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D-MYCIELSKIAN GRAPH OF A GRAPH

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Abstract. The idea of constructing a triangle-free k-chromatic graphs, where $k \geq 3$, was initiated in the middle of the twentieth century. Mycielski who gave a fascinating construction to obtain a triangle-free graph with large chromatic number known as the Mycielskian of a graph. In this paper, we discuss the construction of an interesting transformation graph which is also results in a triangle-free graph with same chromatic number as that of the Mycielskian of a graph. We call this graph as D-Mycielskian graph of a graph. Also, we discuss its basic properties such as connectedness, diameter, covering invariants, connectivity, traversability and domination number. Further, we obtain M-polynomial and Hosoya polynomial of this transformation graph and derive the expressions for some degree-based and distance-based graph indices of this graph.

1. Introduction

Throughout this paper, by a graph $G = (V, E)$ we mean a finite undirected nontrivial graph without loops and multiple edges, where V is the vertex set and E is an edge set. The *open neighbourhood* of a vertex $v \in V(G)$ is defined as the set $N_G(v)$ consisting all the vertices u which are adjacent to v in G and the *closed neighbourhood* of a vertex $v \in V(G)$ is defined as the set $N_G[v]$ consisting v and all the vertices u which are adjacent to v in G i.e., $N_G[v] = N_G(v) \cup \{v\}$. The *degree of a vertex* $v \in V(G)$, denoted by $d_G(v)$ and is defined as $|N_G(v)|$. The minimum and maximum degree of G are defined as $\delta(G) = min\{d_G(x) : x \in V\}$ and $\Delta(G) = max\{d_G(x) : x \in V\}$, respectively. The connectivity [26] $\kappa(G)$ of a connected graph G is the least positive integer k such that there exists $S \subset V$, $|S| = k$ and $G \setminus S$ is disconnected or reduces to the trivial graph K_1. The two graphs G and H are *isomorphic* [26] (written $G \cong H$) if there exists a one-to-one correspondence between their point sets which preserves adjacency.

For a graph $G = (V, E)$, a *dominating set* [28] is a subset D of V such that every vertex of V not in D is adjacent to at least one vertex in D. The dominating set D is *minimal dominating set* of G if no proper subset of D is a dominating set. The *domination number* $\gamma(G)$ *of a graph* is the cardinality of the smallest minimal

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dominating set of G. Basic graph theoretic terminologies and notations can be found in [7, 26, 28].

The idea of constructing a triangle-free *k-chromatic graphs*, where $k \geq 3$, was initiated in the middle of the twentieth century. Mycielski [37] who gave a fascinating construction to obtain a triangle-free graph with large chromatic number known as the *Mycielskian of a graph.*

Definition 1. [37] *For a graph $G = (V, E)$, the Mycielskian of G, denoted by $\mu(G)$, is the graph with vertex set consisting of the disjoint union $V \cup V' \cup \{u\}$, where $V' = \{x' : x \in V\}$, and the edge set $E \cup \{x'y : xy \in E\} \cup \{x'u : x' \in V'\}$. The triad (V, V', u) denote the vertex set of $\mu(G)$. Here, we call x' the twin of x in $\mu(G)$ and vice versa, u is called the root of $\mu(G)$ (See Fig. 1.)*

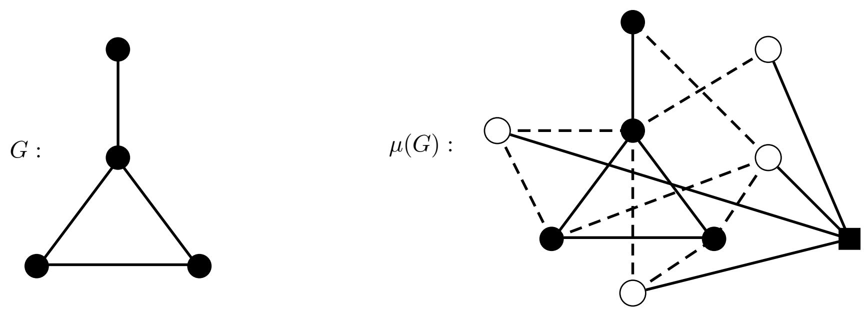

FIGURE 1. The graph G and its Mycielskian.

Now, we define a new transformation graph which is also results in a triangle-free graph with same chromatic number as that of the Mycielskian of a graph. We call this graph as *D-Mycielskian graph of a graph.*

Definition 2. *For a graph $G = (V, E)$, the D-Mycielskian graph of G, denoted by $\mu_\gamma(G)$, is the graph with vertex set consisting of the disjoint union $V \cup V' \cup D$, where $V' = \{x' : x \in V\}, D = \{u_i : 1 \leq i \leq \gamma(G)\}$, and the edge set $E \cup E' \cup D'$, where $E' = \{x'y : xy \in E\}$ and $D' = \{x'u_i : x' \in V'$ and $1 \leq i \leq \gamma(G)\}$. The triad (V, V', D) denote the vertex set of $\mu_\gamma(G)$. Here, we call x' the twin of x in $\mu_\gamma(G)$ (See Fig. 2).*

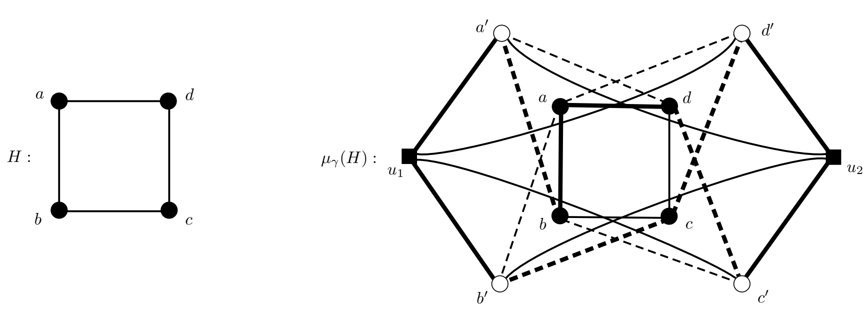

FIGURE 2. The graph H and its D-Mycielskian graph.

Note 1: The dark circles denote the vertices of G while the light circles denote the twin vertices and the dark squares denote the members of D.
Note 2: The word triangle-free construction is used only when the graph (i.e., G) considered for transformation is triangle-free.

Observation 1. *For any graph G, the Mycielskian of G is always an induced subgraph of D-Mycielskian graph of G.*

Remark 1.1. [3] *For any graph G without isolated vertices, the Mycielskian of G is connected.*

Theorem 1.1. [16] *For any nontrivial connected graph G of order n,*

$$\alpha_0 + \beta_0 = \alpha_1 + \beta_1 = n.$$

In this paper, we denote P_n, C_n, K_n, $K_{a,b}$, $K_{1,n}$ and W_n for a path, a cycle, a complete graph, a complete bipartite graph, a star and a wheel, respectively. The symbol $\lceil x \rceil$ denotes the smallest integer that is greater than or equal to x, $\lfloor x \rfloor$ denotes the greatest integer that is smaller than or equal to x.

2. Main results

Theorem 2.1. *Let G be any graph of order n and size m. Then*
(a) $|V(\mu_\gamma(G))| = 2n + \gamma(G)$,
(b) $|E(\mu_\gamma(G))| = 3m + n\gamma(G)$.

Proof. We prove this result by using definition 2 as follows:
(a) We have, $V(\mu_\gamma(G)) = V \cup V' \cup D \implies |V(\mu_\gamma(G))| = 2n + \gamma(G)$,
(b) We have, $E(\mu_\gamma(G)) = E \cup E' \cup D'$, where $E' = \{x'y : xy \in E\}$ and $D' = \{x'u_i : x' \in V' \text{ and } u_i \in D\}$. Therefore,

$$\begin{aligned} |E(\mu_\gamma(G))| &= |E| + |E'| + |D'| \\ &= m + \sum_{v \in V} d_G(v) + n\gamma(G) \\ &= 3m + n\gamma(G). \end{aligned}$$

□

Theorem 2.2. *Let G be any graph of order n. Then*

$$d_{\mu_\gamma(G)}(x) = \begin{cases} 2d_G(x) & if\ \ x \in V, \\ d_G(x) + \gamma(G) & if\ \ x \in V', \\ n & if\ \ x \in D. \end{cases}$$

Proof. The proof follows from the definition 2. □

Theorem 2.3. *Let G be a graph of order n with $\delta(G) = min\{d_G(x) : x \in V\}$ and $\Delta(G) = max\{d_G(x) : x \in V\}$. Then*

$$\begin{aligned} \delta(\mu_\gamma(G)) &= min\{2\delta(G), \delta(G) + \gamma(G)\}, \\ \Delta(\mu_\gamma(G)) &= \begin{cases} 2\Delta(G) & if\ \ \Delta(G) \geq \frac{n}{2}, \\ n & otherwise. \end{cases} \end{aligned}$$

Proof. The proof follows from the definition 2. □

Theorem 2.4. *For any graph G without isolated vertices, the D-Mycielskian graph $\mu_\gamma(G)$ is connected.*

Proof. Suppose G is connected. Then by Observation 1.1, $\mu(G)$ is connected. Therefore from Remark 1, the remaining $\gamma(G)-1$ members of D in $\mu_\gamma(G)$ are joined by the members of V'. Hence, $\mu_\gamma(G)$ is connected.

Suppose G is not connected and has no isolated vertex. Let v_1 and v_2 are any two vertices which have no path connecting them in G. Then they are joined by a path $v_1v_i'u_jv_k'v_2$ in $\mu_\gamma(G)$, where v_i' is a twin vertex of v_i such that $v_1v_i \in E$, v_k' is a twin vertex of v_k such that $v_2v_k \in E$, since G has no isolated vertices, and $u_j \in D$. Hence, $\mu_\gamma(G)$ is connected. □

Theorem 2.5. *For any graph G without isolated vertices,*

$$diam(\mu_\gamma(G)) \leq 4.$$

Proof. For any two adjacent vertices of G, their corresponding vertices are adjacent in $\mu_\gamma(G)$. For any two non adjacent vertices v and w of G we have two situations, (i) if $d_G(v,w) \leq 4$, then there is nothing to prove. (ii) if $d_G(v,w) > 4$, then there exists a path $vv_i'u_jv_k'w$ of length 4 in $\mu_\gamma(G)$, where v_i' is a twin vertex of v_i such that $v_1v_i \in E$, v_k' is a twin vertex of v_k such that $v_2v_k \in E$.

Suppose $v \in V$ and $v_i' \in V'$. If v_i' is a twin vertex of v, then by definition 2, $d_G(v,v_i') = 2$. If v_i' is not a twin vertex of v, then we have two cases as mentioned below:

Case 1. *If v_i' is a twin of a vertex w such that $vw \in E$, then v and w are adjacent in $\mu_\gamma(G)$.*

Case 2. *If v_i' is a twin of a vertex w such that $vw \notin E$, then we have two subcases in the following order:*
Subcase (i). *If v_i' is a twin of a vertex w and $v_kw \in E$, where $vv_k \in E$, then $d_{\mu_\gamma(G)}(v,v_i') = 2$.*
Subcase (ii). *If v_i' is a twin of a vertex w and $v_kw \in E$, where $vv_k \notin E$, then $d_{\mu_\gamma(G)}(v,v_i') = 3$.*

If $v', w' \in V'$, then there exists a path $v'u_jw'$ of length two in $\mu_\gamma(G)$, where $u_j \in D$. If $v' \in V'$ and $u_j \in D$, then by definition 2, they are adjacent in $\mu_\gamma(G)$. For $v \in V$ and $u_j \in D$, there exists a path $vv_i'u_j$ of length two in $\mu_\gamma(G)$, where $v_i' \in V'$ such that $vv_i \in E$, since G has no isolated vertices. Suppose $u_i, u_j \in D$. Then there exists a path $u_iv_k'u_j$ of length two in $\mu_\gamma(G)$. Thus for any two vertices $v, w \in V(\mu_\gamma(G)), d_{\mu_\gamma(G)}(v,w) \leq 4$. □

Corollary 2.6. *If G is any graph without isolated vertices and $diam(G) \geq 4$, then*

$$diam(\mu_\gamma(G)) = 4.$$

Proposition 2.7. *If G is triangle-free, then so is $\mu_\gamma(G)$.*

Remark 2.1. *If $\gamma(G) = 1$, then $\mu_\gamma(G) \cong \mu(G)$.*

Lemma 2.8. [3] *Let $f : G \to H$ be a graph isomorphism of G onto H. Then $f(N_G(x)) = N_H(f(x))$. Furthermore, $G - x \cong H - f(x)$, and $G - N_G[x] \cong H - N_H[f(x)]$ under the restriction maps of f to the respective domains.*

Theorem 2.9. *For any two graphs G and H, $\mu_\gamma(G) \cong \mu_\gamma(H)$ if and only if $G \cong H$.*

Proof. If $G \cong H$, then $\mu_\gamma(G) \cong \mu_\gamma(H)$ is trivial. So assume that G and H are two graphs without isolated vertices such that $\mu_\gamma(G) \cong \mu_\gamma(H)$. For $n = 2$ or 3 the result is trivial. So, assume that $n \geq 4$. If G is of order n, then $\mu_\gamma(G)$ and $\mu_\gamma(H)$ are both of order $2n+\gamma(G)$, hence H is also of order n and $\gamma(H)=\gamma(G)$.

Let $F : \mu_\gamma(G) \to \mu_\gamma(H)$ be the given isomorphism, where $V(\mu_\gamma(G))$ and $V(\mu_\gamma(H))$ are given by the triads (V_1, V_1', D_1) and (V_2, V_2', D_2) respectively.

Now, we look at the possible images of the vertex $u_i \in D_1$ of $\mu_\gamma(G)$ under f. All the vertices $u_i \in D_1$ and $u_j \in D_2$ are of degree n. If $f(u_i)=u_j$, then by Lemma 2.8,

$$G = \mu_\gamma(G) - \bigcup_{i=1}^{\gamma(G)} N[u_i] \cong \mu_\gamma(H) - \bigcup_{j=1}^{\gamma(G)} N[u_j] = H.$$

Next we claim that $f(u_i) \notin V_2$. Suppose $f(u_i) \in V_2$. Since $d_{\mu_\gamma(H)}(f(u_i)) = d_{\mu_\gamma(G)}(u_i) = n$, it follows from the definition 2 that, in $\mu_\gamma(H)$, $\frac{n}{2}$ neighbours of $f(u_i)$ belong to V_2 while another $\frac{n}{2}$ neighbours (the twins) belong to V_2'. (This forces n to be even.) These n neighbours of $f(u_i)$ form an independent subset of $\mu_\gamma(H)$. Then $H' = \mu_\gamma(H) - N_{\mu_\gamma(H)}[f(u_i)] \cong \mu_\gamma(G) - N_{\mu_\gamma(G)}[u_i] = G$. Now , if $x \in V_2$ is adjacent to $f(u_i)$ in $\mu_\gamma(H)$, then x is adjacent to $f(u_i)'$, the twin of $f(u_i)$ belonging to V_2' in $\mu_\gamma(H)$. Further, $d_{H'}(f(u_i)') = 1 = d_G(v)$, where $v \in V_1$ (the vertex set of G) corresponds to $f(u_i)'$ in $\mu_\gamma(H)$, then $d_{\mu_\gamma(G)}(v) = 2$, while $d_{\mu_\gamma(H)}(f(u_i)') = \frac{n}{2}+1 > 2$, as $n \geq 4$. Hence, this case cannot arise.

Finally, suppose $f(u_i) \in V_2'$. Set if $f(u_i) = y'$. Then y, the twin of y' in $\mu_\gamma(H)$, belongs to V_2. As $d_{\mu_\gamma(G)}(u_i) = n, d_{\mu_\gamma(H)}(y') = n$. The vertex y' has $n-1$ neighbours in V_2, say, $x_1, x_2, ..., x_{n-1}$. Then $N_H(y) = \{x_1, x_2, ..., x_{n-1}\}$, and hence y is also adjacent to $x_1', x_2', ..., x_{n-1}'$ in V_2'. Further, as $N_{\mu_\gamma(G)}(u_i)$ is independent, $N_{\mu_\gamma(H)}(y')$ is also independent. Therefore, $H = K_{\gamma(G),n-1}$ consisting of the edges $\{yx_i : 1 \leq i \leq n-1\}$. Moreover, $G = \mu_\gamma(G) - \bigcup\limits_{i=1}^{\gamma(G)} N[u_i] \cong \mu_\gamma(H) - \bigcup N[y'] = K_{\gamma(G),n-1}$ consisting of the edges $\{yx_i' : 1 \leq i \leq n-1\}$. Thus,

$$G \cong K_{\gamma(G),n-1} \cong H.$$

□

Theorem 2.10. *If G is any graph of order n with $\alpha_0(G)$, the point covering number of G, then*

$$\alpha_0(\mu_\gamma(G)) = \begin{cases} n+1 & if \;\; G \cong K_n, \\ 2\alpha_0(G)+\gamma(G) & otherwise. \end{cases}$$

Proof. Suppose $G \cong K_n$. Then it is easy to see that all the members of V together with $u \in D$ forms a minimum cover for $\mu_\gamma(G)$. Thus, $\alpha_0(\mu_\gamma(G)) = n+1$. Suppose $G \ncong K_n$. Let $\alpha_0(G)$ be the point covering number of G and let $S = \{v_1, v_2, ..., v_{\alpha_0(G)}\}$ be the point cover of G with $|S| = \alpha_0(G)$. Then these vertices in $\mu_\gamma(G)$ cover the edges connecting the members of V, and $S' = \{v_1', v_2', ..., v_{\alpha_0(G)}'\}$ consisting the twin vertices of the members of S covers the edges connecting the members of V to the members of V', the remaining edges are covered by $u_i \in D$, for $1 \leq i \leq \gamma(G)$. Thus, $S_\mu = S \cup S' \cup D$ forms a point cover for $\mu_\gamma(G)$. Now, we have to show $|S_\mu| = \alpha_0(\mu_\gamma(G))$. To prove this let S_μ' is a point cover of $\mu_\gamma(G)$ with $|S_\mu'| = \alpha_0(\mu_\gamma(G))$. Here, we need to prove that, $|S_\mu'| = |S_\mu|$. Suppose on contrary assume that $|S_\mu'| \neq |S_\mu|$. Then we have the following cases:

Case 1. If $|S_\mu'| > |S_\mu|$, then S_μ does not cover all the vertices of $\mu_\gamma(G)$. Which is contradiction to the fact that S_μ is a point cover of $\mu_\gamma(G)$.

Case 2. If $\left|S'_\mu\right| < |S_\mu|$, then let us assume that there exists at least one vertex v in S_μ such that $S_\mu - \{v\} = S'_\mu$, the point cover of $\mu_\gamma(G)$. Then we have the following subcases:
Subcase (i). If $v \in S$, then there exists at least one edge $e \in E$ which is not covered by any of the members of $S_\mu - \{v\}$ in $\mu_\gamma(G)$, which is not possible.
Subcase (ii). If $v \in S'$, then there exists at least one edge $e \in E'$ which is not covered by any of the members of $S_\mu - \{v\}$ in $\mu_\gamma(G)$, which is not possible.
Subcase (iii). If $v \in D$, then there exists at least $n - |S'|$ edges which are not covered by any of the members of $S_\mu - \{v\}$ in $\mu_\gamma(G)$, which is not possible. Therefore, our assumption that $\left|S'_\mu\right| \neq |S_\mu|$ is wrong. Thus, $\left|S'_\mu\right| = \alpha_0(\mu_\gamma(G)) = |S_\mu|$. □

Theorem 2.11. *If G is a connected graph of order n with $\beta_0(G)$, the point independence number of G, then*

$$\beta_0(\mu_\gamma(G)) = \begin{cases} n & if\ \ G \cong K_n, \\ 2\beta_0(G) & otherwise. \end{cases}$$

Proof. Suppose $G \cong K_n$. Then it is easy to see that the members of V' forms a maximum point independent set for $\mu_\gamma(G)$. Thus, $\beta_0(\mu_\gamma(G)) = n$. Suppose $G \ncong K_n$. Let $\beta_0(G)$ be the point independence number of G and let $PI = \{v : v \in V\}$ be the point independent set of G with $|PI| = \beta_0(G)$. Then the vertex set $P_\mu = \{v_i : v_i \in V, 1 \leq i \leq \beta_0(G)\} \cup \{v'_i : v'_i \in V', 1 \leq i \leq \beta_0(G)\}$ forms the point independent set of $\mu_\gamma(G)$. Now, we have to show $|P_\mu| = \beta_0(\mu_\gamma(G))$. To prove this let P'_μ is a point independent set of $\mu_\gamma(G)$ with $\left|P'_\mu\right| = \beta_0(\mu_\gamma(G))$. Here, we need to prove that, $\left|P'_\mu\right| = |P_\mu|$. Suppose on contrary assume that $\left|P'_\mu\right| \neq |P_\mu|$. Then we have the following cases:
Case 1. If $\left|P'_\mu\right| < |P_\mu|$, then there exists a vertex v in P_μ such that it is adjacent to at least one member of P'_μ as P'_μ is the maximal point independent set of $\mu_\gamma(G)$. Thus, P_μ is not a point independent set of $\mu_\gamma(G)$. Which is contradiction to the fact that P_μ is a point independent set of $\mu_\gamma(G)$.
Case 2. If $\left|P'_\mu\right| > |P_\mu|$, then for every member $v_i(1 \leq i \leq \beta_0(G))$ of P'_μ there exists at least one vertex v in P_μ such that v and v_i are joined by an edge in $\mu_\gamma(G)$. Which is contradiction to the fact that P'_μ is a point independent set of $\mu_\gamma(G)$. Therefore, our assumption that $\left|P'_\mu\right| \neq |P_\mu|$ is wrong. Thus, $\left|P'_\mu\right| = \beta_0(\mu_\gamma(G)) = |P_\mu|$. □

To prove the following results we consider the set B, the collection of all bipartite graphs and the set $S = \{P_n, C_n, K_n, B : n \geq 4\}$, where P_n, C_n and K_n are a path, a cycle and a complete graph respectively.

Theorem 2.12. *If G is a connected graph of order n with $\alpha_1(G)$, the line covering number of G and $G \notin S$, then*

$$\alpha_1(\mu_\gamma(G)) = \begin{cases} 2\alpha_1(G) + \gamma(G) & \textit{if minimum line cover of } G \textit{ is independent}, \\ 2\alpha_1(G) & \textit{otherwise}. \end{cases}$$

Proof. Suppose minimum line cover of G is not independent. Then let $\alpha_1(G)$ be the line covering number of G and $L = \{e_i : e_i = vw \in E \text{ and } 1 \leq i \leq \alpha_1(G)\}$ be the minimum line cover of G. Clearly, $L_1 = \{e'_i : e'_i = vw', 1 \leq i \leq \alpha_1(G)\}, L_2 = \{e_k : e_k = u_k v_j, 1 \leq k \leq \gamma(G)\}$ and $L_3 = \{e_i : e_i = vw \in E \text{ and } 1 \leq i \leq \alpha_1(G) - \gamma(G)\}$ all together forms a minimum line cover for $\mu_\gamma(G)$ and hence $\alpha_1(\mu_\gamma(G)) = 2\alpha_1(G)$.

Suppose minimum line cover of G is independent and let $L = \{e_i : e_i = vw \in E$ and $1 \leq i \leq \alpha_1(G)\}$ be the line cover of G with $|L| = \alpha_1(G)$. Then the edge sets $L_4 = \{e'_i : e'_i = vw', 1 \leq i \leq \alpha_1(G)\}$ and $L_5 = \{e'_j : e'_j = v'w, 1 \leq j \leq \alpha_1(G)\}$ in $\mu_\gamma(G)$ covers both the members of V and the members of V'. The remaining vertices (i.e., the members of D) are covered by $L_6 = \{e_k : e_k = u_k v_j, 1 \leq k \leq \gamma(G)\}$. Thus, $L_\mu = L_4 \cup L_5 \cup L_6$ forms a line cover for $\mu_\gamma(G)$. Now, we have to show $|L_\mu| = \alpha_1(\mu_\gamma(G))$. To prove this let L'_μ is a line cover of $\mu_\gamma(G)$ with $\left|L'_\mu\right| = \alpha_1(\mu_\gamma(G))$. Here, we need to prove that, $\left|L'_\mu\right| = |L_\mu|$. Suppose on contrary assume that $\left|L'_\mu\right| \neq |L_\mu|$. Then we have the following cases:

Case 1. If $\left|L'_\mu\right| > |L_\mu|$, then L_μ does not cover all the vertices of $\mu_\gamma(G)$. Which is contradiction to the fact that L_μ is a point cover of $\mu_\gamma(G)$.

Case 2. If $\left|L'_\mu\right| < |L_\mu|$, then let us assume that there exists at least one edge e in L_μ such that $L_\mu - \{e\} = L'_\mu$, the line cover of $\mu_\gamma(G)$. Then we have the following subcases:

Subcase (i). If $e \in L_1$ or $e \in L_2$, then there exist at least two vertices $v \in V$ and $v' \in V'$ which are not covered by any of the members of $L_\mu - \{e\}$ in $\mu_\gamma(G)$, which is not possible.

Subcase (ii). If $e \in L_3$, then there exists at least one vertex $u \in D$ which is not covered by any of the members of $L_\mu - \{e\}$ in $\mu_\gamma(G)$, which is not possible. Therefore, our assumption that $\left|L'_\mu\right| \neq |L_\mu|$ is wrong. Thus, $\left|L'_\mu\right| = \alpha_1(\mu_\gamma(G)) = |L_\mu|$.

□

Theorem 2.13. *If G is a connected graph of order n with $\alpha_1(G)$, the line covering number of G and $G \in S \setminus B$, then*

$$\alpha_1(\mu_\gamma(G)) = \begin{cases} n + \left\lceil \frac{\lceil \frac{n}{3} \rceil}{2} \right\rceil & \text{if } G \cong P_n \text{ or } G \cong C_n, \\ n+1 & \text{if } G \cong K_n. \end{cases}$$

Proof. Suppose $G \cong P_n$ or $G \cong C_n$. Then clearly, $L_1 = \{e'_i : e'_i = vw', 1 \leq i \leq n - \lceil \frac{n}{3} \rceil\}$, $L_2 = \{e_k : e_k = u_k v_j, 1 \leq k \leq \gamma(G)\}$ and $L_3 = \left\{e_i : e_i = vw \in E \text{ and } 1 \leq i \leq \left\lceil \frac{\lceil \frac{n}{3} \rceil}{2} \right\rceil\right\}$ altogether form a minimum line cover for $\mu_\gamma(G)$ and hence $\alpha_1(\mu_\gamma(G)) = n + \left\lceil \frac{\lceil \frac{n}{3} \rceil}{2} \right\rceil$. Suppose $G \cong K_n$ and let $L = \{e_i : e_i = vw \in E$ and $1 \leq i \leq \alpha_1(G)\}$ be the line cover of G with $|L| = \alpha_1(G)$. Then the edge sets $L_4 = \{e'_i : e'_i = vw', 1 \leq i \leq \alpha_1(G)\}$ and $L_5 = \{e'_j : e'_j = v'w, 1 \leq j \leq \alpha_1(G)\}$ in $\mu_\gamma(G)$ covers both the members of V and the members of V'. The remaining vertex (i.e., the member of D) is covered by a line $u_k v' \in D'$. Thus, $L_\mu = L_4 \cup L_5 \cup \{u_k v'\}$ forms a minimum line cover for $\mu_\gamma(G)$. □

Theorem 2.14. *If G is a connected graph of order n and $G \in B$, then*

$$\alpha_1(\mu_\gamma(G)) = \begin{cases} 2a + 1 & \text{if } a = b, \\ 2(max\{a, b\}) & \text{otherwise.} \end{cases}$$

Proof. Suppose $G \in B$ and $a = b$. Then clearly, $E'_2 \subset E'$ such that $|E'_2| = 2(a-1)$, $E_3 \subset E$ with $|E_3| = 1$ and $D'_2 \subset D'$ having $|D'_2| = 2$ altogether form a minimum line cover for $\mu_\gamma(G)$ and hence $\alpha_1(\mu_\gamma(G)) = 2a + 1$. Suppose $G \in B$ and $a \neq b$. Then clearly, $E'_1 \subset E'$ such that $|E'_1| = 2(max\{a, b\}) - 3$, $E_2 \subset E$ with $|E_2| = 1$

and $D_1' \subset D'$ having $|D_1'| = 2$ altogether form a minimum line cover for $\mu_\gamma(G)$ and hence $\alpha_1(\mu_\gamma(G)) = 2(max\{a, b\})$. $\square$

The following theorems are immediate from Theorem 1.1, 2.12, 2.13 and 2.14.

Theorem 2.15. *If G is a connected graph of order n with $\beta_1(G)$, the line independence number of G and $G \notin S$, then*

$$\beta_1(\mu_\gamma(G)) = \begin{cases} 2\beta_1(G) & \text{if minimum line cover of } G \text{ is independent,} \\ 2\beta_1(G) + \gamma(G) & \text{otherwise.} \end{cases}$$

Theorem 2.16. *If G is a connected graph of order n with $\beta_1(G)$, the line independence number of G and $G \in S \setminus B$, then*

$$\beta_1(\mu_\gamma(G)) = \begin{cases} n + \lceil \frac{n}{3} \rceil - \left\lceil \frac{\lceil \frac{n}{3} \rceil}{2} \right\rceil & \text{if } G \cong P_n \text{ or } G \cong C_n, \\ n & \text{if } G \cong K_n. \end{cases}$$

Theorem 2.17. *If G is a connected graph of order n and $G \in B$, then*

$$\beta_1(\mu_\gamma(G)) = \begin{cases} 2a + 1 & \text{if } a = b, \\ 2(min\{a, b\} + 1) & \text{otherwise.} \end{cases}$$

Theorem 2.18. *If G has no isolated vertices, then*

$$\kappa(\mu_\gamma(G)) \geq min\{2\kappa(G), \kappa(G) + \gamma(G)\}.$$

Proof. Suppose $V(G) = V$ and $V(\mu_\gamma(G)) = V \cup V' \cup D$, where $V' = \{v' : v \in V\}$ and $D = \{u_i : 1 \leq i \leq \gamma(G)\}$. Let S be a subset of $V(\mu_\gamma(G))$ of size $\kappa(G)$.

If $|V \cap S| < \kappa(G)$, then $G \setminus (V \cap S)$ is connected. Also, for any vertex $v \in V$, v' is adjacent to $\kappa(G)$ vertices of V in $\mu_\gamma(G)$. Therefore, any such vertex v' of $\mu_\gamma(G) \setminus S$ is adjacent to at least one vertex in $G \setminus (V \cap S)$ and also adjacent to u_i (for $1 \leq i \leq \gamma(G)$). Thus, $\mu_\gamma(G) \setminus S$ is connected. If $|V \cap S| = \kappa(G)$, then $S \subseteq V$. Since G has no isolated vertices, any vertex $v \in V \setminus S$ is adjacent to some vertex w' in V', which is in turn adjacent to u_i (for $1 \leq i \leq \gamma(G)$). Thus, $\mu_\gamma(G) \setminus S$ is also connected. Hence, $\kappa(\mu_\gamma(G)) \geq min\{2\kappa(G), \kappa(G) + \gamma(G)\}$. $\square$

Theorem 2.19. *If G has no isolated vertices, then $\kappa(\mu_\gamma(G)) = min\{2\kappa(G), \kappa(G) + \gamma(G)\}$ if and only if $\delta(G) = \kappa(G)$.*

Proof. If $\delta(G) = \kappa(G)$, then we have

$$\begin{aligned} \kappa(\mu_\gamma(G)) &\leq \delta(\mu_\gamma(G)) \\ &= min\{2\delta(G), \delta(G) + \gamma(G)\} \\ &= min\{2\kappa(G), \kappa(G) + \gamma(G)\}. \end{aligned}$$

Further by Theorem 2.18, we have $\kappa(\mu_\gamma(G)) \geq min\{2\kappa(G), \kappa(G) + \gamma(G)\}$. Therefore, $\kappa(\mu_\gamma(G)) = min\{2\kappa(G), \kappa(G) + \gamma(G)\}$.

Conversely, let $\kappa(\mu_\gamma(G)) = min\{2\kappa(G), \kappa(G) + \gamma(G)\} = \eta$. Suppose $\delta(G) \neq \kappa(G)$. Then $1 \leq \kappa(G) < \delta(G)$. Let $S = \{v_1, v_2, ..., v_\eta\}$ be a minimum vertex cut of $\mu_\gamma(G)$. Then we have the following cases:

Case 1. $u_i \notin S$: Suppose $|V \cap S| \geq \kappa(G)$. Then $|V \cap S| = \kappa(G) + i$, $i = 0$ to $\gamma(G)$ and there is a possibility for G to get disconnected. But since $\delta(G) \geq 2$, every vertex in $G \setminus (V \cap S)$ is adjacent to at least two vertices of V' which in turn are adjacent to u_i, for$1 \leq i \leq \gamma(G)$. Hence, even if we remove an additional vertex from V' the resulting graph will remain connected. That is $\mu_\gamma(G) \setminus S$ is connected, a contradiction to the fact that S is a vertex cut. If $(V \cap S) < \kappa(G)$, then $G \setminus (V \cap S)$ is connected and every vertex $v' \in V'$ is adjacent to at least $\kappa(G) + 1$ vertices of G and hence adjacent to at least one vertex of $G \setminus (V \cap S)$. Also, u_i are adjacent to all $v' \in V'$. Then $\mu_\gamma(G) \setminus S$ is connected, which is again contradiction to the fact that S is vertex cut.

Case 2. $u_i \in S$: Now remove u_i from $\mu_\gamma(G)$ and set $G' = \mu_\gamma(G) - U$, where $U = \{u_i\}_{i=1}^{\gamma(G)}$. G' is connected (as $|S| \geq 2$). To disconnect G' we have to remove the remaining $\kappa(G)$ vertices of S. Since $\delta(G) > \kappa(G)$, every vertex in G' is of degree at least $\kappa(G) + 1$.

If $|V \cap (S - U)| < \kappa(G)$, then $G \setminus (V \cap (S - U))$ is connected and every vertex $v' \in V'$ is adjacent to at least $\kappa(G) + 1$ vertices of G and hence to at least one vertex of $G \setminus (V \cap (S - U))$, so that $G' \setminus S$ is connected, a contradiction. If $|V \cap (S - U)| = \kappa(G)$, then there is a possibility for $G \setminus (V \cap (S - U))$ to be disconnected. If $G \setminus (V \cap (S - U))$ is connected, we get a contradiction as in case 1. So, let $G \setminus (V \cap (S - U))$ be disconnected with $G_1, G_2, ..., G_k$ as its components. Since every vertex of $V \cap (S - U)$ is adjacent to all components $G_i, 1 \leq i \leq k$, the twins of $V \cap (S - U)$ are connected and each $v' \in V'$ is adjacent to at least one vertex of $G \setminus (V \cap (S - U))$. Therefore, $\mu_\gamma(G) \setminus S$ is connected, which is again contradiction to the fact that S is a vertex cut. Thus, $\delta(G) = \kappa(G)$. □

Theorem 2.20. *For any graph G,*

$$\gamma(\mu_\gamma(G)) = \begin{cases} \gamma(G) + 1 & if \;\; \gamma(G) = 1, \\ \gamma(G) + 2 & otherwise. \end{cases}$$

Proof. By definition 2, we have G is an induced subgraph of $\mu_\gamma(G)$. Therefore, $\gamma(G)$ vertices dominate all vertices of V in $\mu_\gamma(G)$ and the remaining vertices of $\mu_\gamma(G)$ are dominated either by a member of D if $\gamma(G) = 1$ or by a member of D and a member of V' if $\gamma(G) \geq 2$. □

Theorem 2.21. *For any graph G, $\chi(\mu_\gamma(G)) = \chi(G) + 1$ Where $\chi(G)$ is Chromatic number.*

Proof. By definition 2, there are three types of vertices in $\mu_\gamma(G)$. The members of V receive the same colour as in G. The members of V' receive the same colour that their twin vertices receive in G. The remaining vertices i.e., the members of D receive one colour other than those colours of the vertices in V. Thus, $\chi(\mu_\gamma(G)) = \chi(G) + 1$. □

Theorem 2.22. *If G has no vertex of even degree and $\gamma(G)$ is odd, then $\mu_\gamma(G)$ is eulerian.*

Theorem 2.23. *If the graph G is eulerian of even order and $\gamma(G)$ is even, then $\mu_\gamma(G)$ is eulerian.*

Theorem 2.24. *If G is hamiltonian, then so is $\mu_\gamma(G)$.*

The converse of the Theorem 3.6 is not true always. i.e., If $\mu_\gamma(G)$ is hamiltonian, then G need not be hamiltonian.

An example of non hamiltonian graph whose D-Mycielskian graph is hamiltonian is depicted in Fig. 3, where hamiltonian cycle is shown with dark lines.

Theorem 2.25. [12] *If for all vertices v of G, $d_G(v) \geq \frac{n}{2}$, where $n \geq 3$, then G is hamiltonian.*

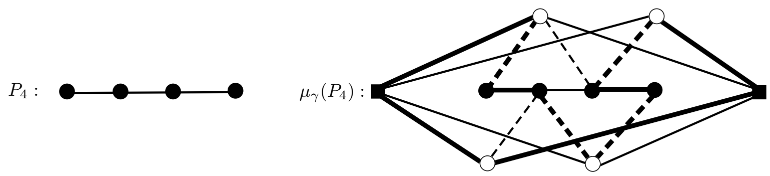

FIGURE 3. A path P_4 and its D-Mycielskian graph.

Theorem 2.26. *If for all vertices v of G, $d_G(v) \geq \frac{n}{2} + \gamma(G)$, where $n \geq 3$, then $\mu_\gamma(G)$ is hamiltonian.*

Proof. The proof follows from Theorem 2.25. □

Theorem 2.27. *If G is k-cyclic graph of order n and $K_{1,2}$ is not an induced subgraph of G having two pendant vertices of G, then $\mu_\gamma(G)$ is hamiltonian.*

An illustrative example of 2-cyclic graph whose D-Mycielskian graph is hamiltonian is depicted in Fig. 4, where hamiltonian cycle is shown with dark lines.

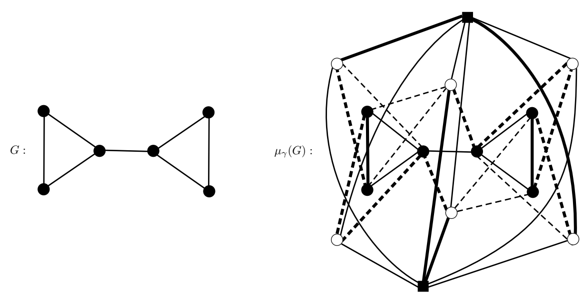

FIGURE 4. A 2-cyclic graph and its D-Mycielskian graph.

3. Graph indices of D-Mycielskian graph of a graph

A graph index is a numerical parameter mathematically derived from the graph structure. It is a graph invariant, thus it does not depend on the labeling or pictorial representation of the graph. The Graph indices play an important role in chemical graph theory. For more details on graph indices refer [19, 20, 33] and references cited there in. It would be interesting that, if all these graph indices are obtained from a single expression. This role is played by polynomials. In fact, there are several graph polynomials like Tutte polynomial [13], matching polynomial [14, 18], Schultz polynomial [17, 27], Zang-Zang polynomial [46], etc., Among them, the Hosoya polynomial [31] is the best and well-known polynomial which plays a vital role in determining distance-based graph indices such as Wiener index [45], hyper Wiener index [9] of graphs. Similarly, M-polynomial which was introduced in 2015 by Deutsch et al., [11] is useful in determining many degree-based graph indices (listed in Table 1 and 2). This motivates us to study M-polynomial of some graph operations and some wheel related graphs. Recently, the study of M-polynomial are reported in [5, 34, 35, 36].

Definition 3. [11] *Let G be a graph. Then M-polynomial of G is defined as*

$$M(G;x,y) = \sum_{i \leq j} m_{ij}(G) x^i y^j, \tag{1}$$

where $m_{ij}, i,j \geq 1$, is the number [23] *of edges uv of G such that $\{d_G(u), d_G(v)\} = \{i,j\}$.*

Table 1. Operations to Derive degree-based graph indices from M-polynomial [11].

Notation	Graph Index	$f(x,y)$	Derivation from $M(G;x,y)$
$M_1(G)$	First Zagreb	$x+y$	$(D_x + D_y)(M(G;x,y))\|_{x=y=1}$
$M_2(G)$	Second Zagreb	xy	$(D_x D_y)(M(G;x,y))\|_{x=y=1}$
${}^mM_2(G)$	Second modified Zagreb	$\frac{1}{xy}$	$(S_x S_y)(M(G;x,y))\|_{x=y=1}$
$S_D(G)$	Symmetric division index	$\frac{x^2+y^2}{xy}$	$(D_x S_y + D_y S_x)(M(G;x,y))\|_{x=y=1}$
$H(G)$	Harmonic	$\frac{2}{x+y}$	$2S_x J(M(G;x,y))\|_{x=1}$
$I_n(G)$	Inverse sum index	$\frac{xy}{x+y}$	$S_x J D_x D_y(M(G;x,y))\|_{x=1}$

where $D_x = x\frac{\partial f(x,y)}{\partial x}$, $D_y = y\frac{\partial f(x,y)}{\partial y}$, $S_x = \int_0^x \frac{f(t,y)}{t}dt$, $S_y = \int_0^y \frac{f(x,t)}{t}dt$ and $J(f(x,y)) = f(x,x)$ are the operators. Two more operators are given in Table 2, to calculate general sum connectivity index and first general Zagreb index.

Table 2. [6] New operators to derive degree-based graph indices from M-polynomial.

Notation	Graph Index	$f(x,y)$	Derivation from $M(G;x,y)$
$\chi_\alpha(G)$	General sum connectivity	$(x+y)^\alpha$	$D_x^\alpha(J(M(G;x,y)))\|_{x=1}$
$M_1^\alpha(G)$	First general Zagreb	$x^\alpha + y^\alpha$	$(D_x^\alpha + D_y^\alpha)(M(G;x,y))\|_{x=y=1}$

Note 3: Hyper Zagreb index is obtained by taking $\alpha = 2$ in general sum connectivity index.

Note 4: Taking $\alpha = 1, 2$, in first general Zagreb index, first Zagreb index and forgotten graph index are obtained respectively.

The first and second Zagreb indices are amongst the oldest and best known graph indices defined in 1972 by Gutman [25] as follows:

$$M_1(G) = \sum_{v \in V(G)} d_G^2(v), \tag{2}$$

$$\text{and } M_2(G) = \sum_{vw \in E(G)} d_G(v) \cdot d_G(w), \text{ respectively.} \tag{3}$$

Ashrafi et al. [2] defined the first and second Zagreb coindices as

$$\overline{M_1}(G) = \sum_{uv \notin E(G)} [d_G(u) + d_G(v)] \quad \textit{and} \quad \overline{M_2}(G) = \sum_{uv \notin E(G)} d_G(u) \cdot d_G(v),$$

respectively.
The vertex-degree-based graph invariant,

$$F(G) = \sum_{v \in V(G)} d_G^3(v) \tag{4}$$

was encountered in [25]. This index is called "forgotten graph index"[15].

Recently, Shirdel et al. [4, 42] introduced a new version of Zagreb index called hyper-Zagreb index, which is defined for a graph G as

$$HM(G) = \sum_{vw \in E(G)} (d_G(v) + d_G(w))^2. \tag{5}$$

Recently, Gutman [22] put forward a new coindex called hyper-Zagreb coindex, which is defined as

$$\overline{HM}(G) = \sum_{vw \notin E(G)} (d_G(v) + d_G(w))^2. \tag{6}$$

Recently, Gutman et al. [24] put forward a new index called sigma index, which is defined as

$$\sigma(G) = \sum_{vw \in E(G)} (d_G(v) - d_G(w))^2. \tag{7}$$

The following results are useful to prove our results.

Theorem 3.1. [24] *If G is any graph, then*

$$\sigma(G) = F(G) - 2M_2(G).$$

Theorem 3.2. [29] *Let G be any graph of order n and size m. Then*

$$M_1(G) + \overline{M_1}(G) = 2m(n-1).$$

Theorem 3.3. [21] *Let G be a graph of order n and size m. Then*

$$\overline{M_2}(G) = 2m^2 - \frac{1}{2}M_1(G) - M_2(G).$$

Theorem 3.4. [22] *Let G be a graph of order n and size m. Then*

$$\overline{HM}(G) = 4m^2 + (n-2)M_1(G) - HM(G).$$

Theorem 3.5. [41] *The Hosoya polynomial satisfies the following conditions:*

(i) *$deg(W(G;q))$ equals the diameter of G.*
(ii) *$[q^o]W(G;q) = 0$.*
(iii) *$[q^1]W(G;q) = |E(G)|$, where $E(G)$ is an edge set of G.*
(iv) *$W(G;1) = \binom{|V(G)|}{2}$, where $V(G)$ is the vertex set of G.*
(v) *$W'(G;1) = W(G)$.*

One of the oldest and most thoroughly studied distance-based graph index is Wiener index [45] and it has numerous chemical applications. In 1947, American physical chemist H. Wiener introduced this index. The Wiener index (or Wiener number) [45] of a graph G, denoted by $W(G)$ is the sum of distances between all (unordered) pairs of vertices of G.

$$W(G) = \sum_{i<j} d_G(v_i, v_j).$$

The Wiener index of a graph belongs to the molecular structure-descriptors called graph indices, which are used for the design of molecules with desired properties [39]. Its mathematical properties are well established. The Wiener polarity index [45] of a graph G, denoted by $W_p(G)$, is equal to the number of unordered pairs of vertices of distance three in G.

$$W_p(G) = |\{(u,v)/d_G(u,v) = 3\}|.$$

In [45], Wiener used a linear formula involving $W(G)$ and $W_p(G)$ to obtain the boiling points t_B of the paraffins, that is

$$t_B = aW(G) + bW_p(G) + c$$

where a, b and c are constants for a given isomeric group.

In 1988, Hosoya [31] introduced a new distance-based graph polynomial called Wiener polynomial. For more details refer [1, 9, 11, 23, 30, 43]. The Wiener polynomial of a connected graph G is denoted by $W(G;q)$ and is defined by,

$$W(G;q) = \sum_{i<j} q^{d_G(v_i,v_j)}$$

where q is a parameter. Nowadays, the majority of researchers uses the name Hosoya polynomial instead of Wiener polynomial. The relation between Wiener polynomial and Wiener index is,

$$W(G) = \frac{d}{dq}(W(G;q))\Big|_{q=1}. \tag{8}$$

Hence, we can derive the expression for the Wiener index of G from that of the Hosoya polynomial of G. We denote the number of unordered pairs of vertices of distance four and more than four in G by $W_{F'}(G)$ (i.e., $W_{F'}(G)) = \big|\{(u,v)/d_G(u,v) \geq 4\}\big|$).

In 1990, Tratch et al. [44] introduced the distance-based index called Tratch-Stankevitch-Zefirov index, denoted by $TSZ(G)$ and is defined as

$$TSZ(G) = \sum_{u,v \in V(G)} \left(\frac{1}{3} d_G(u,v) + \frac{1}{2} d_G^2(u,v) + \frac{1}{6} d_G^3(u,v) \right).$$

In 1993, Plavšić et al. [38] introduced another distance-based index called Harary index [32, 38], which is denoted by $H(G)$ and defined as

$$H_a(G) = \sum_{i<j} \frac{1}{d_G(v_i, v_j)}. \tag{9}$$

In the same year Randić [40] introduced another distance-based index which is generalization of Wiener index and he named it as hyper-Wiener index [10, 40], which is denoted and defined as follows

$$WW(G) = \frac{1}{2} \sum_{u,v \in V(G)} (d_G(u,v) + d_G^2(u,v)).$$

The relationship between Hosoya polynomial and different distane-based indices [8] are given below:

$$\begin{aligned} H_a(G) &= \int_0^1 \frac{W(G:q)}{q}dq, & (10)\\ WW(G) &= \frac{1}{2}\frac{d^2}{dq^2}\left(qW(G:q)\right)\Big|_{q=1}, & (11)\\ TSZ(G) &= \frac{1}{3!}\frac{d^3}{dq^3}\left(q^2W(G:q)\right)\Big|_{q=1}. & (12)\end{aligned}$$

3.1. **Degree-based indices.**

Theorem 3.6. *If G is a graph with n vertices and m edges, then*

$$M_1(\mu_\gamma(G)) = 5M_1(G) + n\gamma(G)(n+\gamma(G)) + 4m\gamma(G).$$

Proof. By using Eq. (2), we have

$$\begin{aligned} M_1(\mu_\gamma(G)) &= \sum_{v\in V(\mu_\gamma(G))} d^2_{\mu_\gamma(G)}(v)\\ &= \sum_{v\in V(G)} 4d_G^2(v) + \sum_{v\in V(G)}(d_G(v)+\gamma(G))^2 + \sum_{i=1}^{\gamma(G)} n^2\\ &= 4M_1(G) + \sum_{v\in V(G)}\left[d_G^2(v) + (\gamma(G))^2 + 2d_G(v)\gamma(G)\right] + n^2\gamma(G)\\ &= 5M_1(G) + n\gamma(G)(n+\gamma(G)) + 4m\gamma(G).\end{aligned}$$

□

Now, using Theorems 3.2 and 3.6, we have the following theorem.

Theorem 3.7. *If G is a graph with n vertices and m edges, then*

$$\overline{M_1}(\mu_\gamma(G)) = 12mn + 2m\gamma(G) - 6m + 3n^2\gamma(G) + n(\gamma(G))^2 - 2n\gamma(G) - 5M_1(G).$$

Theorem 3.8. *If G is a graph with n vertices and m edges, then*

$$M_2(\mu_\gamma(G)) = 8M_2(G) + 2\gamma(G)M_1(G) + 4mn\gamma(G).$$

Proof. By using Eq. (3), we have

$$\begin{aligned} M_2(\mu_\gamma(G)) &= \sum_{vw\in E(\mu_\gamma(G))} d_{\mu_\gamma(G)}(v)\cdot d_{\mu_\gamma(G)}(w)\\ &= \sum_{vw\in E} 4d_G(v)\cdot d_G(w) + \sum_{vx'\in E'} 2d_G(v)(d_G(x)+\gamma(G))\\ &\quad + \sum_{u_ix'\in D'} n(d_G(x)+\gamma(G))\\ &= 4M_2(G) + 2\sum_{vx'\in E'} d_G(v)\cdot d_G(x) + 2\gamma(G)\sum_{vx'\in E'} d_G(v)\\ &\quad + n\sum_{u_ix'\in D'} d_G(x) + (n\gamma(G))^2\\ &= 8M_2(G) + 2\gamma(G)M_1(G) + 4mn\gamma(G).\end{aligned}$$

□

Now, using Theorems 3.3, 3.6 and 3.8, we have the following theorem.

Theorem 3.9. *If G is a graph with n vertices and m edges, then*

$$\begin{aligned}\overline{M_2}(\mu_\gamma(G)) &= 18m^2+2n^2(\gamma(G))^2+8mn\gamma(G)-\left(\frac{5+4\gamma(G)}{2}\right)M_1(G)\\ &\quad -\frac{1}{2}n(\gamma(G))^2-\frac{1}{2}n^2\gamma(G)-2m\gamma(G)-8M_2(G).\end{aligned}$$

Theorem 3.10. *If G is a graph with n vertices and m edges, then*

$$F(\mu_\gamma(G)) = 9F(G)+\gamma(G)(3M_1(G)+n^3)+(\gamma(G))^2(n\gamma(G)+6m).$$

Proof. By using Eq. (4), we have

$$\begin{aligned}F(\mu_\gamma(G)) &= \sum_{v\in V(\mu_\gamma(G))} d^3_{\mu_\gamma(G)}(v)\\ &= \sum_{v\in V(G)} 8d^3_G(v)+\sum_{v\in V(G)}(d_G(v)+\gamma(G))^3+\sum_{i=1}^{\gamma(G)} n^3\\ &= 8F(G)+\sum_{v\in V(G)}\left[d^3_G(v)+(\gamma(G))^3+3d_G(v)\gamma(G)(d_G(v)+\gamma(G))\right]\\ &\quad +n^3\gamma(G)\\ &= 9F(G)+\gamma(G)(3M_1(G)+n^3)+(\gamma(G))^2(n\gamma(G)+6m).\end{aligned}$$

□

Theorem 3.11. *If G is a graph with n vertices and m edges, then*

$$\begin{aligned}HM(\mu_\gamma(G)) &= 4HM(G)+5F(G)+7\gamma(G)M_1(G)+8M_2(G)+2m(3(\gamma(G))^2\\ &\quad +2n\gamma(G))+n\gamma(G)(n+\gamma(G))^2.\end{aligned}$$

Proof. By using Eq. (5), we have

$$\begin{aligned}HM(\mu_\gamma(G)) &= \sum_{vw\in E(\mu_\gamma(G))}(d_{\mu_\gamma(G)}(v)+d_{\mu_\gamma(G)}(w))^2\\ &= \sum_{vw\in E}4(d_G(v)+d_G(w))^2+\sum_{vx'\in E'}(2d_G(v)+d_G(x)+\gamma(G))^2\\ &\quad +\sum_{u_ix'\in D'}(n+d_G(x)+\gamma(G))^2\\ &= 4HM(G)+\sum_{vx'\in E'}\left[4d^2_G(v)+(d_G(x)+\gamma(G))^2+4d_G(v)(d_G(x)+\gamma(G))\right]\\ &\quad +\sum_{u_ix'\in D'}\left[(n+\gamma(G))^2+d^2_G(x)+2(n+\gamma(G))d_G(x)\right]\\ &= 4HM(G)+5F(G)+7\gamma(G)M_1(G)+8M_2(G)+2m(3(\gamma(G))^2\\ &\quad +2n\gamma(G))+n\gamma(G)(n+\gamma(G))^2.\end{aligned}$$

□

Now, using Theorems 3.4, 3.6 and 3.11, we have the following theorem.

Theorem 3.12. *If G is a graph with n vertices and m edges, then*

$$\begin{aligned}\overline{HM}(\mu_\gamma(G)) &= 12m^2(3+\gamma(G))+4(\gamma(G))^2(n^2+m)+2m\gamma(G)(14n\\ &\quad -3\gamma(G)-4)+(15m+5n\gamma(G)-7\gamma(G)-10)M_1(G)\\ &\quad +n\gamma(G)(n+\gamma(G))(3m+n\gamma(G)-\gamma(G)-n-2)\\ &\quad -4HM(G)-5F(G)-8M_2(G).\end{aligned}$$

Theorem 3.13. *If G is a graph with n vertices and m edges, then*

$$\sigma(\mu_\gamma(G)) = 9F(G)-\gamma(G)M_1(G)-16M_2(G)+n\gamma(G)(n^2-8m)+(\gamma(G))^2(6m+n\gamma(G)).$$

Proof. By using Theorems 3.1, 3.8 and 3.10, we can have the desired result. □

Now, we obtain the M-polynomial of D-Mycielskian graph of a graph from which one can obtain the expressions for degree-based graph indices (as listed in Tables 1 and 2) of D-Mycielskian graph of a graph, The following theorem gives the M-polynomial of D-Mycielskian graph of a graph.

Theorem 3.14. *If G is a graph of order n and size m with the M-polynomial $M(G;x,y)=\sum_{i\le j} m_{ij}(G)x^iy^j$, then*

$$M(\mu_\gamma(G);x,y)=\sum_{i\le j} m_{ij}(G)x^{2i}y^{2j}+\sum_{k'\le l'} m_{k'l'}(G)x^{k'}y^{l'},$$

where $k'=min\{k,l\}, l'=max\{k,l\}$, and for $i'=min\{i,j\}, j'=max\{i,j\}$

$$m_{k'l'}(G)=\begin{cases} m_{i'j'}(G) & if\ k=i+\gamma(G), l=2j \quad and\ \ i\neq j,\\ 2m_{i'j'}(G) & if\ k=i+\gamma(G), l=2j \quad and\ \ i=j,\\ \gamma(G)|\{v:d_v=i\}| & if\ k=i+\gamma(G), l=n \quad for\ \ i=1,2,...,n-1.\end{cases}$$

Proof. By definition of D-Mycielskian graph of a graph, we have the degree of the original vertices of G in $\mu_\gamma(G)$ is twice the degree of that vertex in G, the degree $d_{\mu_\gamma(G)}(v_i')=d_G(v_i)+\gamma(G)$ of the duplicates v_i' of $v_i\in V(G)$ and the degree of the vertices $u_i\in D$ is n. Therefore, we have the following:

$$m_{2i2j}(\mu_\gamma(G))=m_{ij}(G)$$

and

$$m_{k'l'}(G)=\begin{cases} m_{i'j'}(G) & if\ k=i+\gamma(G), l=2j \quad and\ \ i\neq j,\\ 2m_{i'j'}(G) & if\ k=i+\gamma(G), l=2j \quad and\ \ i=j,\\ \gamma(G)|\{v:d_v=i\}| & if\ k=i+\gamma(G), l=n \quad for\ \ i=1,2,...,n-1.\end{cases}$$

Thus, we get the desired result by substituting these values in Eq. (1). □

Corollary 3.15. *If M-polynomial of D-Mycielskian of a graph G is*

$$M(\mu_\gamma(G);x,y)=\sum_{i\le j} m_{ij}(G)x^{2i}y^{2j}+\sum_{k'\le l'} m_{k'l'}(G)x^{k'}y^{l'},$$

then

$$\begin{aligned} M_1(\mu_\gamma(G)) &= 2\sum_{i\le j}(i+j)m_{ij}(G)+\sum_{k'\le l'}(k'+l')m_{k'l'}(G),\\ M_2(\mu_\gamma(G)) &= 4\sum_{i\le j} ij m_{ij}(G)+\sum_{k'\le l'} k'l' m_{k'l'}(G),\\ {}^mM_2(\mu_\gamma(G)) &= \frac{1}{4}\sum_{i\le j}\frac{m_{ij}(G)}{ij}+\sum_{k'\le l'}\frac{m_{k'l'}(G)}{k'l'},\end{aligned}$$

$$\begin{aligned} S_D(\mu_\gamma(G)) &= \sum_{i\leq j}\frac{(i^2+j^2)m_{ij}(G)}{ij}+\sum_{k'\leq l'}\frac{(k'^2+l'^2)m_{k'l'}(G)}{k'l'}, \\ H(\mu_\gamma(G)) &= \sum_{i\leq j}\frac{m_{ij}(G)}{(i+j)}+2\sum_{k'\leq l'}\frac{m_{k'l'}(G)}{(k'+l')}, \\ I_n(\mu_\gamma(G)) &= \sum_{i\leq j} ij(i+j)m_{ij}(G)+\sum_{k'\leq l'}k'l'(k'+l')m_{k'l'}(G). \end{aligned}$$

Proof. We get the desired results by applying the appropriate operators from Tables 1 and 2 to M-polynomial of $\mu_\gamma(G)$. □

3.2. **Distance-based indices.**

Theorem 3.16. *If G is any graph without isolated vertices, then*

$$\begin{aligned} W(\mu_\gamma(G);q) &= W_{F'}(G)q^4+(3W_p(G)+2W_{F'}(G))q^3 \\ &\quad +\left[\binom{2n+\gamma(G)}{2}-3W_{F'}(G)-3W_p(G)-3m-n\gamma(G)\right]q^2 \\ &\quad +(3m+n\gamma(G))q \end{aligned}$$

and

$$W(\mu_\gamma(G)) = 4W_{F'}(G)+3W_p(G)-3m-n\gamma(G)+2\binom{2n+\gamma(G)}{2}.$$

Proof. Let G be a graph of order n and size m without isolated vertices. Therefore, from the definition of Hosoya polynomial,

$$W(\mu_\gamma(G);q) = \sum_{u,v\in V(\mu_\gamma(G))} q^{d_{\mu_\gamma(G)}(u,v)}.$$

Since $diam(\mu_\gamma(G)) \leq 4$ for any graph G without isolated vertices. Therefore, from Theorem 3.5, the highest power of Hosoya polynomial $\mu_\gamma(G)$ is equal to the diameter of $\mu_\gamma(G)$. Let $A_i(G) = |\{(u,v)/d_G(u,v)=i\}|$. Thus, the expected Hosoya polynomial for $\mu_\gamma(G)$ is

$$W(\mu_\gamma(G);q) = \sum_{i=1}^{4} A_i(\mu_\gamma(G))q^i.$$

By definition of $A_i(G)$, we have
$A_1(\mu_\gamma(G)) = 3m+n\gamma(G)$, $A_4(\mu_\gamma(G)) = W_{F'}(G)$, $A_3(\mu_\gamma(G)) = 3W_p(G)+2W_{F'}(G)$ and
$A_2(\mu_\gamma(G)) = \binom{2n+\gamma(G)}{2}-3W_{F'}(G)-3W_p(G)-3m-n\gamma(G)$.
Therefore,

$$\begin{aligned} W(\mu_\gamma(G);q) &= W_{F'}(G)q^4+(3W_p(G)+2W_{F'}(G))q^3 \\ &\quad +\left[\binom{2n+\gamma(G)}{2}-3W_{F'}(G)-3W_p(G)-3m-n\gamma(G)\right]q^2 \\ &\quad +(3m+n\gamma(G))q. \end{aligned}$$

TABLE 3.

Graph (G)	$W_p(\mu_\gamma(G))$	$W_{F'}(\mu_\gamma(G))$
Path ($P_n, n \geq 4$)	n^2-4n+3	$\frac{n^2-7n+12}{2}$
Cycle ($C_n, n \geq 7$)	$n(n-4)$	$\frac{n^2-7n}{2}$
Complete (K_n)	0	0
Complete bipartite ($K_{a,b}, 1 < a < b$)	0	0
Star ($K_{1,n}$)	0	0
Wheel (W_n)	0	0

Now, from Eq. (8), the Wiener index for $\mu_\gamma(G)$ is

$$\begin{aligned} W(\mu_\gamma(G)) &= \frac{d}{dq}(W(\mu_\gamma(G);q))\Big|_{q=1} \\ &= 4W_{F'}(G) + 3W_p(G) - 3m - n\gamma(G) + 2\binom{2n+\gamma(G)}{2}. \end{aligned}$$

□

Using Theorem 3.16 and Table 3, we obtain the following corollaries.

Corollary 3.17. *If $P_n, n \geq 4$ is a path of order n, then*

$$\begin{aligned} W(\mu_\gamma(P_n);q) &= \left(\frac{n^2-7n+12}{2}\right)q^4 + (n^2-4n+3)q^3 \\ &+ \left[\binom{2n+\lceil\frac{n}{3}\rceil}{2} - n\left\lceil\frac{n}{3}\right\rceil - \left(\frac{3n^2-9n+12}{2}\right)\right]q^2 \\ &+ \left(3(n-1) + n\left\lceil\frac{n}{3}\right\rceil\right)q \end{aligned}$$

and

$$W(\mu_\gamma(P_n)) = \left(2n + \left\lceil\frac{n}{3}\right\rceil\right) + \left(2n + \left\lceil\frac{n}{3}\right\rceil - 1\right) - n\left\lceil\frac{n}{3}\right\rceil + 2n^2 - 14n + 18.$$

Corollary 3.18. *If $C_n, n \geq 7$ is a cycle of order n, then*

$$\begin{aligned} W(\mu_\gamma(C_n);q) &= \left(\frac{n^2-7n}{2}\right)q^4 + n(n-4)q^3 \\ &+ \left[\binom{2n+\lceil\frac{n}{3}\rceil}{2} - n\left\lceil\frac{n}{3}\right\rceil - \frac{3}{2}(n^2-3n)\right]q^2 + \left(3n + n\left\lceil\frac{n}{3}\right\rceil\right)q \end{aligned}$$

and

$$W(\mu_\gamma(C_n)) = 2\binom{2n+\lceil\frac{n}{3}\rceil}{2} + -n\left\lceil\frac{n}{3}\right\rceil + 2n^2 - 14n.$$

Corollary 3.19. *If K_n is a complete graph of order n, then*

$$W(\mu_\gamma(K_n);q) = \left(\frac{n^2+3n}{2}\right)q^2 + \left(\frac{3n^2-n}{2}\right)q$$

and

$$W(\mu_\gamma(K_n)) = \frac{5n(n+1)}{2}.$$

Corollary 3.20. *If $K_{1,n}$ is a star graph of order $n+1$, then*

$$W(\mu_\gamma(K_{1,n}); q) = (2n^2+n+2)q^2 + (4n+1)q$$

and

$$W(\mu_\gamma(K_{1,n})) = 4n^2+6n+5.$$

Corollary 3.21. *If $K_{a,b}, (1 < a < b)$ is a complete bipartite graph of order $a+b$, then*

$$W(\mu_\gamma(K_{a,b}); q) = (2a^2+2b^2+ab+a+b+1)q^2 + (3ab+2a+2b)q$$

and

$$W(\mu_\gamma(K_{a,b})) = 4(a^2+b^2+a+b)+5ab+2.$$

Corollary 3.22. *If $W_n, n \geq 3$ is a wheel of order $n+1$, then*

$$W(\mu_\gamma(W_n); q) = (2n^2-2n+2)q^2 + (7n+1)q$$

and

$$W(\mu_\gamma(W_n)) = 4n^2+3n+5.$$

Using the relation given in Eq. (10), we obtain the following theorem.

Theorem 3.23. *If G is any graph without isolated vertices, then*

$$H_a(\mu_\gamma(G)) = \frac{1}{2}\binom{2n+\gamma(G)}{2} + \frac{3}{2}m + \frac{n}{2}\gamma(G) - \frac{7}{12}W_{F'}(G) - \frac{1}{2}W_p(G).$$

Using Theorem 3.23 and Table 3, we obtain the following corollaries.

Corollary 3.24. *If $P_n, n \geq 4$ is a path of order n, then*

$$H_a(\mu_\gamma(P_n)) = \frac{1}{2}\binom{2n+\lceil\frac{n}{3}\rceil}{2} + \frac{1}{2}n\left\lceil\frac{n}{3}\right\rceil - \left(\frac{7n^2-73n+84}{24}\right).$$

Corollary 3.25. *If $C_n, n \geq 7$ is a cycle of order n, then*

$$H_a(\mu_\gamma(C_n)) = \frac{1}{2}\binom{2n+\lceil\frac{n}{3}\rceil}{2} + \frac{1}{2}n\left\lceil\frac{n}{3}\right\rceil - \left(\frac{7n^2-73n}{24}\right).$$

Corollary 3.26. *If K_n is a complete graph of order n, then*

$$H_a(\mu_\gamma(K_n)) = \frac{7n^2+n}{4}.$$

Corollary 3.27. *If $K_{1,n}$ is a star graph of order $n+1$, then*

$$H_a(\mu_\gamma(K_{1,n})) = \frac{2n^2+9n+4}{2}.$$

Corollary 3.28. *If $K_{a,b}, (1 < a < b)$ is a complete bipartite graph of order $a+b$, then*

$$H_a(\mu_\gamma(K_{a,b})) = \frac{2a^2+2b^2+7ab+5a+5b+1}{2}.$$

Corollary 3.29. *If $W_n, n \geq 3$ is a wheel of order $n+1$, then*

$$H_a(\mu_\gamma(W_n)) = n^2+6n+2.$$

Using the relation given in Eq. (11), we obtain the following theorem.

Theorem 3.30. *If G is any graph without isolated vertices, then*

$$WW(\mu_\gamma(G)) = 13W_{F'}(G) + 9W_p(G) + 3\binom{2n+\gamma(G)}{2} - 2n\gamma(G) - 6m.$$

Using the relation given in Eq. (12), we obtain the following theorem.

Theorem 3.31. *If G is any graph without isolated vertices, then*

$$TSZ(\mu_\gamma(G)) = 28W_{F'}(G) + 18W_p(G) + 4\binom{2n+\gamma(G)}{2} - 3n\gamma(G) - 9m.$$

One can easily obtain the expressions for the hyper-Wiener index and Tratch-Stankevitch-Zefirov index of the above mentioned standard graph families using Theorems 3.30 and 3.31 and Table 3, respectively.

4. Conclusion

In this paper, we have introduced a new transformation graph called D-Mycielskian graph of a graph which is triangle-free with large chromatic number. The basic properties of this new graph are investigated. In addition, we have obtained M-polynomial and Hosoya polynomial of D-Mycielskian graph of a graph and derived the expressions for both degree-based and distance-based graph indices. Now, we conclude with the following open problems:

Problem 1. Necessary and sufficient conditions for $\mu_\gamma(G)$ to be eulerian.

Problem 2. Necessary and sufficient conditions for hamiltonicity of $\mu_\gamma(G)$.

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