{R}^{d_2})))$ (also see [12, Proposition II.1]). By extracting a subsequence if necessary, it converges weakly in $L^\infty([0,T];L^p_{loc}(\mathbb{R}^{d_1};L^1_{loc}(\mathbb{R}^{d_2})))$ to some u, and now $u \in L^\infty([0,T];L^p(\mathbb{R}^{d_1};L^1(\mathbb{R}^{d_2})))$ which satisfies (5). $\Box$

If the velocity field is more regular, the weak solution is unique. Before establishing the uniqueness for weak solutions, we requires some useful lemmas and firstly one appeals a lemma [12, Lemma 2.1].

Lemma 2.2 Let $B \in L^1([0,T]; W^{1,\zeta}_{loc}(\mathbb{R}^d;\mathbb{R}^d))$, $S \in L^\infty([0,T]; L^{\zeta'}_{loc}(\mathbb{R}^d))$, with $1 \le \zeta \le \infty$, $1/\zeta + 1/\zeta' = 1$. Then

$$(B \cdot \nabla S) * \varrho_\varepsilon - B \cdot \nabla (S * \varrho_\varepsilon) \longrightarrow 0 \ \text{ in } \ L^1([0,T]; L^1_{loc}(\mathbb{R}^d)) \ \text{ as } \ \varepsilon \to 0,$$

where

$$\varrho_\varepsilon = \frac{1}{\varepsilon^d} \varrho(\frac{\cdot}{\varepsilon}) \ \text{ with } \varrho \in \mathcal{D}_+(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} \varrho dx = 1, \quad \varepsilon > 0.$$

From above lemma, one gains:

Lemma 2.3 Suppose $p \in [1,\infty)$ and $q \in (1,\infty]$ such that $1/p + 1/q = 1$. Let $b(t,x) = (b_1(t,x_1), b_2(t,x))$ such that

$$b_1 \in L^1([0,T]; W^{1,q}_{loc}(\mathbb{R}^{d_1};\mathbb{R}^{d_1})), \quad b_2 \in L^1([0,T]; L^q_{loc}(\mathbb{R}^{d_1}; W^{1,\infty}_{loc}(\mathbb{R}^{d_2};\mathbb{R}^{d_2}))). \quad (14)$$

Assume that $u \in L^\infty([0,T]; L^p_{loc}(\mathbb{R}^{d_1}; L^1_{loc}(\mathbb{R}^{d_2})))$ and

$$\partial_t u(t,x) + b(t,x) \cdot \nabla u(t,x) \in L^1([0,T]; L^1_{loc}(\mathbb{R}^d)). \quad (15)$$

Then

(i) for any Borel set $K \subset \mathbb{R}$ with $\mu_1(K) = 0$,

$$\mu_{d+1}(\{(t,x) \in [0,T] \times \mathbb{R}^d; \ u(t,x) \in K \ \text{and} \ \partial_t u(t,x) + b(t,x) \cdot \nabla u(t,x) \neq 0\}) = 0; \quad (16)$$

(ii) for any Lipschitz function β,

$$\partial_t[\beta(u)] + b(t,x) \cdot \nabla[\beta(u)] = \beta'(u)[\partial_t u(t,x) + b(t,x) \cdot \nabla u(t,x)], \quad (17)$$

where μ_1 and μ_{d+1} denote the standard Lebesgue measure in $\mathbb{R}$ and $\mathbb{R}^{d+1}$, respectively.

Proof. Obviously, (16) is equivalent to

$$1_K(u)[\partial_t u(t,x) + b(t,x) \cdot \nabla u(t,x)] = 0, \ a.e. \ (t,x) \in [0,T] \times \mathbb{R}^d. \quad (18)$$

Notice that $1_K(\tau) = \frac{d}{d\tau}(\int_0^\tau 1_K(s)ds)$, it suffices to show

$$\beta'(u)[\partial_t u(t,x) + b(t,x) \cdot \nabla u(t,x)] = 0, \ a.e. \ (t,x) \in [0,T] \times \mathbb{R}^d,$$

if one fetches $\beta(\tau) = \int_0^\tau 1_K(s)ds$.

Firstly, we assume K is compact. There exist open sets $V_1, V_2, \dots, V_n, \dots \subset \mathbb{R}$ such that $V_{n+1} \subset V_n$ and $K = \bigcap_n V_n$. By the Urysohn lemma, there exist $\vartheta_n \in \mathcal{D}(V_n)$ such that $0 \le \vartheta_n \le 1, \vartheta_n = 1$ on K. Define

$$\beta_n(\tau) = \int_0^\tau \vartheta_n(s)ds, \ \tau \in \mathbb{R},$$

then β_n is smooth, $|\beta_n(\tau)| \le |\tau|$ and $\beta_n'(\tau) \to 1_K(\tau)$ as $n \to \infty$. By making use of the Lebesgue dominated convergence theorem, we get

$$\beta_n(\tau) \longrightarrow \int_0^\tau 1_K(s)ds = \beta(\tau), \ \text{ as } \ n \to \infty.$$

Now we claim that

$$\partial_t[\beta_n(u)] + b(t,x) \cdot \nabla[\beta_n(u)] = \beta_n'(u)[\partial_t u(t,x) + b(t,x) \cdot \nabla u(t,x)]. \quad (19)$$

We prove it by two steps.

Step 1 : u is smooth in t.
Let ρ_1 and ρ_2 be as in (6). Then $u_{\varepsilon_1,\varepsilon_2} = (u(t) * \rho_{\varepsilon_1,1}) * \rho_{\varepsilon_2,2}$ satisfies

$$\begin{aligned} & \partial_t[\beta_n(u_{\varepsilon_1,\varepsilon_2})] + b(t,x)\cdot\nabla[\beta_n(u_{\varepsilon_1,\varepsilon_2})] \\ = \; & \beta_n'(u_{\varepsilon_1,\varepsilon_2})[\partial_t u_{\varepsilon_1,\varepsilon_2} + (b(t,x)\cdot\nabla u)_{\varepsilon_1,\varepsilon_2}(t,x) - \epsilon_\varepsilon], \end{aligned} \tag{20}$$

where ε_1 and ε_2 are positive real numbers,

$$\epsilon_\varepsilon(t,x) = (b\cdot\nabla u)_{\varepsilon_1,\varepsilon_2}(t,x) - b(t,x)\cdot\nabla u_{\varepsilon_1,\varepsilon_2}(t,x) =: I_{1,\varepsilon}(t,x) + I_{2,\varepsilon}(t,x),$$

and

$$\begin{aligned} I_{1,\varepsilon}(t,x) &= (b_1\cdot\nabla_{x_1}u)_{\varepsilon_1,\varepsilon_2}(t,x) - b_1(t,x)\cdot\nabla_{x_1}u_{\varepsilon_1,\varepsilon_2}(t,x), \\ I_{2,\varepsilon}(t,x) &= (b_2\cdot\nabla_{x_2}u)_{\varepsilon_1,\varepsilon_2}(t,x) - b_2(t,x)\cdot\nabla_{x_2}u_{\varepsilon_1,\varepsilon_2}(t,x). \end{aligned}$$

For $\varepsilon_2 > 0$ be fixed, by (14) and Lemma 2.2,

$$\lim_{\varepsilon_1\to 0} I_{1,\varepsilon}(t,x) = 0 \text{ in } \; L^1([0,T]; L^1_{loc}(\mathbb{R}^{d_1})), \text{ for } \; a.e. \; x_2 \in \mathbb{R}^{d_2}.$$

A subtle argument analogue of Lemma 2.2, also hints that

$$\lim_{\varepsilon_1\to 0} I_{1,\varepsilon}(t,x) = 0 \;\text{ in }\; L^1([0,T]; L^1_{loc}(\mathbb{R}^{d})). \tag{21}$$

At the same time,

$$\begin{aligned} & I_{2,\varepsilon}(t,x) \\ = \; & -b_2(t,x)\cdot\nabla_{x_2}\int_{\mathbb{R}^d} u(t,y_1,y_2)\rho_{\varepsilon_1,1}(x_1-y_1)\rho_{\varepsilon_2,2}(x_2-y_2)dy_1dy_2 \\ & +\langle b_2(t,\cdot,\cdot)\cdot\nabla_{x_2}u(t,\cdot,\cdot), \rho_{\varepsilon_1,1}(x_1-\cdot)\rho_{\varepsilon_2,2}(x_2-\cdot)\rangle \\ = \; & \int_{\mathbb{R}^d} u(t,y_1,y_2)\rho_{\varepsilon_1,1}(x_1-y_1)[b_2(t,y_1,y_2) - b_2(t,x_1,x_2)] \\ & \quad \cdot\nabla_{x_2}\rho_{\varepsilon_2,2}(x_2-y_2)dy_1dy_2 \\ & -\int_{\mathbb{R}^d} u(t,y_1,y_2)\rho_{\varepsilon_1,1}(x_1-y_1)\mathrm{div}_{y_2}b_2(t,y_1,y_2)\rho_{\varepsilon_2,2}(x_2-y_2)dy_1dy_2 \\ \longrightarrow \; & \int_{\mathbb{R}^{d_2}} u(t,x_1,y_2)[b_2(t,x_1,y_2) - b_2(t,x_1,x_2)]\cdot\nabla_{x_2}\rho_{\varepsilon_2,2}(x_2-y_2)dy_2 \\ & -\int_{\mathbb{R}^{d_2}} u(t,x_1,y_2)\mathrm{div}_{y_2}b_2(t,x_1,y_2)\rho_{\varepsilon_2,2}(x_2-y_2)dy_2, \end{aligned} \tag{22}$$

for almost all $(t,x) \in [0,T]\times\mathbb{R}^d$, if we tend ε_1 to 0 for fixed ε_2. Setting the limit by $I_{2,\varepsilon_2}(t,x)$, then Lemma 2.2 uses again (with a slight change), one concludes that

$$I_{2,\varepsilon_2}(t,x) \to 0 \text{ in } L^1([0,T]; L^1_{loc}(\mathbb{R}^d)), \text{ as } \varepsilon_2 \to 0.$$

On the other hand, for any $r > 0$, if we denote $\tilde{B}_r(0)$ by the product of two balls in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ with the same radius r, i.e. $\tilde{B}_r(0) = B_{1,r}(0)\times B_{2,r}(0)$, then one has the following estimate

$$\begin{aligned}
&\Big\|\int_{\mathbb{R}^d} u(t,y_1,y_2)\rho_{\varepsilon_1,1}(x_1-y_1)[b_2(t,y)-b_2(t,x)] \\
&\qquad \cdot\nabla_{x_2}\rho_{\varepsilon_2,2}(x_2-y_2)dy_1dy_2\Big\|_{L^1([0,T];L^1(\tilde{B}_r(0)))} \\
= \ & \int_0^T\int_{\tilde{B}_r(0)}\Big|\int_{\mathbb{R}^d} u(t,y_1,y_2)\rho_{\varepsilon_1,1}(x_1-y_1)[b_2(t,y)-b_2(t,x)] \\
&\qquad \cdot\nabla_{x_2}\rho_{\varepsilon_2,2}(x_2-y_2)dy_1dy_2\Big|dxdt \\
\leq \ & C\int_0^T dt\int_{\tilde{B}_r(0)}dx\int_{\mathbb{R}^{d_1}}dy_1\int_{|y_2-x_2|\leq C\varepsilon_2}\rho_{\varepsilon_1,1}(x_1-y_1)|u(t,y_1,y_2)| \\
&\qquad \times\frac{|b_2(t,y)-b_2(t,x)|}{\varepsilon_2}dy_2 \\
\leq \ & C\Big[\int_0^T dt\int_{\tilde{B}_r(0)}dx\int_{\mathbb{R}^{d_1}}dy_1\int_{|y_2-x_2|\leq C\varepsilon_2}\rho_{\varepsilon_1,1}(x_1-y_1)|u(t,y_1,y_2)| \\
&\qquad \times\frac{|b_2(t,y)-b_2(t,y_1,x_2)|}{\varepsilon_2}dy_2 \\
&+\int_0^T dt\int_{\tilde{B}_r(0)}dx\int_{\mathbb{R}^{d_1}}dy_1\int_{|y_2-x_2|\leq C\varepsilon_2}\rho_{\varepsilon_1,1}(x_1-y_1)|u(t,y_1,y_2)| \\
&\qquad \times\frac{|b_2(t,y_1,x_2)-b_2(t,x)|}{\varepsilon_2}dy_2\Big] \\
\leq \ & C\Big[\int_0^T dt\int_{B_{1,r+1}(0)}dy_1\int_{B_{2,r+C+1}(0)}|u(t,y)|\|\nabla_{y_2}b_2(t,y_1,\cdot)\|_{L^\infty(B_{2,r+C+1})}dy_2 \\
&+\frac{1}{\varepsilon_2}\int_0^T dt\int_{B_{1,r+1}(0)}dy_1\int_{B_{2,r+C+1}(0)}|u(t,y)|\|b_2(t,y_1,\cdot)\|_{L^\infty(B_{2,r+C+1})}dy_2\Big]. \quad (23)
\end{aligned}$$

From (22) and (23), in view of the Lebesgue dominated convergence theorem, we gain

$$\lim_{\varepsilon_2\to 0}\lim_{\varepsilon_1\to 0} I_{2,\varepsilon}(t,x)=0, \quad \text{in} \quad L^1([0,T];L^1_{loc}(\mathbb{R}^d)). \qquad (24)$$

Observing that β_n' is bounded, by letting ε_1 tend to 0 first, ε_2 tend to 0 next, from (20), (21) and (24), one obtains the identity (19) for smooth (in t) u.

Step 2 : $u\in L^\infty([0,T];L^p_{loc}(\mathbb{R}^{d_1};L^1_{loc}(\mathbb{R}^{d_2})))$.

Let ρ_3 be a standard smoothing kernel in t. Then $u_{\varepsilon_3}=u*\rho_{\varepsilon_3,3}(t)$ ($\rho_{\varepsilon_3,3}(t)=\frac{1}{\varepsilon_3}\rho_3(\frac{t}{\varepsilon_3})$) is smooth in t for $t\in(3\varepsilon_3,T-3\varepsilon_3)$ and

$$u_{\varepsilon_3}(\cdot,x)\longrightarrow u(\cdot,x) \ \text{ in } \ L^1[0,T] \ \text{ for } \ a.e.\ x\in\mathbb{R}^d. \qquad (25)$$

Now by Step 1,

$$\begin{aligned}
&\partial_t[\beta_n(u_{\varepsilon_3})]+b(t,x)\cdot\nabla[\beta_n(u_{\varepsilon_3})] \\
= \ & \beta_n'(u_{\varepsilon_3})[\partial_t u_{\varepsilon_3}(t,x)+(b(t,x)\cdot\nabla u)_{\varepsilon_3}(t,x)-\epsilon_{\varepsilon_3}] \qquad (26)
\end{aligned}$$

for a.e. $(t,x)\in(3\varepsilon_3,T-3\varepsilon_3)\times\mathbb{R}^d$, where

$$\epsilon_{\varepsilon_3}=(b(t,x)\cdot\nabla u)_{\varepsilon_3}(t,x)-b(t,x)\cdot\nabla u_{\varepsilon_3}(t,x). \qquad (27)$$

The calculations used in Step 1 from (21) to (24) is applicable here again, from (25) to (27) , by letting ε_3 approach to 0, we get (19), and which suggests that

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} \beta_n'(u)[\partial_t u(t,x) + b(t,x)\cdot\nabla u(t,x)]\phi(t,x)dxdt \\
= \quad & \langle \partial_t[\beta_n(u)] + b\cdot\nabla[\beta_n(u)],\ \phi\rangle \\
= \quad & -\int_0^T \int_{\mathbb{R}^d} \beta_n(u)[\partial_t\phi(t,x) + \mathrm{div}(b(t,x)\phi(t,x))]dxdt, \qquad (28)
\end{aligned}$$

for every $\phi \in \mathcal{D}((0,T)\times\mathbb{R}^d)$. Note that for a.e. $(t,x)\in \mathrm{supp}\phi$,

$$\lim_{n\to\infty}\beta_n(u(t,x)) = \lim_{n\to\infty}\int_0^{u(t,x)}\vartheta_n(\tau)d\tau = \int_0^{u(t,x)} 1_K(\tau)d\tau = 0 \qquad (29)$$

and

$$\lim_{n\to\infty}\beta_n'(u(t,x)) = \lim_{n\to\infty}\vartheta_n(u(t,x) = 1_K(u(t,x)). \qquad (30)$$

From (28), (29) and (30), we derive

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} 1_K(u)[\partial_t u(t,x) + b(t,x)\cdot\nabla u(t,x)]\phi(t,x)dxdt \\
= \quad & \lim_{n\to\infty}\int_0^T \int_{\mathbb{R}^d} \beta_n'(u)[\partial_t u(t,x) + b(t,x)\cdot\nabla u(t,x)]\phi(t,x)dxdt \\
= \quad & \lim_{n\to\infty}\langle \partial_t[\beta_n(u)] + b(t,x)\cdot\nabla[\beta_n(u)], \phi\rangle \\
= \quad & -\lim_{n\to\infty}\int_0^T \int_{\mathbb{R}^d} \beta_n(u)[\partial_t\phi(t,x) + \mathrm{div}(b(t,x)\phi(t,x))]dxdt = 0.
\end{aligned}$$

Since $\phi \in \mathcal{D}((0,T)\times\mathbb{R}^d)$ is arbitrary, one proves the conclusion (18) for compact set K.

For a general Borel set K, we choose a compact set $L\subset\mathbb{R}$ and define the regular Borel measure θ on $\mathbb{R}$ by

$$\theta(K) = \int_{u^{-1}(K)} |\partial_t u(t,x) + b(t,x)\cdot\nabla u(t,x)|1_L dtdx.$$

Since for any compact set $K\subset\mathbb{R}$ with zero Lebesgue measure, $\theta(K)=0$. One gains

$$\theta(K) = 0,\ \text{for any zero Lebesgue measure set } K.$$

Therefore

$$1_K(u)[\partial_t u(t,x) + b(t,x)\cdot\nabla u(t,x)] = 0, \quad a.e.\ (t,x)\in[0,T]\times\mathbb{R}^d,$$

for L is arbitrary.

It remains to show the chain rule (17) for Lipschitz function β. In fact, if we approximate β by a sequence of smooth functions β_k, such that $|\beta_k'|\le C$, then from (19),

$$\partial_t[\beta_k(u)] + b(t,x)\cdot\nabla[\beta_k(u)] = \beta_k'(u)[\partial_t u(t,x) + b(t,x)\cdot\nabla u(t,x)].$$

Notice that

$$\lim_{k\to\infty}\beta_k'(u)[\partial_t u(t,x) + b(t,x)\cdot\nabla u(t,x)] = \beta'(u)[\partial_t u(t,x) + b(t,x)\cdot\nabla u(t,x)],$$

and

$$\lim_{k\to\infty}\Big[\partial_t[\beta_k(u)]+b(t,x)\cdot\nabla[\beta_k(u)]\Big]=\partial_t[\beta(u)]+b(t,x)\cdot\nabla[\beta(u)],$$

in distributional sense. Thus (17) is valid. □

We are now in a position to state and prove the uniqueness.

Theorem 2.2 (Uniqueness). Let p, b, c, u_0 and f be as in Theorem 2.1, and let (14) hold. We assume further that $b_1=b_{1,1}+b_{1,2}$, $b_2=b_{2,1}+b_{2,2}$, and

$$b_{1,2}\cdot x_1\ge 0,\quad \frac{|b_{1,1}|}{1+|x_1|}\in L^1([0,T];L^1(\mathbb{R}^{d_1}))+L^1([0,T];L^\infty(\mathbb{R}^{d_1})), \tag{31}$$

$$b_{2,2}\cdot x_2\ge 0,\ \frac{|b_{2,1}|}{1+|x_2|}\in L^1([0,T];L^1_{loc}(\mathbb{R}^{d_1};L^1(\mathbb{R}^{d_2}))+L^q_{loc}(\mathbb{R}^{d_1};L^\infty(\mathbb{R}^{d_2}))). \tag{32}$$

Then the weak solution of the Cauchy problem (1), (2) is unique.

Proof. Assume for the time being that, we have two solutions u_1 and u_2 to (1) sharing the same inhomogeneous condition, the same initial data, then the difference u of u_1 and u_2 solves the homogeneous problem (1) supplied with initial data vanishes, so it suffices to show that a weak solution with $u_0=0$ and $f=0$ vanishes identically.

For any real number $M_1>0$, we take $\beta(\tau)=|\tau|\wedge M_1$. By virtue of Lemma 2.3, then

$$\begin{aligned}&\partial_t[|u|\wedge M_1]+b_1\cdot\nabla_{x_1}[|u|\wedge M_1]+b_2\cdot\nabla_{x_2}[|u|\wedge M_1]\\&+c[|u|\wedge M_1]1_{[0,M_1]}(|u|)\mathrm{sgn}(u)=0.\end{aligned}$$

For any $\varphi_2\in\mathcal{D}(\mathbb{R}^{d_2})$, one has

$$\partial_t v(t,x_1)+b_1(t,x_1)\cdot\nabla_{x_1}v(t,x_1)=g(t,x_1), \tag{33}$$

where

$$v(t,x_1)=\int_{\mathbb{R}^{d_2}}[|u|\wedge M_1](t,x_1,x_2)\varphi_2(x_2)dx_2$$

and

$$g(t,x_1)=\int_{\mathbb{R}^{d_2}}[|u|\wedge M_1][\mathrm{div}_{x_2}(b_2(t,x)\varphi_2(x_2))-c(t,x)1_{[0,M_1]}(|u|)\varphi_2(x_2)\mathrm{sgn}(u)]dx_2.$$

Using Lemma 2.3 again for $\beta(\tau)=(|\tau|\wedge M_2)^p$ with some real number $M_2>0$, it follows from (33) that

$$\partial_t[|v|\wedge M_2]^p+b_1(t,x_1)\cdot\nabla_{x_1}[|v|\wedge M_2]^p=p[|v|\wedge M_2]^{p-1}g(t,x_1)1_{[0,M_2]}(|v|)\mathrm{sgn}(v),$$

i.e.

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^{d_1}}[|v|\wedge M_2]^p(t,x_1)\varphi_1(x_1)dx_1\\=&\int_{\mathbb{R}^{d_1}}[|v|\wedge M_2]^p(t,x_1)\mathrm{div}_{x_1}(b_1(t,x_1)\varphi_1(x_1))dx_1\\&+p\int_{\mathbb{R}^{d_1}}[|v|\wedge M_2]^{p-1}1_{[0,M_2]}(|v|)\mathrm{sgn}(v)g(t,x_1)\varphi_1dx_1,\forall\varphi_1\in\mathcal{D}(\mathbb{R}^{d_1}).\end{aligned} \tag{34}$$

In particular, we choose φ_1 and φ_2 above being two cut off functions with respect to variables x_1 and x_2, respectively, i.e. $\varphi_i \in \mathcal{D}(\mathbb{R}^{d_i}),\ 0 \leq \varphi_i \leq 1$ and

$$\varphi_i(x_i) = \left\{ \begin{array}{l} 1,\ \text{on } |x_i| \leq 1, \\ 0,\ \text{on } |x_i| \geq 2, \end{array} \right. \quad \varphi_i(x_i) = \varphi_i(|x_i|),\ \varphi_i' \leq 0,\ i = 1, 2. \tag{35}$$

Let

$$\varphi_{i,r}(x_i) = \varphi_i(\frac{x_i}{r}), \quad \text{for any} \quad r > 0, \quad i = 1, 2. \tag{36}$$

If one replaces φ_1 and φ_2 in (34) by $\varphi_{1,n}$ and $\varphi_{2,k}$ in (35) and (36), respectively, it yields that

$$\begin{aligned}
& \frac{d}{dt}\int_{\mathbb{R}^{d_1}} [v_k \wedge M_2]^p(t, x_1)\varphi_{1,n}(x_1)dx_1 \\
= \quad & \int_{\mathbb{R}^{d_1}} [v_k \wedge M_2]^p(t, x_1)[\mathrm{div}_{x_1} b_1(t, x_1)\varphi_{1,n}(x_1) \\
& + b_{1,2} \cdot \frac{x_1}{n|x_1|}\varphi_{1,n}' + b_{1,1}(t, x_1) \cdot \nabla_{x_1}\varphi_{1,n}(x_1)]dx_1 \\
& + p\int_{\mathbb{R}^{d_1}} [v_k \wedge M_2]^{p-1}(t, x_1) 1_{[0,M_2]}(v_k)\varphi_{1,n}(x_1)dx_1 \\
& \quad \times \int_{\mathbb{R}^{d_2}} [|u| \wedge M_1](t, x)[\mathrm{div}_{x_2} b_2(t, x)\varphi_{2,k}(x_2) \\
& + \ b_{2,2} \cdot \frac{x_2}{k|x_2|}\varphi_{2,k}' + b_{2,1}(t, x) \cdot \nabla_{x_2}\varphi_{2,k}(x_2) \\
& - c(t, x) 1_{[0,M_1]}(|u|)\varphi_{2,k}(x_2)\mathrm{sgn}(u)]dx_2 \\
\leq \quad & \Big[\|\mathrm{div}_{x_1} b_{1,1}(t)\|_{L^\infty(\mathbb{R}^{d_1})} + p\|\mathrm{div}_{x_2} b_2(t)\|_{L^\infty(\mathbb{R}^d)} + p\|c(t)\|_{L^\infty(\mathbb{R}^d)}\Big] \\
& \quad \times \int_{\mathbb{R}^{d_1}} [v_k \wedge M_2]^p(t, x_1)\varphi_{1,n}dx_1 \\
& + \int_{\mathbb{R}^{d_1}} [v_k \wedge M_2]^p(t, x_1) b_{1,1}(t, x_1) \cdot \nabla_{x_1}\varphi_{1,n}(x_1)dx_1 \\
& + p\int_{\mathbb{R}^{d_1}} [v_k \wedge M_2]^{p-1}(t, x_1)\varphi_{1,n}(x_1)dx_1 \\
& \quad \times \int_{\mathbb{R}^{d_2}} [|u| \wedge M_1](t, x)|b_{2,1}(t, x) \cdot \nabla_{x_2}\varphi_{2,k}(x_2)|dx_2,
\end{aligned} \tag{37}$$

where

$$v_k(t, x_1) = \int_{\mathbb{R}^{d_2}} [|u| \wedge M_1](t, x_1, x_2)\varphi_{2,k}(x_2)dx_2, \tag{38}$$

and in the fifth line in (37) we have used (31) and (32).

By conditions (H_1) and (H_2), it follows that

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^{d_1}}[v_k\wedge M_2]^p(t,x_1)\varphi_{1,n}(x_1)dx_1\\
\leq\;& C(t)\int_{\mathbb{R}^{d_1}}[v_k\wedge M_2]^p(t,x_1)\varphi_{1,n}(x_1)dx_1\\
&+C\int_{n\leq|x_1|\leq 2n}[v_k\wedge M_2]^p(t,x_1)\frac{|b_{1,1}(t,x_1)|}{1+|x_1|}dx_1\\
&+C\int_{\mathbb{R}^{d_1}}[v_k\wedge M_2]^{p-1}(t,x_1)\varphi_{1,n}(x_1)dx_1\\
&\quad\times\int_{k\leq|x_2|\leq 2k}[|u|\wedge M_1](t,x)\frac{|b_{2,1}(t,x)|}{1+|x_2|}dx_2. \qquad (39)
\end{aligned}$$

For M_1, M_2 and n being fixed, if one approaches k to infinity, from (38) and (39), by condition (32), one gains

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^{d_1}}\Big[\int_{\mathbb{R}^{d_2}}[|u|\wedge M_1](t,x_1,x_2)dx_2\wedge M_2\Big]^p\varphi_{1,n}(x_1)dx_1\\
\leq\;& C(t)\int_{\mathbb{R}^{d_1}}\Big[\int_{\mathbb{R}^{d_2}}[|u|\wedge M_1](t,x_1,x_2)dx_2\wedge M_2\Big]^p\varphi_{1,n}(x_1)dx_1\\
&+C\int_{n\leq|x_1|\leq 2n}\Big[\int_{\mathbb{R}^{d_2}}[|u|\wedge M_1](t,x_1,x_2)dx_2\wedge M_2\Big]^p\frac{|b_{1,1}(t,x_1)|}{1+|x_1|}dx_1.
\end{aligned}$$

By (31), if we tend n to infinity in the above inequality, then

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^{d_1}}\Big[\int_{\mathbb{R}^{d_2}}[|u|\wedge M_1]dx_2\wedge M_2\Big]^p dx_1\\
\leq\;& C(t)\int_{\mathbb{R}^{d_1}}\Big[\int_{\mathbb{R}^{d_2}}[|u|\wedge M_1]dx_2\wedge M_2\Big]^p dx_1. \qquad (40)
\end{aligned}$$

From (40), by applying a Grönwall type argument, it follows that

$$\int_{\mathbb{R}^{d_1}}\Big[\int_{\mathbb{R}^{d_2}}[|u|\wedge M_1]dx_2\wedge M_2\Big]^p dx_1=0,$$

for $u_0=0$. Hence

$$\int_{\mathbb{R}^{d_2}}[|u|\wedge M_1]dx_2\wedge M_2=0.$$

Since $M_2>0$ is arbitrary, we conclude $\int_{\mathbb{R}^{d_2}}[|u|\wedge M_1]dx_2=0$. The same argument procedure used again, one finishes that $u=0$ and this achieves the proof. □

More general, we have the following comparison principle.

Theorem 2.3 (Comparison principle). Let b, c, u_0 and f be as in Theorem 2.2. If $u_0\leq 0$, $f\leq 0$, then $u\leq 0$.

Proof. Let u be the unique solution of (1). By Lemma 2.3, for any Lipschitz function β, $\beta(u)$ solves

$$\begin{aligned}
&\partial_t[\beta(u)]+b(t,x)\cdot\nabla[\beta(u)]+c(t,x)\beta'(u)u\\
=\;&\beta'(u)[\partial_t u+b(t,x)\cdot\nabla u(t,x)+c(t,x)u]=\beta'(u)f,
\end{aligned}$$

supplied with $\beta(u)|_{t=0}=\beta(u_0)$.

In particular, if we choose $\beta(\tau)=\tau^+\wedge M_1$, then it follows that $[u^+\wedge M_1](t=0)\le 0$ and

$$\partial_t[u^+\wedge M_1]+b(t,x)\cdot\nabla[u^+\wedge M_1]+c(t,x)1_{[0,M_1]}(u)[u^+\wedge M_1]\le 0,\quad \forall\ t>0.$$

The argument applied in Theorem 2.2 for $|u|\wedge M_1$ and $([\int_{\mathbb{R}^{d_2}}|u|\wedge M_1dx_2]\wedge M_2)^p$ adapted to $u^+\wedge M_1$ and $([\int_{\mathbb{R}^{d_2}}u^+\wedge M_1dx_2]\wedge M_2)^p$ here, combing a Grönwall type argument, suggests that

$$\int_{\mathbb{R}^{d_1}}\Big[\int_{\mathbb{R}^{d_2}}[u^+\wedge M_1]dx_2\wedge M_2\Big]^p dx_1\le 0.$$

Employing the same technique discussed in Theorem 2.2, one derives $u^+=0$, this completes the proof. □

It is time for us to give a regularity result.

Theorem 2.4 (Regularity). Let b,c,u_0 and f be as in Theorem 2.2. Then the unique solution u satisfies

$$u\in\mathcal{C}([0,T];L^p(\mathbb{R}^{d_1};L^1(\mathbb{R}^{d_2}))). \tag{41}$$

Proof. Since the proof for $p>2$ is analogue of the issue of $1\le p\le 2$, we only concentrate our attention on $1\le p\le 2$. We approximate u_0 and f by u_0^n and f_n which are in class of $L^2(\mathbb{R}^d)\cap L^p(\mathbb{R}^{d_1};L^1(\mathbb{R}^{d_2}))$ and $L^2([0,T]\times\mathbb{R}^d)\cap L^p([0,T]\times\mathbb{R}^{d_1};L^1(\mathbb{R}^{d_2}))$, respectively, such that (7) holds. For any n, there exists a unique $u_n\in L^\infty(0,T;L^2(\mathbb{R}^d))$ solving the Cauchy problem

$$\left\{\begin{array}{l}\partial_t u_n(t,x)+b(t,x)\cdot\nabla u_n(t,x)+c(t,x)u_n(t,x)\\ \quad =f_n(t,x),\ (t,x)\in(0,T]\times\mathbb{R}^d,\\ u_n(t=0,x)=u_0^n(x),\ x\in\mathbb{R}^d.\end{array}\right. \tag{42}$$

Moreover, for any $\varphi\in\mathcal{D}(\mathbb{R}^d)$, (5) holds true. Therefore,

$$\int_{\mathbb{R}^d}u_n(t,x)\varphi(x)dx\in\mathcal{C}([0,T]). \tag{43}$$

By Lemma 2.3, with a cumbersome approximation discussion which is akin to the computation from (33) to (40), for any $t_0,t\in[0,T]$, we end up with

$$\begin{aligned}&\Big|\|u_n(t)\|^2_{L^2(\mathbb{R}^d)}-\|u_n(t_0)\|^2_{L^2(\mathbb{R}^d)}\Big|\\ \le\ & 2\int_{t_0}^t\|f_n(s)u_n(s)\|_{L^1(\mathbb{R}^d)}ds\\ &+\int_{t_0}^t\|u_n(s)\|^2_{L^2(\mathbb{R}^d)}[\|\mathrm{div}b(s)-2c(s)\|_{L^\infty(\mathbb{R}^d)}]ds,\end{aligned} \tag{44}$$

hence $\|u_n(t)\|_{L^2(\mathbb{R}^d)}\in\mathcal{C}([0,T])$.

The same tools used in Theorem 2.2 applies again, one also concludes

$$\begin{aligned}&\|u_n(t)-u(t)\|_{L^p(\mathbb{R}^{d_1};L^1(\mathbb{R}^{d_2}))}\\ \le\ & C\Big[\int_0^t\|f_n(s)-f(s)\|_{L^p(\mathbb{R}^{d_1};L^1(\mathbb{R}^{d_2}))}ds+\|u_0^n-u_0\|_{L^p(\mathbb{R}^{d_1};L^1(\mathbb{R}^{d_2}))}\Big].\end{aligned} \tag{45}$$

From (43) to (45), in order to show (41), it is sufficient to prove $u_n\in\mathcal{C}([0,T];L^2(\mathbb{R}^d))$.

Indeed, for any $t_0 \in [0,T]$ and $t \in [0,T]$,

$$\begin{aligned}\limsup_{t\to t_0} \|u_n(t)-u_n(t_0)\|^2_{L^2(\mathbb{R}^d)} &= \limsup_{t\to t_0} \langle u_n(t)-u_n(t_0),\ u_n(t)-u_n(t_0)\rangle \\ &= 2\limsup_{t\to t_0} \langle u_n(t_0)-u_n(t),\ u_n(t_0)\rangle,\end{aligned}$$

where the notation $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(\mathbb{R}^d)$. Therefore, for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$,

$$\begin{aligned} & \limsup_{t\to t_0} \|u_n(t)-u_n(t_0)\|^2_{L^2(\mathbb{R}^d)} \\ \le\ & 2\limsup_{t\to t_0} \langle u_n(t_0)-u_n(t),\ u_n(t_0)-\varphi\rangle + 2\limsup_{t\to t_0} \langle u_n(t_0)-u_n(t),\ \varphi\rangle \\ =\ & 2\limsup_{t\to t_0} \langle u_n(t_0)-u_n(t),\ u_n(t_0)-\varphi\rangle \\ \le\ & 4\|u_n\|_{L^\infty(0,T;L^2(\mathbb{R}^d))}\|u_n(t_0)-\varphi\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

It follows that

$$\begin{aligned} & \limsup_{t\to t_0} \|u_n(t)-u_n(t_0)\|^2_{L^2(\mathbb{R}^d)} \\ \le\ & 4\|u_n\|_{L^\infty(0,T;L^2(\mathbb{R}^d))} \inf_{\varphi\in\mathcal{D}(\mathbb{R}^d)} \|u_n(t_0)-\varphi\|_{L^2(\mathbb{R}^d)} = 0,\end{aligned}$$

and from this, we accomplish the proof. □

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