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LINEARIZED OSCILLATION THEORY OF SECOND ORDER NEUTRAL IMPULSIVE DIFFERENCE EQUATIONS

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ABSTRACT. This article studies the oscillation of solutions of a class of second order nonlinear neutral impulsive difference equations of the form:

$$\begin{cases} \Delta^2[u(n)-p(n)f(u(n-\alpha))]+q(n)h(u(n-\beta))=0,\ n\neq m_j \\ \underline{\Delta}[\Delta(u(m_j-1)-p(m_j-1)f(u(m_j-\alpha-1)))] \\ \qquad\qquad +r(m_j-1)h(u(m_j-\beta-1))=0,\ j\in\mathbb{N} \end{cases}$$

for the various ranges of the neutral coefficient. The technique employed here is due to the linearization method by using the Banach contraction principle and Knaster-Tarski fixed point theorem. In addition, some illustrative examples are given to verify our main results.

1. Introduction

In this work, we consider a second order nonlinear neutral impulsive difference equations of the form:

$$(E)\begin{cases} \Delta^2[u(n)-p(n)f(u(n-\alpha))]+q(n)h(u(n-\beta))=0,\ n\neq m_j & (1) \\ \underline{\Delta}[\Delta(u(m_j-1)-p(m_j-1)f(u(m_j-\alpha-1)))] \\ \qquad\qquad +r(m_j-1)h(u(m_j-\beta-1))=0,\ j\in\mathbb{N}, & (2) \end{cases}$$

where α, β are positive integers, $p\in\mathbb{R}-\{0\}$, $q,r\in\mathbb{R}_+$, $f,h\in C(\mathbb{R},\mathbb{R})$ and m_j, $j\in\mathbb{N}$ are the discrete moments of impulsive effect such that $m_1<m_2<\cdots<m_j$ with the properties $\lim_{j\to\infty} m_j=\infty$ and $\rho=\max\{\alpha,\beta\}\leq\max\{m_j-m_{j-1}\}<\infty$. Here, Δ is the forward difference operator defined by $\Delta u(n)=u(n+1)-u(n)$ and $\underline{\Delta}$ is the difference operator defined by $\underline{\Delta}u(m_j-1)=u(m_j)-u(m_j-1)$.

By a solution of (E) we mean a real valued function $u(n)$ defined on $\mathbb{N}(n_0-\rho)=\{n_0-\rho,,...n_0,n_0+1,...\}$ which satisfy (E) for $n\geq n_0$ with the initial conditions $u(i)=\phi(i)$, $i=n_0-\rho,\cdots,n_0$, where $\phi(i)$, $i=n_0-\rho,\cdots,n_0$ are given. A nontrivial solution $u(n)$ of (E) is said to be nonoscillatory, if it is either eventually positive or

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eventually negative. Otherwise, the solution is said to be oscillatory. (E) is said to be oscillatory if all its solutions are oscillatory.

The objective of this work is to establish linearized oscillation theory for highly nonlinear neutral impulsive difference equations (E) by using Banach contraction principle and Knaster-Tarski fixed point theorem in the ranges $-\infty < p(n) < -1$, $-1 < p(n) \leq 0$ and $0 \leq p(n) < 1$. To understand the theory, we refer the monographs [8] and [9], and about the development of impulsive equations we refer [10], [15] and [16].

Indeed, (1) is called as the nonimpulsive difference equation which is so called as difference equation and to its solution $u(n)$ when we apply impulse $m_j, j \in \mathbb{N}$, we find an impulsive solution $u(m_j)$ satisfying (2) and together we have our impulsive difference equation (E). It is a challenge to study $(1)/(E)$ with and without fixed point theory via the Qualitative Behaviour of Solutions method.

Let the linear impulsive system associated with the nonlinear impulsive system (E) be

$$(E_l) \begin{cases} \Delta^2[y(n) - py(n-\alpha)] + qy(n-\beta) = 0, \; n \neq m_j \\ \underline{\Delta}[\Delta(y(m_j - 1) - py(m_j - \alpha - 1))] + ry(m_j - \beta - 1) = 0, \; j \in \mathbb{N} \end{cases}$$

and in [20], the authors have predicted the possible solution of (E_l) as

$$y(n) = \lambda^n A^{i(n_0,n)}, \; n_0 > \rho = \max\{\alpha, \beta\}, \tag{3}$$

where $i(n_0, n) = j =$ number of impulsive points $m_j, j \in \mathbb{N}$ between n_0 to n and $A \neq 0$ is a real number which is called as the *pulsatile constant*. But, it is not that much simple to predict the solution of (E) when nothing is known about (E). In this work, we establish the linearized oscillation results (E) through its limiting equation of type (3). Of course, some results of [20] are our state of art along with fixed point theory. As long as (3) is concerned, the study of (1) is a special case study of (E) and the approach of neutral equation is a general discussion comparing to the study of nonneutral equations (see for e.g.[2, 3, 13], [23]-[26]). We study (E) with a general set up, and in this direction we refer some of the works [1, 4, 5, 6, 7, 12, 14], [17]-[21] and [22] and the references cited there in.

2. Preliminaries

In this section, we present some existing results from [20] for our discussion in which we have the following notations:

- $i(n-\beta, n) = l_1$ is the number of impulsive points between $n - \beta$ to n,
- $i(n-\alpha, n) = l_2$ is the number of impulsive points between $n - \alpha$ to n.

Theorem 2.1. Let $\alpha > \beta$ and $r \neq q \neq 0$. Then (E_l) admits an oscillatory solution in the impulsive form (3) if and only if the algebraic equation

$$\left[\frac{1}{\lambda}\Big(1 - \frac{r}{q}\Big) + \frac{r}{q}\right]^{l_1} (\lambda - 1)^2 - p\lambda^{-\alpha}\left[\frac{1}{\lambda}\Big(1 - \frac{r}{q}\Big) + \frac{r}{q}\right]^{l_1 - l_2} (\lambda - 1)^2 + q\lambda^{-\beta} = 0 \tag{4}$$

has at least one real root λ with $\lambda < 1 - \frac{q}{r}$ for $\frac{r}{q} > 1$ and $\lambda > 1 - \frac{q}{r}$ for $\frac{r}{q} < 1$.

Remark 2.2. In Theorem 2.1, we may note that

$$i(n_0, n-\alpha) - i(n_0, n-\beta) = -i(n-\alpha, n-\beta) = -[i(n-\alpha, n) - i(n-\beta, n)] = l_1 - l_2$$

for $\alpha > \beta$. If $\alpha < \beta$, then we find

$$i(n_0, n-\alpha) - i(n_0, n-\beta) = i(n-\beta, n-\alpha) = i(n-\beta, n) - i(n-\alpha, n) = l_1 - l_2$$

and hence $i(n_0, n-\alpha) - i(n_0, n-\beta) = l_1 - l_2$. Therefore, Theorem 2.1 holds for any $\alpha, \beta \in \mathbb{R}_+$.

Theorem 2.3. Let the assumptions of Theorem 2.1 hold. Then (E_l) admits an eventually positive solution in the form of (3) if and only if (4) has at least one real root λ with $\lambda > 1 - \frac{q}{r}$ for $\frac{r}{q} > 1$ and $\lambda < 1 - \frac{q}{r}$ for $\frac{r}{q} < 1$.

Theorem 2.4. Let $q, r > 0$ such that $r > q$. Then

(1) for $p > 0$ and $\alpha < \beta$, (E_l) has an oscillation in the form of (3) if and only if (4) has no positive real root in $[1 - \frac{q}{r}, \infty)$;
(2) for $p < 0$, (E_l) has an oscillation in the form of (3) if and only if (4) has no positive real root in $[1 - \frac{q}{r}, \infty)$.

3. Linearized Oscillation

This section deals with the linearized oscillation criteria for the system (E). Of course, the criteria are obtained by means of its limiting impulsive equation in the form of (E_l) for different ranges of the neutral coefficient $p(n)$.

Theorem 3.1. Let $p(n) < -1$ be such that $\lim_{n\to\infty} p(n) = p_0 \in (-\infty, -1)$. Assume that

(H_1) $\lim_{n\to\infty} q(n) = q_0 \in (0,\infty)$ *and* $\liminf_{n\to\infty} r(n) = r_0 \in (0, \infty)$,
(H_2) $uf(u) > 0, \frac{f(u)}{u} \geq 1$ *for* $u \neq 0$ *and* $\lim_{u\to 0} \frac{f(u)}{u} = 1$,
(H_3) $vh(v) > 0$ *for* $v \neq 0, \liminf_{|v|\to\infty} |h(v)| \geq v_0 > 0$ *and* $\lim_{v\to 0} \frac{h(v)}{v} = 1$,
(H_4) $\sum_{s=n^*}^{\infty} q(s) + \sum_{j=1}^{\infty} r(m_j - 1) = \infty$
and
(H_5) $\sum_{s=n^*}^{\infty} \left[\sum_{t=n^*}^{s-1} q(t) + \sum_{n^* \leq m_j - 1 \leq s-1} r(m_j - 1) \right] = \infty, \ s > n^* + 1$

hold. If the limiting impulsive system of (E)

$$\begin{cases} \Delta^2[w(n) - (\varepsilon - p_0)w(n-\alpha)] + (q_0 - \varepsilon)w(n-\beta) = 0, \ n \neq m_j \\ \underline{\Delta}[\Delta(w(m_j - 1) - (\varepsilon - p_0)w(m_j - \alpha - 1))] + r_0 w(m_j - \beta - 1) = 0, \ j \in \mathbb{N} \end{cases}$$

has no positive real root in $[1 - \frac{q_0}{r_0}, \infty)$ for $r_0 > q_0$ and some $\varepsilon > 0$, then every solution of the system (E) oscillates.

Proof. Let $u(n)$ be a nonoscillatory solution of (E). Without loss of generality, we may assume that $u(n) > 0$, $u(n-\alpha) > 0$ and $u(n-\beta) > 0$ for $n \geq n_0 > \max\{\alpha, \beta\}$ due to (H_2) and (H_3). Setting

$$\begin{aligned} z(n) &= u(n) - p(n) f(u(n-\alpha)), \\ z(m_j - 1) &= u(m_j - 1) - p(m_j - 1) f(u(m_j - \alpha - 1)) \end{aligned}$$

in (E), we obtain

$$(E_1) \begin{cases} \Delta^2 z(n) = -q(n) h(u(n-\beta)) \leq 0, \, n \neq m_j \\ \underline{\Delta}(\Delta z(m_j - 1)) = -r(m_j - 1) h(u(m_j - \beta - 1)) \leq 0, \, j \in \mathbb{N}. \end{cases}$$

Hence, we can find an $n_1 > n_0 + \beta + 1$ such that $\Delta z(n)$ is nonincreasing for $n \geq n_1$. Let there exists $n_2 > n_1$ such that $\Delta z(n) > 0$ for $n \geq n_2$. Summing the impulsive system (E_1) from n_2 to $n - 1$ $(n > n_2 + 1)$, we get

$$\Delta z(n) - \Delta z(n_2) - \sum_{n_2 \leq m_j - 1 \leq n-1} \underline{\Delta}(\Delta z(m_j - 1)) = -\sum_{s=n_2}^{n-1} q(s) h(u(n-\beta)),$$

that is,

$$\sum_{s=n_2}^{n-1} q(s)h(u(n-\beta)) + \sum_{n_2 \le m_j - 1 \le n-1} r(m_j-1)h(u(m_j-\beta-1)) = -\Delta z(n) + \Delta z(n_2).$$

Using (H_3), it follows that

$$\begin{aligned} h_0 \left[\sum_{s=n_2}^{n-1} q(s) + \sum_{n_2 \le m_j - 1 \le n-1} r(m_j-1) \right] &\le -\Delta z(n) + \Delta z(n_2) \\ &\le \Delta z(n_2) \end{aligned}$$

which is a contradiction to (H_4). Ultimately, $\Delta z(n) < 0$ for $n \ge n_2$ and hence $z(n) > 0$ is nonincreasing for $n \ge n_2$. Now, summing the impulsive system (E_1) from n_2 to $n-1$, we get

$$\begin{aligned} \sum_{s=n_2}^{n-1} q(s)h(u(n-\sigma)) + \sum_{n_2 \le m_j - 1 \le n-1} r(m_j-1)h(u(m_j-\sigma-1)) \\ = \Delta z(n_2) - \Delta z(n) \le -\Delta z(n), \end{aligned}$$

that is,

$$\begin{aligned} \sum_{s=n_2}^{n-1} \left[\sum_{t=n_2}^{s-1} q(t)h(u(t-\sigma)) + \sum_{n_2 \le m_j - 1 \le s-1} r(m_j-1)h(u(m_j-\sigma-1)) \right] &\le z(n_2) - z(n) \\ &\le z(n_2) \\ &< \infty \end{aligned}$$

due to (H_1), (H_3) and (H_5) implies that $\liminf_{n\to\infty} u(n) = 0$. Applying [11, Lemma 2.1.], it follows that $\lim_{n\to\infty} z(n) = 0$. Consequently,

$$\begin{aligned} 0 = \lim_{n\to\infty} z(n) &= \limsup_{n\to\infty} z(n) \\ &\ge \limsup_{n\to\infty} (-p(n)f(u(n-\tau))) \\ &\ge \limsup_{n\to\infty} (-p(n)u(n-\tau)) \\ &= -p_0 \limsup_{n\to\infty} u(n) \end{aligned}$$

leads to the fact that $\lim_{n\to\infty} u(n) = 0$. Because $m_j - 1, m_j - \tau - 1, \cdots$ are the nonimpulsive points, then $\lim_{j\to\infty} u(m_j - 1) = 0$. Since $m_j - 1 < m_j < n$, then an application of Sandwich theorem shows that $\lim_{j\to\infty} u(m_j) = 0$. Denote

$$\begin{aligned} &P(n) = -p(n)\frac{f(u(n-\alpha))}{u(n-\alpha)}, \qquad Q(n) = q(n)\frac{h(u(n-\beta))}{u(n-\beta)} \\ &\text{and } \ R(m_j-1) = r(m_j-1)\frac{h(u(m_j-\beta-1))}{u(m_j-\beta-1)}. \end{aligned}$$

Indeed,

$$\lim_{n\to\infty} Q(n) = \lim_{n\to\infty} q(n) \lim_{n\to\infty} \frac{h(u(n-\beta))}{u(n-\beta)} = q_0,$$

$$\liminf_{n\to\infty} R(n) \ge \liminf_{n\to\infty} r(n) \lim_{n\to\infty} \frac{h(u(n-\beta))}{u(n-\beta)} = r_0$$

and

$$\limsup_{n\to\infty} P(n) = \limsup_{n\to\infty}(-p(n)) \lim_{n\to\infty} \frac{f(u(n-\alpha))}{u(n-\alpha)} = -p_0$$

due to (H_1), (H_2) and (H_3). Using above substitutions in (E), we obtain

$$(E_2) \begin{cases} \Delta^2[u(n) + P(n)u(n-\alpha)] + Q(n)u(n-\beta) = 0, \; n \neq m_j, \; j \in \mathbb{N} \\ \underline{\Delta}[\Delta(u(m_j-1) + P(m_j-1)u(m_j-\alpha-1))] + R(m_j-1)u(m_j-\beta-1) = 0. \end{cases}$$

Taking sum to (E_2) from n_2 to $n-1$, we get

$$\Delta Z(n) - \Delta Z(n_2) + \sum_{s=n_2}^{n-1} Q(s)u(s-\beta) + \sum_{n_2 \le m_j - 1 \le n-1} R(m_j-1)u(m_j-\beta-1) = 0,$$

where $Z(n) = u(n)+P(n)u(n-\alpha)$. Similar to the argument for the case $\Delta z(n) < 0$, it is easy to show that $\Delta Z(n) < 0$ due to (H_3) and (H_4). As $\Delta Z(n)$ is nonincreasing, the above relation reduces to

$$\Delta Z(n) + \sum_{s=n_2}^{n-1} Q(s)u(s-\beta) + \sum_{n_2 \le m_j - 1 \le n-1} R(m_j-1)u(m_j-\beta-1) \le 0,$$

that is,

$$Z(l) - Z(n) + \sum_{s=n}^{l-1} \left[\sum_{t=n_2}^{s-1} Q(t)u(t-\beta) + \sum_{n_2 \le m_j - 1 \le s-1} R(m_j-1)u(m_j-\beta-1) \right] \le 0.$$

As a result

$$Z(n) \ge \sum_{s=n}^{\infty} \left[\sum_{t=n_2}^{s-1} Q(t)u(t-\beta) + \sum_{n_2 \le m_j - 1 \le s-1} R(m_j-1)u(m_j-\beta-1) \right]$$

which is equivalent to

$$u(n) \ge \frac{1}{P(n+\alpha)} \Big[-u(n+\alpha) + \sum_{s=n+\alpha}^{\infty} \Big[\sum_{t=n_2}^{s-1} Q(t)u(t-\beta) + \sum_{n_2 \le m_j - 1 \le s-1} R(m_j-1)u(m_j-\beta-1) \Big] \Big]. \quad (5)$$

Let $\gamma > 1$ and $\varepsilon \in (0, q_0)$ be given such that $(1-\gamma)p_0 < \varepsilon$. Suppose that there exists $n_3 > n_2 + 1$ such that

$$P(n) < \frac{\varepsilon - p_0}{\gamma} \quad , \quad Q(n) > q_0 - \varepsilon, \; R(m_j - 1) > r_0.$$

Then for $n \ge n_3$, (5) reduces to

$$u(n) \ge \frac{\gamma}{\varepsilon - p_0} \times \Big[-u(n+\alpha) + \sum_{s=n+\alpha}^{\infty} \Big[(q_0 - \varepsilon) \sum_{s=n_3}^{s-1} u(t-\beta) + r_0 \sum_{n_3 \le m_j - 1 \le s-1} u(m_j - \beta - 1) \Big] \Big]. \quad (6)$$

Let $X = l_\infty^{n_3}$ be the Banach space of all real valued bounded functions $y(n)$ with sup norm defined by

$$\|y\| = \sup\{|y(n)| : n \geq n_3\}.$$

Consider a closed subset Ω of X such that

$$\Omega = \{y \in X : 0 \leq y(n) \leq 1, n \geq n_3\},$$

and for $y \in \Omega$, $n \geq n_3$ define

$$(Ty)(n) = \begin{cases} Ty(n_3+\rho), \ n_3 \leq n \leq n_3+\rho, \\ \frac{1}{(\varepsilon - p_0)u(n)}\Big[-u(n+\alpha)y(n+\alpha) + \sum_{s=n+\alpha}^{\infty}\big[(q_0-\varepsilon)\sum_{t=n_3}^{s-1} u(t-\beta)y(t-\beta) \\ +r_0\sum_{n_3\leq m_j-1\leq s-1} u(m_j-\beta-1)y(m_j-\beta-1)\big]\Big], \ n > n_3+\rho. \end{cases}$$

For $y \in \Omega$ and due to (6), we have

$$\begin{aligned} Ty(n) &\leq \frac{1}{(\varepsilon-p_0)u(n)}\Big[-u(n+\alpha) + \sum_{s=n+\alpha}^{\infty}\big[(q_0-\varepsilon)\sum_{t=n_3}^{s-1}u(t-\beta) \\ &\quad + r_0 \sum_{n_3\leq m_j-1\leq s-1} u(m_j-\beta-1)\big]\Big] \\ &\leq \frac{1}{\gamma} < 1, \end{aligned}$$

and $Ty(n) \geq 0$ implies that $Ty(n) \in \Omega$ for every $n \geq n_3$. Now, for $y_1, y_2 \in \Omega$

$$\begin{aligned} |Ty_1(n) - Ty_2(n)| &\leq \frac{1}{(\varepsilon-p_0)u(n)}\Big[-u(n+\alpha)|y_1(n+\alpha)-y_2(n+\alpha)| \\ &\quad + \sum_{s=n+\alpha}^{\infty}\big[(q_0-\varepsilon)\sum_{t=n_3}^{s-1}u(t-\beta)|y_1(t-\beta)-y_2(t-\beta)| \\ &\quad + r_0\sum_{n_3\leq m_j-1\leq s-1} u(m_j-\beta-1)|y_1(m_j-\beta-1)-y_2(m_j-\beta-1)|\big]\Big] \end{aligned}$$

implies that

$$\begin{aligned} |Ty_1(n) - Ty_2(n)| &\leq \frac{1}{(\varepsilon-p_0)u(n)}\Big[-u(n+\alpha) + \sum_{s=n+\alpha}^{\infty}\big[(q_0-\varepsilon)\sum_{t=n_3}^{s-1}u(t-\beta) \\ &\quad + r_0\sum_{n_3\leq m_j-1\leq s-1} u(m_j-\beta-1)\big]\Big]\|y_1-y_2\| \\ &\leq \frac{1}{\gamma}\|y_1-y_2\|, \end{aligned}$$

that is,

$$\|Ty_1 - Ty_2\| \leq \frac{1}{\gamma}\|y_1-y_2\|.$$

Since $\frac{1}{\gamma} < 1$, then T is a contraction. By Banach's fixed point theorem [8], T has a unique fixed point $y \in \Omega$ such that $Ty = y$, that is,

$$y(n) = \begin{cases} y(n_3+\rho), \ n_3 \leq n \leq n_3+\rho, \\ \frac{1}{(\varepsilon - p_0)u(n)}\Big[-u(n+\alpha)y(n+\alpha) + \sum_{s=n+\alpha}^{\infty}\big[(q_0-\varepsilon)\sum_{t=n_3}^{s-1} u(t-\beta)y(t-\beta) \\ +r_0\sum_{n_3\leq m_j-1\leq s-1} u(m_j-\beta-1)y(m_j-\beta-1)\big]\Big], \ n > n_3+\rho. \end{cases}$$

Setting $w(n) = u(n)y(n)$ for $n \geq n_3 + \rho$, we obtain

$$w(n) = \frac{1}{(\varepsilon - p_0)}$$
$$\Big[- w(n+\alpha) + \sum_{s=n+\alpha}^{\infty} \big[(q_0 - \varepsilon) \sum_{t=n_3}^{s-1} w(t-\beta) + r_0 \sum_{n_3 \leq m_j - 1 \leq s-1} w(m_j - \beta - 1)\big]\Big]$$

which is a positive solution of the impulsive system

$$(E_3) \begin{cases} \Delta^2[w(n) - (\varepsilon - p_0)w(n-\alpha)] + (q_0 - \varepsilon)w(n-\beta) = 0, \ n \neq m_j \\ \underline{\Delta}[\Delta(w(m_j - 1) - (\varepsilon - p_0)w(m_j - \alpha - 1))] + r_0 w(m_j - \beta - 1) = 0, \ j \in \mathbb{N} \end{cases}$$

whose characteristic equation is given by

$$\Big[\frac{1}{\lambda}(1 - \frac{r_0}{(q_0 - \varepsilon)}) + \frac{r_0}{(q_0 - \varepsilon)}\Big]^{l_1} (\lambda - 1)^2$$
$$- (\varepsilon - p_0)\lambda^{-\alpha}\Big[\frac{1}{\lambda}(1 - \frac{r_0}{(q_0 - \varepsilon)}) + \frac{r_0}{(q_0 - \varepsilon)}\Big]^{l_1 - l_2} (\lambda - 1)^2 + (q_0 - \varepsilon)\lambda^{-\beta} = 0.$$

By Theorem 2.3, $w(n)$ is a positive solution of (E_3) if and only if

$$\lambda > 1 - \frac{(q_0 - \varepsilon)}{r_0} > 1 - \frac{q_0}{r_0}$$

for $\frac{r_0}{q_0} > 1$, a contradiction due to Theorem 2.4. This completes the proof of the theorem.

Remark 3.2. The prototype of the functions f and h in Theorem 3.1 satisfying (H_2) and (H_3) could be of the form

$$G(u) = u(1 + |u|^{\gamma}), \ u \in \mathbb{R}, \ \gamma > 0.$$

Theorem 3.3. Let $-1 < p(n) \leq 0$ and $\lim_{n\to\infty} p(n) = p_0 \in (-1, 0]$. Assume that $(H_1) - (H_5)$ hold. If the limiting impulsive system

$$\begin{cases} \Delta^2[y(n) - (p_0 - \varepsilon_1)y(n-\alpha)] + (q_0 - \varepsilon)y(n-\beta) = 0, \ n \neq m_j \\ \underline{\Delta}[\Delta(y(m_j - 1) - (p_0 - \varepsilon_1)y(m_j - \alpha - 1))] + r_0 y(m_j - \beta - 1) = 0, \ j \in \mathbb{N}. \end{cases}$$

has no positive real roots in $[1 - \frac{q_0}{r_0}, \infty)$ for $r_0 > q_0$ and for some $\varepsilon_1, \varepsilon > 0$, then every solution of the system (E) oscillates.

Proof. Let $u(n)$ be a nonoscillatory solution of (E). Proceeding as in the proof of Theorem 3.1, we have $\lim_{n\to\infty} u(n) = 0$ and $\lim_{j\to\infty} u(m_j - 1) = 0$ and then an application of Sandwich theorem gives $\lim_{j\to\infty} u(m_j) = 0$. Denote

$$P(n) = p(n)\frac{f(u(n-\alpha))}{u(n-\alpha)}, \qquad Q(n) = q(n)\frac{h(u(n-\beta))}{u(n-\beta)}$$
$$\text{and } \ R(m_j - 1) = r(m_j - 1)\frac{h(u(m_j - \beta - 1))}{u(m_j - \beta - 1)}.$$

Due to (H_1), (H_2) and (H_3), it follows that $\lim_{n\to\infty} Q(n) = q_0$, $\liminf_{j\to\infty} R(m_j - 1) \geq r_0$ and $\limsup_{n\to\infty} P(n) = -p_0$. Therefore, (E) can be written as

$$(E_4) \begin{cases} \Delta^2[u(n) + P(n)u(n-\alpha)] + Q(n)u(n-\beta) = 0, \ n \neq m_j \\ \underline{\Delta}[\Delta(u(m_j - 1) + P(m_j - 1)u(m_j - \alpha - 1))] \\ \qquad\qquad\qquad + R(m_j - 1)u(m_j - \beta - 1) = 0, \ j \in \mathbb{N}. \end{cases}$$

Summing (E_4) twice and then using the argument as in Theorem 3.1, we find

$$\begin{aligned} u(n) \geq & - P(n)u(n-\alpha) \\ & + \sum_{s=n}^{\infty}\Big[\sum_{t=n_2}^{s-1} Q(t)u(t-\beta) + \sum_{n_2\leq m_j-1\leq s-1} R(m_j-1)u(m_j-\beta-1)\Big]. \end{aligned} \tag{7}$$

Let $\varepsilon \in (0, q_0)$ and $\varepsilon_1 \in (0, 1+p_0)$. Suppose that there exists $n_3 > n_2+1$ such that

$$Q(n) > q_0 - \varepsilon,\ R(m_j-1) > r_0,\ P(n) < \varepsilon_1 - p_0$$

for $n \geq n_3$. Then for $n \geq n_3$, (7) reduces to

$$\begin{aligned} u(n) \geq & (p_0-\varepsilon_1)u(n-\alpha) \\ & + \sum_{s=n}^{\infty}\Big[(q_0-\varepsilon)\sum_{s=n_2}^{s-1} u(t-\beta) + r_0 \sum_{n_2\leq m_j-1\leq s-1} u(m_j-\beta-1)\Big]. \end{aligned} \tag{8}$$

Let $X = l_\infty^{n_3}$ be the Banach space of all real valued bounded functions $y(n)$ with sup norm defined by

$$\|y\| = \sup\{|y(n)| : n \geq n_3\}.$$

Set

$$\Omega = \{y \in X : y(n) = u(n) \text{ for } n_3 \leq n \leq n_3+\rho \text{ and } 0 \leq y(n) \leq u(n), n > n_3+\rho\}.$$

For $y_1, y_2 \in \Omega$, we define a partial order $y_1 \leq y_2$ on Ω means that $y_1(n) \leq y_2(n)$ for all $n \geq n_3$. Because $\lim_{n\to\infty} u(n) = 0$ and by the definition Ω, it follows that $\inf \Omega$ exist in Ω. Let $\phi \subset \Omega^* \subset \Omega$ be such that

$$\Omega^* = \{y \in X : y_1(n) \leq y(n) \leq y_2(n), 0 \leq y_1(n), y_2(n) \leq u(n), n \geq n_3\}.$$

Since $\limsup_{n\to\infty} u(n) = 0$, then we can find $\{n_k\} \subset \{n\}$ such that $\sup \Omega^* \in \Omega$ as long as $y_2(n_k) \leq u(n_k)$ holds for $n_k \geq n_3, k \in \mathbb{N}$. Next, we define

$$(Ty)(n) = \begin{cases} Ty(n_3+\rho),\ n_3 \leq n \leq n_3+\rho, \\ (p_0-\varepsilon_1)y(n-\alpha) + \sum_{s=n}^{\infty}\big[(q_0-\varepsilon)\sum_{t=n_3}^{s-1} y(t-\beta) \\ \qquad\qquad +r_0\sum_{n_3\leq m_j-1\leq s-1} y(m_j-\beta-1)\big],\ n > n_3+\rho. \end{cases}$$

Then for $y \in \Omega$ and using (8), it follows that

$$\begin{aligned} Ty(n) &\leq (p_0-\epsilon_1)y(n-\alpha) + \sum_{s=n}^{\infty}\big[(q_0-\varepsilon)\sum_{t=n_3}^{s-1} y(t-\beta) + r_0\sum_{n_3\leq m_j-1\leq s-1} y(m_j-\beta-1)\big] \\ &\leq (p_0-\epsilon_1)u(n-\alpha) + \sum_{s=n}^{\infty}\big[(q_0-\varepsilon)\sum_{t=n_3}^{s-1} u(t-\beta) + r_0\sum_{n_3\leq m_j-1\leq s-1} u(m_j-\beta-1)\big] \\ &\leq u(n) \end{aligned}$$

and $Ty(n) \geq 0$ for $n \geq n_3$ implies that $Ty(n) \in \Omega$ for every $n \geq n_3$. For $y_1, y_2 \in \Omega$ with $y_1 \leq y_2$, it is easy to verify that $Ty_1 \leq Ty_2$. Being a closed subspace of X, Ω is a partially ordered Banach space. Therefore, by Knaster-Tarski fixed point theorem [8], T has a unique fixed point $y \in \Omega$ such that $Ty = y$, that is,

$$y(n) = \begin{cases} y(n_3+\rho), & n_3 \leq n \leq n_3+\rho, \\ (p_0-\varepsilon_1)y(n-\alpha) + \sum_{s=n}^{\infty}\big[(q_0-\varepsilon)\sum_{t=n_3}^{s-1} y(t-\beta) & \\ \qquad\qquad +r_0\sum_{n_3\leq m_j-1\leq s-1} y(m_j-\beta-1)\big], & n > n_3+\rho. \end{cases}$$

Therefore, for $n \geq n_3 + \rho$, we obtain

$$y(n) =$$

$$(p_0 - \varepsilon_1)y(n-\alpha) + \sum_{s=n}^{\infty} \Big[(q_0 - \varepsilon) \sum_{t=n_3}^{s-1} y(t-\beta) + r_0 \sum_{n_3 \leq m_j - 1 \leq s-1} y(m_j - \beta - 1)\Big].$$

We may note that, $y(n)$ is a positive solution of the impulsive system

$$(E_5) \begin{cases} \Delta^2[y(n) - (p_0 - \varepsilon_1)y(n-\alpha)] + (q_0 - \varepsilon)y(n-\beta) = 0, \ n \neq m_j \\ \underline{\Delta}[\Delta(y(m_j - 1) - (p_0 - \varepsilon_1)y(m_j - \alpha - 1))] + r_0 y(m_j - \beta - 1) = 0, \ j \in \mathbb{N}. \end{cases}$$

Indeed, its characteristic equation is given by

$$\Big[\frac{1}{\lambda}(1 - \frac{r_0}{(q_0 - \varepsilon)}) + \frac{r_0}{(q_0 - \varepsilon)}\Big]^{l_1} (\lambda - 1)^2$$

$$- (p_0 - \varepsilon_1)\lambda^{-\alpha} \Big[\frac{1}{\lambda}(1 - \frac{r_0}{(q_0 - \varepsilon)}) + \frac{r_0}{(q_0 - \varepsilon)}\Big]^{l_1 - l_2} (\lambda - 1)^2 + (q_0 - \varepsilon)\lambda^{-\beta} = 0.$$

Due to Theorem 2.3, $y(n)$ is a positive solution of (E_5) if and only if

$$\lambda > 1 - \frac{(q_0 - \varepsilon)}{r_0} > 1 - \frac{q_0}{r_0}$$

for $\frac{r_0}{q_0} > 1$, a contradiction due to Theorem 2.4. This completes the proof of the theorem.

Theorem 3.4. Let $\alpha \leq \beta$. Let $0 \leq p(n) < 1$ be such that $\lim_{n\to\infty} p(n) = p_0 \in [0, 1)$. Assume that (H_1), (H_3), (H_4), (H_5) and
(H_6) $uf(u) > 0, \frac{f(u)}{u} \leq 1 \ for \ u \neq 0 \ and \ \lim_{u\to 0} \frac{f(u)}{u} = 1$
hold. If the limiting impulsive system

$$\begin{cases} \Delta^2[w(n) - \gamma(p_0 - \epsilon)w(n-\alpha)] + \gamma(q_0 - \varepsilon)w(n-\beta) = 0, \ n \neq m_j \\ \underline{\Delta}[\Delta(w(m_j - 1) - \gamma(p_0 - \epsilon)w(m_j - \alpha - 1))] + \gamma r_0 w(m_j - \beta - 1) = 0, \ j \in \mathbb{N}. \end{cases}$$

has no positive real root in $[1 - \frac{q_0}{r_0}, \infty)$ for $r_0 > q_0$ and for some $\epsilon, \gamma, \varepsilon > 0$, then every solution of (E) oscillates.

Proof. Let $u(n)$ be a nonoscillatory solution of (E). Then proceeding as in the proof of Theorem 3.1, we have that $\Delta z(n) < 0$ and $z(n)$ is monotonic for $n \geq n_1$. Suppose there exists $n_2 > n_1$ such that $z(n) < 0$ for $n \geq n_2$. Since $\Delta z(n)$ is nonincreasing, then we can find a $n_3 > n_2 + 1$ and a constant $C > 0$ such that $\Delta z(n) \leq -C$ for $n \geq n_3$ and hence $\Delta z(m_j - 1) \leq -C$. Therefore,

$$z(n) - z(n_3) - \sum_{n_3 \leq m_j - 1 \leq n-1} \Delta z(m_j - 1) \leq - \sum_{s=n_3}^{n-1} C$$

implies that

$$z(n) \leq z(n_3) - \Big[\sum_{s=n_3}^{n-1} C + \sum_{n_3 \leq m_j - 1 \leq n-1} C\Big]$$

that is, $\lim_{n\to\infty} z(n) = -\infty$. By Sandwich theorem and due to $m_j - 1 < m_j < n$, we obtain $\lim_{j\to\infty} z(m_j) = -\infty$. On the other hand, $z(n) < 0$ for $n \geq n_3$ implies that

$$u(n) \leq p(n)f(u(n-\alpha)) \leq u(n-\alpha) \leq u(n-2\alpha) \leq u(n-3\alpha) \cdots \leq u(n_3)$$

and analogously,

$$u(m_j-1)\le u(m_j-\alpha-1)\le u(m_j-2\alpha-1)\le u(m_j-3\alpha-1)\cdots\le u(n_3)$$

due to the nonimpulsive points $m_j-1, m_j-\alpha-1, m_j-2\alpha-1,\cdots$. Therefore, $u(n)$ is bounded for all nonimpulsive points. We assert that $u(m_j)$ is bounded. If not, let it be $\lim_{j\to\infty} u(m_j)=+\infty$. Therefore,

$$\begin{aligned}z(m_j)&=u(m_j)+p(m_j)f(u(m_j-\alpha))\\&\ge u(m_j)-f(u(m_j-\alpha))\\&\ge u(m_j)-u(m_j-\alpha)\ge u(m_j)-b,\end{aligned}$$

implies that $z(m_j)>0$ as $j\to\infty$, a contradiction, where $u(m_j-\alpha)\le b$. So, our assertation holds. Ultimately, $z(n)$ is bounded for every n, a contradiction. Thus, $z(n)>0$ for $n\ge n_2$. Proceeding as in the proof of Theorem 2.1, we obtain $\liminf_{n\to\infty} u(n)=0$ and $\lim_{n\to\infty} z(n)=0$. Clearly,

$$\begin{aligned}0=\lim_{n\to\infty} z(n)&=\limsup_{n\to\infty} z(n)\\&\ge\limsup_{n\to\infty}(u(n)-p(n)f(u(n-\alpha)))\\&\ge\limsup_{n\to\infty} u(n)-\liminf_{n\to\infty}(p(n)u(n-\alpha)\\&=(1-p_0)\limsup_{n\to\infty} u(n)\end{aligned}$$

implies that $\limsup_{n\to\infty} u(n)=0$ and hence $\lim_{n\to\infty} u(n)=0$. Also, $\lim_{j\to\infty} u(m_j-1)=0$ for nonimpulsive points $m_j-1, m_j-\tau-1,\cdots$. Due to Sandwich theorem and for $m_j-1<m_j<n$, we have $\lim_{j\to\infty} u(m_j)=0$. Denote

$$\begin{aligned}P(n)=&p(n)\frac{f(u(n-\alpha))}{u(n-\alpha)},\qquad Q(n)=q(n)\frac{h(u(n-\beta))}{u(n-\beta)}\\&\textit{and}\;\; R(m_j-1)=r(m_j-1)\frac{h(u(m_j-\beta-1))}{u(m_j-\beta-1)}.\end{aligned}$$

Then, the system (E) can be written as

$$(E_6)\begin{cases}\Delta^2[u(n)-P(n)u(n-\alpha)]+Q(n)u(n-\beta)=0,\; n\ne m_j\\ \underline{\Delta}[\Delta(u(m_j-1)-P(m_j-1)u(m_j-\alpha-1))]\\ \qquad\qquad\qquad +R(m_j-1)u(m_j-\beta-1)=0,\; j\in\mathbb{N}.\end{cases}$$

By (H_1), (H_3) and (H_6), it follows that

$$\lim_{n\to\infty} Q(n)=\lim_{n\to\infty} q(n)\lim_{n\to\infty}\frac{h(u(n-\beta))}{u(n-\beta)}=q_0,$$

$$\liminf_{j\to\infty} R(m_j-1)\ge\liminf_{j\to\infty} r(m_j-1)\lim_{j\to\infty}\frac{h(u(m_j-\beta-1))}{u(m_j-\beta-1)}=r_0$$

and

$$\liminf_{n\to\infty} P(n)\ge\liminf_{n\to\infty} p(n)\lim_{n\to\infty}\frac{f(u(n-\alpha))}{u(n-\alpha)}=p_0.$$

Let $\varepsilon\in(0,q_0)$, $\epsilon\in(0,1-p_0)$ and $0<\gamma<1$. Suppose that there exists $n_3\ge n_2+1$ such that

$$Q(n)>q_0-\varepsilon,\; R(m_j-1)>r_0,\; P(n)>(p_0-\epsilon).$$

Summing (E_6) twice, we get

$$u(n) = P(n)u(n-\alpha) + \sum_{s=n}^{\infty}\Big[\sum_{t=n_3}^{s-1} Q(t)u(t-\beta) + \sum_{n_3\le m_j-1\le s-1} R(m_j-1)u(m_j-\beta-1)\Big]$$
$$\ge (p_0-\epsilon)u(n-\alpha) + \sum_{s=n}^{\infty}\Big[(q_0-\varepsilon)\sum_{t=n_3}^{s-1} u(t-\beta) + r_0 \sum_{n_3\le m_j-1\le t-1} u(m_j-\beta-1)\Big]. \tag{9}$$

Let $X = l_{\infty}^{n_3}$ be the Banach space of all real valued bounded functions $y(n)$ with the sup norm defined by

$$\|y\| = \sup\{|y(n)| : n \ge n_3\}.$$

Consider a closed subset Ω of X such that

$$\Omega = \{y \in X : 0 \le y(n) \le 1, n \ge n_3\}.$$

For $y \in \Omega$ and $n \ge n_3$, we define

$$(Ty)(n) = \begin{cases} Ty(n_3+\rho), \ n_3 \le n \le n_3+\rho, \\ \frac{\gamma}{u(n)}\Big[(p_0-\epsilon)u(n-\alpha)y(n-\alpha) + \sum_{s=n}^{\infty}\Big[(q_0-\varepsilon)\sum_{t=n_3}^{s-1} u(t-\beta)y(t-\beta) \\ +r_0 \sum_{n_3\le m_j-1\le t-1} u(m_j-\beta-1)y(m_j-\beta-1)\Big]\Big], \ n > n_3+\rho. \end{cases}$$

For $y \in \Omega$ and using (9), we have

$$Ty(n)$$
$$\le \frac{\gamma}{u(n)}\Big[(p_0-\epsilon)u(n-\alpha) + \sum_{s=n}^{\infty}\Big[(q_0-\varepsilon)\sum_{t=n_3}^{s-1} u(t-\beta) + r_0 \sum_{n_3\le m_j-1\le t-1} u(m_j-\beta-1)\Big]\Big]$$
$$\le \gamma < 1,$$

and $Ty(n) \ge 0$ implies that $Ty(n) \in \Omega$ for $n \ge n_3$. For $y_1, y_2 \in \Omega$, we have

$$|Ty_1(n) - Ty_2(n)| \le \frac{\gamma}{|u(n)|}\Big[(p_0-\epsilon)u(n-\alpha)|y_1(n-\alpha) - y_2(n-\alpha)|$$
$$+ \sum_{s=n}^{\infty}\Big[(q_0-\varepsilon)\sum_{t=n_3}^{s-1} u(n-\beta)|y_1(t-\beta) - y_2(t-\beta)|$$
$$+ r_0 \sum_{n_3\le m_j-1\le t-1} u(m_j-\beta-1)|y_1(m_j-\beta-1) - y_2(m_j-\beta-1)|\Big]\Big]$$

which implies

$$|Ty_1(n) - Ty_2(n)| \le \frac{\gamma}{|u(n)|}\Big[(p_0-\epsilon)u(n-\alpha) + \sum_{s=n}^{\infty}\Big[(q_0-\varepsilon)\sum_{s=n_3}^{s-1} u(t-\beta)$$
$$+ r_0 \sum_{n_3\le m_j-1\le t-1} u(m_j-\beta-1)\Big]\Big]\|y_1 - y_2\|$$
$$\le \gamma\|y_1 - y_2\|,$$

that is,

$$\|Ty_1 - Ty_2\| \le \gamma\|y_1 - y_2\|.$$

Since $\gamma < 1$, then T is a contraction. By Banach's fixed point theorem [8], T has a unique fixed point $y \in \Omega$ such that $Ty = y$. Thus,

$$y(n) = \begin{cases} y(n_3+\rho),\ n_3 \le n \le n_3+\rho, \\ \frac{\gamma}{u(n)}\Big[(p_0-\epsilon)u(n-\alpha)y(n-\alpha) + \sum_{s=n}^{\infty}\Big[(q_0-\varepsilon)\sum_{s=n_3}^{s-1} u(t-\beta)y(t-\beta) \\ +r_0\sum_{n_3\le m_j-1\le t-1} u(m_j-\beta-1)y(m_j-\beta-1)\Big]\Big],\ n > n_3+\rho. \end{cases}$$

If we set $w(n) = u(n)y(n)$ for $n \ge n_3+\rho$, then

$$w(n) =$$

$$\gamma\Big[(p_0-\epsilon)w(n-\alpha) + \sum_{s=n}^{\infty}\Big[(q_0-\varepsilon)\sum_{t=n_3}^{s-1} w(t-\beta) + r_0 \sum_{n_3\le m_j-1\le t-1} w(m_j-\beta-1)\Big]\Big]$$

which is a positive solution of the impulsive system

$$(E_7)\begin{cases} \Delta^2[w(n)-\gamma(p_0-\epsilon)w(n-\alpha)] + \gamma(q_0-\varepsilon)w(n-\beta) = 0,\ n \ne m_j \\ \underline{\Delta}[\Delta(w(m_j-1)-\gamma(p_0-\epsilon)w(m_j-\alpha-1))] + \gamma r_0 w(m_j-\beta-1) = 0,\ j\in\mathbb{N}. \end{cases}$$

Indeed, its characteristic equation is given by

$$\Big[\frac{1}{\lambda}\Big(1-\frac{r_0}{q_0-\varepsilon}\Big)+\frac{r_0}{q_0-\varepsilon}\Big]^{\mu}(\lambda-1)^2$$
$$-\gamma(p_0-\epsilon)\lambda^{-\alpha}\Big[\frac{1}{\lambda}\Big(1-\frac{r_0}{q_0-\varepsilon}\Big)+\frac{r_0}{q_0-\varepsilon}\Big]^{\mu-\nu}(\lambda-1)^2 + \gamma(q_0-\varepsilon)\lambda^{-\beta} = 0.$$

Because of Theorem 2.3, $w(n)$ is a positive solution of (E_7) if and only if

$$\lambda > 1 - \frac{\gamma(q_0-\varepsilon)}{\gamma r_0} = 1 - \frac{(q_0-\varepsilon)}{r_0} > 1 - \frac{q_0}{r_0}$$

for $\frac{r_0}{q_0} > 1$, a contradiction due to Theorem 2.4. This completes the proof of the theorem.

Remark 3.5. The prototype of the functions h and f in Theorem 3.4 satisfying (H_2) and (H_6) respectively could be of the form

$$h(u) = u(1+|u|^{\gamma}),\ u\in\mathbb{R},\ \gamma > 0$$

and

$$f(u) = \frac{u}{(1+|u|^{\gamma})},\ u\in\mathbb{R},\ \gamma>0.$$

4. Discussion and Examples

The solutions of nonlinear impulsive equations behave in peculiar ways and these ways can be developed by means of different techniques incorporated in the method. Linearized oscillation is one of them in which fixed point theory is a key. An attempt was made here to establish the sufficient conditions with the fact that the solution space of nonlinear equation is reducing to the solution space of its limiting equation. But, we guess that *under what condition the converse will be true.* May be due to our method, we could not view this technique for the critical points $p(n) = 1, -1$.

We conclude this section with the following examples to illustrate our main results:
Example 4.1. For $n > 2$, consider

$$(E_8)\begin{cases}\Delta^2[u(n)-p(n)f(u(n-1))]+q(n)h(u(n-2))=0,\ n\neq 3j\\ \underline{\Delta}[\Delta(u(m_j-1)-p(m_j-1)f(u(m_j-2)))]\\ \qquad\qquad\qquad +r(m_j-1)h(u(m_j-3))=0,\ j\in\mathbb{N},\end{cases}$$

where $p(n) = -2 + e^{-(n+1)}$, $q(n) = 0.1 + e^{-(n^2+1)}$, $r(m_j - 1) = 6(2 + \cos(m_j - 1))$, $m_j = 3j$, $j \in \mathbb{N}$, $f(u) = u(1 + |u|)$ and $h(u) = u$. The limiting equation of (E_8) is given by

$$(E_9)\begin{cases}\Delta^2[y(n)-p_0y(n-1)]+q_0y(n-2)=0,\ n\neq 3j\\ \underline{\Delta}[\Delta(y(m_j-1)-p_0y(m_j-2))]+r_0y(m_j-3)=0,\ j\in\mathbb{N},\end{cases}$$

where $p_0 = -2$, $q_0 = 0.1$, $r_0 = 6$. Clearly, (E_8) has no positive real roots in $[1-\frac{q_0}{r_0},\infty) = [0.983,\infty)$ and hence by Theorem 3.1, every solution of (E_9) oscillates. Let $l_1 = 5$ and $l_2 = 2$. We may note that (E_8) has an oscillatory solution $y(n) = (0.967213)^n(-1)^{i(3,n)}$ due to Theorem 2.1.
Example 4.2. For $n > 3$, consider

$$(E_{10})\begin{cases}\Delta^2[u(n)-p(n)f(u(n-1))]+q(n)h(u(n-3))=0,\ n\neq 3j\\ \underline{\Delta}[\Delta(u(m_j-1)-p(m_j-1)f(u(m_j-2)))]\\ \qquad\qquad\qquad +r(m_j-1)h(u(m_j-4))=0,\ j\in\mathbb{N},\end{cases}$$

where $p(n) = \frac{1}{2}(1 + \frac{1}{n})$, $q(n) = \frac{6n^3+16n^2+10n+2}{n(2n+2)(2n+4)}$, $r(m_j - 1) = \frac{1}{2}(4 + \frac{1}{2m_j-2} - \frac{1}{2m_j} - \frac{1}{2m_j+4} - \frac{1}{2m_j+6})$, $m_j = 3j$, $j \in \mathbb{N}$, $f(u) = u$ and $h(u) = u(1 + u^2)$. The limiting equation of (E_{10}) is given by

$$(E_{11})\begin{cases}\Delta^2[y(n)-p_0y(n-1)]+q_0y(n-3)=0,\ n\neq 3j\\ \underline{\Delta}[\Delta(y(m_j-1)-p_0y(m_j-2))]+r_0y(m_j-4)=0,\ j\in\mathbb{N},\end{cases}$$

where $p_0 = \frac{1}{2}$, $q_0 = \frac{3}{2}$, $r_0 = 2$. Clearly, (E_{11}) has no positive real roots in $[1-\frac{q_0}{r_0},\infty) = [0.25,\infty)$ and hence by Theorem 3.4, every solution of (E_{10}) oscillates. In particular, $u(n) = (-1)^n$ is an oscillatory solution of first equation of (E_{10}). We may note that the second equation of (E_{10}) has a solution $2^{i(3,n)}$. Let $l_1 = 3$ and $l_2 = 1$, then by Theorem 2.4 every solution of (E_{11}) oscillates.

References

[1] L. Berezansky and E. Braverman, Linearized oscillation theory for a nonlinear delay impulsive equation, J. Comput. Appl. Math., **161** (2003), 477–495.
[2] L. Berezansky and E. Braverman, Linearized oscillation theory for a nonlinear equation with a distributed delay, Math. Comput. Model. **48** (2008), 287–304.
[3] E. Braverman and B. Karpuz, Linearized oscillation theory for a nonlinear nonautonomous difference equations, 3–19 (2020), Difference Equations and Discrete Dynamical Systems with Applications, Springer Proceeding in Mathematics and Statistics 312.
[4] G. N. Chhatria, Nonoscillation of first order neutral impulsive difference equations, Int. J. Difference Equ., **14** (2019), 115–125.
[5] G. N. Chhatria, On oscillatory second order impulsive neutral difference equations, AIMS Math., **5** (2020), 2433–2447.
[6] G. N. Chhatria, Application of characteristic equation of first order neutral impulsive difference equations, J. Anal., (2020).
https://doi.org/10.1007/s41478-020-00255-9.

[7] Y. Duan, W. Feng and J. Yan, Linearized oscillation of nonlinear impulsive delay differential equations, Comput. Math. Appl. **44** (2002), 1267–1274.

[8] L. H. Erbe, Q. Kong and B. G. Zhang, Oscillation Theory for Functional Differential Equations, Marcel Dekker, New York, 1995.

[9] I. Gyori and G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, Claredon Press, New York, 1991.

[10] V. Lakshmikantham, D. D. Bainov and P. S. Simieonov, Oscillation Theory of Impulsive Differential Equations World Scientific, Singapore, 1989.

[11] N. Parhi and A. K. Tripathy, Oscillation of forced nonlinear neutral delay difference equations of first order, Czech. Math. J. **53** (2002), 83–101.

[12] M. Peng, Oscillation theorems of second order nonlinear neutral delay difference equations with impulse, Comput. Math. Appl. **44** (2002), 741–748.

[13] D. H. Peng, M. A. Han and H. Y. Wang, Linearized oscillations of first order nonlinear neutral delay difference equations, Comput, Math. Appl. **45** (2003), 1785–1796.

[14] M. Peng, Oscillation criteria for second order impulsive delay difference equations, Appl. Math. Comput. **146** (2003), 227–235.

[15] A. M. Samoilenko and N. A. Perestynk, Differential Equations with Impulse Effect, Visca Skola, Kiev, 1987.

[16] I. Stamova, G. Stamov, Applied Impulsive Mathematical Models, CMS Books in Mathematics, Springer, Switzerland, (2016).

[17] A. K. Tripathy and G. N. Chhatria, Oscillation criteria for forced first order nonlinear neutral impulsive difference system, Tatra Mt. Math. Publ. **71** (2018), 175–193.

[18] A. K. Tripathy and G. N. Chhatria, Oscillation criteria for first order neutral impulsive difference equations with constant coefficients, Differ. Equ. Dyn. Syst., (2019). https://doi.org/10.1007/s12591-019-00495-7.

[19] A. K. Tripathy and G. N. Chhatria, On oscillatory first order neutral impulsive difference equations, Math. Bohem., **145** (2020), 361–375.

[20] A. K. Tripathy and G. N. Chhatria, On the behaviour of solutions of neutral impulsive difference equations of second order, Math. Commun., **25** (2020), 297–314.

[21] A. K. Tripathy and G. N. Chhatria, Nonlinear second order impulsive difference equations and their oscillation properties, Tatra Mt. Math. Publ. **76** (2020), 171–190.

[22] G. P. Wei, The persistance of nonoscillatory solutions of difference equation under impulsive perturbations, Comput. Math. Appl. **50** (2005), 1579–1586.

[23] Y. Xiao, S. Tang and L Chen, A linearized oscillation result for odd order neutral difference equations, Indian J. Pure Appl. Math. **33** (2002), 277–286.

[24] Z. Zhou and J. S. Yu, Linearized oscillation theorems for neutral difference equations, Israel J. Math. **114** (1999), 149–156.

[25] Z. Zhou, J. Yu and G. Lei, Oscillations for even order neutral difference equations, Korean J. Comput. and Math. **7** (2000), 601–610.

[26] Z. Zhou and Z. Lin, Some results on the linearized oscillation of the odd order neutral difference equations, Appl. Anal. **82** (2003), 401–409.

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