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SOME PROPERTIES OF THE SUBCLASSES OF JANOWSKI TYPE ALPHA-QUASI-CONVEX FUNCTIONS

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Abstract. The present investigation is concerned with certain generalized subclasses of Janowski type alpha-quasi-convex functions in the open unit disc $E = \{z : |z| < 1\}$. We discuss some geometric properties such as the coefficient estimates, distortion theorems, growth theorems and radius of quasi convexity for these classes. The results proved earlier by various authors will follow as special cases.

1. Introduction

Let A denote the class of functions f, analytic in the unit disc $E = \{z : |z| < 1\}$, normalized by $f(0) = f'(0) - 1 = 0$ and with the Taylor series expansion of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

By U, we denote the class of Schwarzian functions of the form

$$w(z) = \sum_{k=1}^{\infty} c_k z^k,$$

which are analytic in the unit disc E and satisfying the conditions

$$w(0) = 0, |w(z)| < 1.$$

Let the functions f and g be analytic in E. Then we say that f is subordinate to g in E, if a Schwarzian function $w(z) \in U$ can be found such that $f(z) = g(w(z))$, denoted by $f \prec g$. This result is known as principle of subordination.
If g is univalent in E, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

We denote by S, S^* and K, the classes of functions $f \in A$, which are respectively univalent, starlike and convex in the unit disc E. Clearly $f(z) \in K$ if and only if $zf'(z) \in S^*$.

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Kaplan [5], introduced the concept of close-to-convex functions. A function $f \in A$ is said to be close-to-convex if there exists a convex function $h \in K$ such that

$$Re\left(\frac{f'(z)}{h'(z)}\right) > 0, z \in E.$$

The class of close-to-convex functions is denoted by C.

Subsequently, Noor [8] introduced the class of quasi-convex functions as

$$C^* = \left\{f : f \in A, Re\left(\frac{(zf'(z))'}{h'(z)}\right) > 0, h \in K, z \in E\right\}.$$

Note that every quasi-convex function is convex and so univalent. It is obvious that $f(z) \in C^*$ if and only if $zf'(z) \in C$. Also $K \subset C^* \subset C \subset S$.

Mocanu [7], established the class $M_\alpha(0 \leq \alpha \leq 1)$ of alpha-convex functions $f \in A$ with $f(z)f'(z) \neq 0$ and satisfying the condition

$$Re\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\right\} > 0, z \in E.$$

Obviously $M_0 \equiv S^*$ and $M_1 \equiv K$. It was shown in [6], that all alpha-convex functions are univalent and the class M_α unify the classes S^* and K.

Following the concept of alpha-convex functions, Noor [9] introduced the class $Q_\alpha(0 \leq \alpha \leq 1)$ of alpha-quasi-convex functions $f \in A$ and satisfying the condition

$$Re\left\{(1-\alpha)\frac{f'(z)}{h'(z)} + \alpha\frac{(zf'(z))'}{h'(z)}\right\} > 0, h \in K, z \in E.$$

It is obvious that $Q_0 \equiv C$ and $Q_1 \equiv C^*$. Q_α is a linear combination of the classes C of close-to-convex functions and C^* of quasi-convex functions.

The class $P[A, B]$ consists of the functions $p(z)$ analytic in E with $p(0) = 1$ and subordinate to $\dfrac{1+Az}{1+Bz}, (-1 \leq B < A \leq 1)$. This class was established by Janowski [4] and so the functions in the class $P[A, B]$ are also known as Janowski-type functions.

Various subclasses of close-to-convex functions and quasi-convex functions such as $C^*(\alpha, \beta)$, $C_s^*(\alpha, \beta)$, $C^*(A, B)$, $C_s^*(A, B)$, $C^*(A, B; C, D)$, $C_s^*(A, B; C, D)$ were studied respectively by Selvaraj and Stelin [13], Selvaraj et al. [14], Xiong and Liu [21], Singh and Singh [16]. Also certain subclasses of alpha-quasi-convex functions such as $Q_\alpha(1-2\beta, -1; 1-2\gamma, -1)$, $C_*(\beta, \gamma)$, $Q_\alpha(A, B)$, $C_\lambda^*(\alpha, \beta)$ were investigated respectively by Noor [9], Noor and Al Aboudi [11], Selvaraj and Thirupathi [15] and Selvaraj and Logu [12]. Recently some interesting subclasses of analytic functions were also studied in [18, 19, 20].

To avoid repetition, it is laid down once for all that

$$-1 \leq D \leq B < A \leq C \leq 1, z \in E.$$

Motivated by the work of the above mentioned authors, we introduce the following subclasses of Janowski type alpha-quasi-convex functions:

Definition 1 $Q_\alpha(A,B;C,D)$ be the class of functions $f \in A$ of the form (1) which satisfy the condition

$$(1-\alpha)\frac{f'(z)}{h'(z)} + \alpha\frac{(zf'(z))'}{h'(z)} \prec \frac{1+Cz}{1+Dz},$$

where $h(z) = z + \sum_{k=2}^{\infty} b_k z^k \in K(A,B)$.
The following observations are obvious:
(i) $Q_\alpha(1,-1;1,-1) \equiv Q_\alpha$, the class introduced by Noor [9].
(ii) For $A = 1-2\beta$, $B = -1$, $C = 1-2\gamma$ and $D = -1$, the class $Q_\alpha(A,B;C,D)$ reduces to $Q_\alpha(1-2\beta,-1;1-2\gamma,-1)$, the class introduced by Noor [9].
(iii) $Q_\alpha(1,-1;C,D) \equiv Q_\alpha(C,D)$, the class studied by Selvaraj and Thirupathi [15].
(iv) $Q_1(A,B;C,D) \equiv C^*(A,B;C,D)$, the class introduced and studied by Singh and Singh [16].
(v) $Q_1(1,-1;(2\alpha-1)\beta,\beta) \equiv C^*(\alpha,\beta)$, the class studied by Selvaraj and Stelin [13].
(vi) $Q_1(1,-1;C,D) \equiv C^*(C,D)$, the class studied by Xiong and Liu [21].

Definition 2 Let $Q^*_\alpha(A,B;C,D)$ denote the class of functions $f \in A$ of the form (1) and satisfying the condition that

$$(1-\alpha)\frac{f'(z)}{g'(z)} + \alpha\frac{(zf'(z))'}{g'(z)} \prec \frac{1+Cz}{1+Dz},$$

where $g(z) = z + \sum_{k=2}^{\infty} d_k z^k \in S^*(A,B)$.
We have the following observations:
(i) $Q^*_\lambda(1,-1;(2\alpha-1)\beta,\beta) \equiv C^*_\lambda(\alpha,\beta)$, the class studied by Selvaraj and Logu [12].
(ii) $Q^*_1(A,B;C,D) \equiv C^*_s(A,B;C,D)$, the class introduced and studied by Singh and Singh [16].
(iii) $Q^*_1(A,B;1-2\beta,-1) \equiv C^*_\beta(A,B)$, the class introduced by Noor [10].

The paper is concerned with the study of some geometric properties of the classes $Q_\alpha(A,B;C,D)$ and $Q^*_\alpha(A,B;C,D)$. We obtain the coefficient estimates, distortion theorems, growth theorems and radius of quasi convexity for the functions in these classes. By giving the particular values to the parameters A,B,C,D and α, the results of several earlier works follows as special cases.

2. Preliminary Results

Lemma 1 [3] If $P(z) = \dfrac{1+Cw(z)}{1+Dw(z)} = 1 + \sum_{k=1}^{\infty} p_k z^k$, then

$$|p_n| \leq (C-D), n \geq 1.$$

Lemma 2 [2] If $g(z) \in S^*(A,B)$, then for $A-(n-1)B \geq (n-2), n \geq 3$,

$$|d_n| \leq \frac{1}{(n-1)!}\prod_{j=2}^{n}(A-(j-1)B).$$

Lemma 3 [2] If $g(z) \in S^*(A,B)$, then for $|z| = r < 1$,

$$r(1-Br)^{\frac{A-B}{B}} \leq |g(z)| \leq r(1+Br)^{\frac{A-B}{B}}, B \neq 0;$$

$$re^{-Ar} \leq |g(z)| \leq re^{Ar}, B = 0.$$

Lemma 4 [17] If $h(z) \in K(A,B)$, then for $A-(n-1)B \geq (n-2), n \geq 3$,

$$|b_n| \leq \frac{1}{n!}\prod_{j=2}^{n}(A-(j-1)B).$$

Lemma 5 [17] If $h(z) \in K(A,B)$, then for $|z| = r < 1$,

$$\frac{1}{A}\left[1-(1-Br)^{\frac{A}{B}}\right] \leq |h(z)| \leq \frac{1}{A}\left[(1+Br)^{\frac{A}{B}}-1\right], B \neq 0;$$

$$\frac{1}{A}\left[1-e^{-Ar}\right] \leq |h(z)| \leq \frac{1}{A}\left[e^{Ar}-1\right], B = 0.$$

Lemma 6 [1] If $P(z) = \dfrac{1+Cw(z)}{1+Dw(z)}, -1 \leq D < C \leq 1, w(z) \in U$,
then for $|z| = r < 1$, we have

$$Re\frac{zP'(z)}{P(z)} \geq \begin{cases} -\dfrac{(C-D)r}{(1-Cr)(1-Dr)}, & \text{if } R_1 \leq R_2, \\ 2\dfrac{\sqrt{(1-D)(1-C)(1+Cr^2)(1+Dr^2)}-(1-CDr^2)}{(C-D)(1-r^2)} & \\ +\dfrac{C+D}{C-D}, & \text{if } R_1 \geq R_2, \end{cases}$$

where $R_1 = \sqrt{\dfrac{(1-C)(1+Cr^2)}{(1-D)(1+Dr^2)}}$ and $R_2 = \dfrac{1-Cr}{1-Dr}$.

3. The class $Q_\alpha(A,B;C,D)$

Theorem 1 Let $f(z) \in Q_\alpha(A,B;C,D)$, then for $A-(n-1)B \geq (n-2), n \geq 2$,

$$\begin{aligned} |a_n| \leq \frac{1}{[(1-\alpha)n+\alpha n^2]}\Bigg\{&\frac{1}{(n-1)!}\prod_{j=2}^{n}(A-(j-1)B) \\ &+(C-D)\left[1+\sum_{k=2}^{n-1}\frac{1}{(k-1)!}\prod_{j=2}^{k}(A-(j-1)B)\right]\Bigg\}. \end{aligned} \tag{2}$$

The bounds are sharp.

Proof. In Definition 1, using Principle of subordination, we have

$$(1-\alpha)f'(z)+\alpha(zf'(z))' = h'(z)\left(\frac{1+Cw(z)}{1+Dw(z)}\right), w(z) \in U. \tag{3}$$

On expanding (3), it yields

$$(1-\alpha)[1+2a_2z+3a_3z^2+...+na_nz^{n-1}+...]+\alpha[1+4a_2z+9a_3z^2+...+n^2a_nz^{n-1}+...]$$
$$= (1+2b_2z+3b_3z^2+...+nb_nz^{n-1}+...)(1+p_1z+p_2z^2+...+p_{n-1}z^{n-1}+...). \tag{4}$$

Equating the coefficients of z^{n-1} in (4), we have

$$[(1-\alpha)n+\alpha n^2]a_n = nb_n+(n-1)p_1b_{n-1}+(n-2)p_2b_{n-2}...+2p_{n-2}b_2+p_{n-1}. \tag{5}$$

Applying triangle inequality and using Lemma 1 in (5), it gives

$$[(1-\alpha)n+\alpha n^2]|a_n| \leq n|b_n|+(C-D)\left[(n-1)|b_{n-1}|+(n-2)|b_{n-2}|...+2|b_2|+1\right]. \tag{6}$$

Using Lemma 4 in (6), the result (2) is obvious.
For $n=2$, equality sign in (2) hold for the functions $f_n(z)$ defined as

$$(1-\alpha)f_n'(z)+\alpha(zf_n'(z))' = (1+B\delta_1 z)^{\frac{(A-B)}{B}}\left(\frac{1+C\delta_2 z^n}{1+D\delta_2 z^n}\right), B\neq 0, |\delta_1|=1, |\delta_2|=1. \tag{7}$$

Remark 1
(i) For $A=1, B=-1, C=1, D=-1$, Theorem 1 gives the result due to Noor [9].
(ii) For $A=1-2\beta$, $B=-1$, $C=1-2\gamma$ and $D=-1$, Theorem 1 agrees with the result due to Noor [9].
(iii) On putting $A=1$, $B=-1$ in Theorem 1, we can obtain the result proved by Selvaraj and Thirupathi [15].
(iv) For $\alpha=1$, Theorem 1 gives the result proved by Singh and Singh [16].
(v) On putting $\alpha=1, A=1, B=-1, C=(2\alpha-1)\beta, D=\beta$, Theorem 1 agrees with the result due to Selvaraj and Stelin [13].
(vi) For $\alpha=1, A=1, B=-1$, Theorem 1 gives the result proved by Xiong and Liu [21].

Theorem 2 If $f(z)\in Q_\alpha(A,B;C,D)$, then for $|z|=r, 0<r<1$, we have
for $\alpha=0, B\neq 0$,

$$\int_0^r\left(\frac{1-Ct}{1-Dt}\right)(1-Bt)^{\frac{A-B}{B}}dt\leq|f(z)|\leq\int_0^r\left(\frac{1+Ct}{1+Dt}\right)(1+Bt)^{\frac{A-B}{B}}dt; \tag{8}$$

for $\alpha=0, B=0$,

$$\int_0^r\frac{1}{A}\left(\frac{1-Ct}{1-Dt}\right)(1+Ae^{-At})dt\leq|f(z)|\leq\int_0^r\frac{1}{A}\left(\frac{1+Ct}{1+Dt}\right)(Ae^{At}-1)dt, \tag{9}$$

and for $0<\alpha\leq 1, B\neq 0$,

$$\frac{1}{\alpha}\int_0^r\left[\frac{1}{s}\int_0^s\left(\frac{1-Ct}{1-Dt}\right)(1-Bt)^{\frac{A-B}{B}}dt\right]ds\leq|f(z)|$$
$$\leq\frac{1}{\alpha}\int_0^r\left[\frac{1}{s}\int_0^s\left(\frac{1+Ct}{1+Dt}\right)(1+Bt)^{\frac{A-B}{B}}dt\right]ds; \tag{10}$$

for $0<\alpha\leq 1, B=0$,

$$\frac{1}{A\alpha}\int_0^r\left[\frac{1}{s}\int_0^s\left(\frac{1-Ct}{1-Dt}\right)(1+Ae^{-At})dt\right]ds\leq|f(z)|$$
$$\leq\frac{1}{A\alpha}\int_0^r\left[\frac{1}{s}\int_0^s\left(\frac{1+Ct}{1+Dt}\right)(Ae^{At}-1)dt\right]ds. \tag{11}$$

Estimates are sharp.
Proof. From (3), we have

$$|(1-\alpha)f'(z)+\alpha(zf'(z))'| = |h'(z)|\left|\frac{1+Cw(z)}{1+Dw(z)}\right|, w(z)\in U. \tag{12}$$

It is easy to show that the transformation

$$\frac{(zf'(z))'}{h'(z)}=\frac{1+Cw(z)}{1+Dw(z)}$$

maps $|w(z)|\leq r$ onto the circle

$$\left|\frac{(zf'(z))'}{h'(z)}-\frac{1-CDr^2}{1-D^2r^2}\right|\leq\frac{(C-D)r}{(1-D^2r^2)}, |z|=r.$$

This implies that

$$\frac{1-Cr}{1-Dr} \leq \left|\frac{1+Cw(z)}{1+Dw(z)}\right| \leq \frac{1+Cr}{1+Dr}. \tag{13}$$

Let $F'(z) = (1-\alpha)f'(z) + \alpha(zf'(z))'$.
As $h(z) \in K(A, B)$, so from Lemma 5, we have

$$\begin{cases} (1-Br)^{\frac{A-B}{B}} \leq |h'(z)| \leq (1+Br)^{\frac{A-B}{B}}, & \text{if } B \neq 0; \\ \dfrac{1}{A}\left[1+Ae^{-Ar}\right] \leq |h'(z)| \leq \dfrac{1}{A}\left[Ae^{Ar}-1\right], & \text{if } B = 0. \end{cases} \tag{14}$$

Using (13) and (14) in (12), it yields

$$\begin{cases} \left(\dfrac{1-Cr}{1-Dr}\right)(1-Br)^{\frac{A-B}{B}} \leq |F'(z)| \leq \left(\dfrac{1+Cr}{1+Dr}\right)(1+Br)^{\frac{A-B}{B}}, & \text{if } B \neq 0; \\ \left(\dfrac{1-Cr}{1-Dr}\right)\dfrac{1}{A}\left[1+Ae^{-Ar}\right] \leq |F'(z)| \leq \left(\dfrac{1+Cr}{1+Dr}\right)\dfrac{1}{A}\left[Ae^{Ar}-1\right], & \text{if } B = 0. \end{cases} \tag{15}$$

On integrating, (15) yields

$$\begin{cases} \int_0^r \left(\dfrac{1-Ct}{1-Dt}\right)(1-Bt)^{\frac{A-B}{B}}dt \leq |F(z)| \leq \int_0^r \left(\dfrac{1+Ct}{1+Dt}\right)(1+Bt)^{\frac{A-B}{B}}dt, & \text{if } B \neq 0; \\ \int_0^r \left(\dfrac{1-Ct}{1-Dt}\right)\dfrac{1}{A}\left[1+Ae^{-At}\right]dt \leq |F(z)| \leq \int_0^r \left(\dfrac{1+Ct}{1+Dt}\right)\dfrac{1}{A}\left[Ae^{At}-1\right]dt, & \text{if } B = 0. \end{cases} \tag{16}$$

This implies

$$\begin{cases} \int_0^r \left(\dfrac{1-Ct}{1-Dt}\right)(1-Bt)^{\frac{A-B}{B}}dt \leq |(1-\alpha)f(z) + \alpha zf'(z)| & \\ \leq \int_0^r \left(\dfrac{1+Ct}{1+Dt}\right)(1+Bt)^{\frac{A-B}{B}}dt, & \text{if } B \neq 0; \\ \int_0^r \left(\dfrac{1-Ct}{1-Dt}\right)\dfrac{1}{A}\left[1+Ae^{-At}\right]dt \leq |(1-\alpha)f(z) + \alpha zf'(z)| & \\ \leq \int_0^r \left(\dfrac{1+Ct}{1+Dt}\right)\dfrac{1}{A}\left[Ae^{At}-1\right]dt, & \text{if } B = 0. \end{cases} \tag{17}$$

For $\alpha = 0$, the resluts (8) and (9) are obvious from (17).
Also for $0 < \alpha \leq 1$ and on integrating (17), the results (10) and (11) are obvious.
Sharpness follows for the function $f_n(z)$ defined in (7).

Remark 2
(i) On putting $A = 1$, $B = -1$ in Theorem 1, we can obtain the result proved by Selvaraj and Thirupathi [15].
(ii) For $\alpha = 1$, Theorem 1 gives the result proved by Singh and Singh [16].
(iii) On putting $\alpha = 1, A = 1, B = -1, C = (2\alpha - 1)\beta, D = \beta$, Theorem 1 agrees with the result due to Selvaraj and Stelin [13].
(iv) For $\alpha = 1, A = 1, B = -1$, Theorem 1 gives the result proved by Xiong and Liu [21].

Theorem 3 Let $F'(z)=(1-\alpha)f'(z)+\alpha(zf'(z))'$, where $f(z)\in Q_\alpha(A,B;C,D)$, then

$$Re\frac{(zF'(z))'}{F'(z)}\geq\begin{cases}\dfrac{1-Ar}{1-Br}-\dfrac{(C-D)r}{(1-Cr)(1-Dr)}, & \text{if } R_1\leq R_2,\\ \dfrac{1-Ar}{1-Br}+\dfrac{C+D}{C-D} & \\ +2\dfrac{\sqrt{(1-D)(1-C)(1+Cr^2)(1+Dr^2)}-(1-CDr^2)}{(C-D)(1-r^2)}, & \text{if } R_1\geq R_2,\end{cases} \tag{18}$$

where R_1 and R_2 are defined in Lemma 6.
Proof. As $f(z)\in Q_\alpha(A,B;C,D)$, we have

$$(1-\alpha)f'(z)+\alpha(zf'(z))'=h'(z)\left(\frac{1+Cw(z)}{1+Dw(z)}\right)=h'(z)P(z).$$

Here $F'(z)=(1-\alpha)f'(z)+\alpha(zf'(z))'$. So on differentiating it logarithmically, we get

$$\frac{(zF'(z))'}{F'(z)}=\frac{(zh'(z))'}{h'(z)}+\frac{zP'(z)}{P(z)}. \tag{19}$$

Now for $h\in K(A,B)$, we have

$$Re\left(\frac{(zh'(z))'}{h'(z)}\right)\geq\frac{1-Ar}{1-Br}. \tag{20}$$

So using Lemma 6 and inequality (20) in equation (19), the result (18) is obvious. Sharpness follows if we take the function $f_n(z)$ to be same as in (7).

Remark 3
(i) For $\alpha=1$, Theorem 1 gives the result proved by Singh and Singh [16].
(ii) On putting $\alpha=1, A=1, B=-1, C=(2\alpha-1)\beta, D=\beta$, Theorem 1 agrees with the result due to Selvaraj and Stelin [13].
(iii) For $\alpha=1, A=1, B=-1$, Theorem 1 gives the result proved by Xiong and Liu [21].

4. The class $Q^*_\alpha(A,B;C,D)$

Theorem 4 Let $f(z)\in Q^*_\alpha(A,B;C,D)$, then for $A-(n-1)B\geq(n-2), n\geq 2$,

$$\begin{aligned}|a_n|\leq\frac{1}{[(1-\alpha)n+\alpha n^2]}&\left\{\frac{n}{(n-1)!)}\prod_{j=2}^{n}(A-(j-1)B)\right.\\ &\left.+(C-D)\left[1+\sum_{k=2}^{n-1}\frac{k}{(k-1)!}\prod_{j=2}^{k}(A-(j-1)B)\right]\right\}.\end{aligned} \tag{21}$$

The results are sharp.
Proof. From Definition 2, using Principle of subordination, we have

$$(1-\alpha)f'(z)+\alpha(zf'(z))'=g'(z)\left(\frac{1+Cw(z)}{1+Dw(z)}\right), w(z)\in U. \tag{22}$$

On expanding (22), it yields

$$(1-\alpha)[1+2a_2z+3a_3z^2+...+na_nz^{n-1}+...]+\alpha[1+4a_2z+9a_3z^2+...+n^2a_nz^{n-1}+...]$$
$$=(1+2d_2z+3d_3z^2+...+nd_nz^{n-1}+...)(1+p_1z+p_2z^2+...+p_{n-1}z^{n-1}+...). \quad (23)$$

Equating the coefficients of z^{n-1} in (23), we have

$$[(1-\alpha)n+\alpha n^2]a_n = nd_n+(n-1)p_1d_{n-1}+(n-2)p_2d_{n-2}...+2p_{n-2}d_2+p_{n-1}. \quad (24)$$

Applying triangle inequality and Lemma 1 in (24), it gives

$$[(1-\alpha)n+\alpha n^2]|a_n| \leq n|d_n|+(C-D)\left[(n-1)|d_{n-1}|+(n-2)|d_{n-2}|...+2|d_2|+1\right]. \quad (25)$$

Using Lemma 2 in (25), the result (21) is obvious.
For $n=2$, equality sign in (21) hold for the functions $f_n(z)$ defined as

$$(1-\alpha)f_n'(z)+\alpha(zf_n'(z))'$$
$$=(1+B\delta_1 z)^{\frac{(A-B)}{B}}\left(\frac{1+A\delta_1 z^n}{1+B\delta_1 z^n}\right)\left(\frac{1+C\delta_2 z^n}{1+D\delta_2 z^n}\right), B\neq 0, |\delta_1|=1, |\delta_2|=1. \quad (26)$$

Remark 4

(i) For $\alpha=\lambda, A=1, B=-1, C=(2\alpha-1)\beta, D=\beta$, Theorem 4 agrees with the result due to Selvaraj and Logu [12].
(ii) On putting $\alpha=1$, Theorem 4 gives the result proved by Singh and Singh [16].

Theorem 5 If $f(z)\in Q_\alpha^*(A,B;C,D)$, then for $|z|=r, 0<r<1$, we have for $\alpha=0, B\neq 0$,

$$\int_0^r\left(\frac{1-Ct}{1-Dt}\right)(1-At)(1-Bt)^{\frac{A-2B}{B}}dt \leq |f(z)| \leq \int_0^r\left(\frac{1+Ct}{1+Dt}\right)(1+At)(1+Bt)^{\frac{A-2B}{B}}dt; \quad (27)$$

for $\alpha=0, B=0$,

$$\int_0^r\frac{1}{A}\left(\frac{1-Ct}{1-Dt}\right)e^{-At}(1-At)dt \leq |f(z)| \leq \int_0^r\frac{1}{A}\left(\frac{1+Ct}{1+Dt}\right)e^{At}(1+At)dt, \quad (28)$$

and for $0<\alpha\leq 1, B\neq 0$,

$$\frac{1}{\alpha}\int_0^r\left[\frac{1}{s}\int_0^s\left(\frac{1-Ct}{1-Dt}\right)(1-At)(1-Bt)^{\frac{A-2B}{B}}dt\right]ds \leq |f(z)|$$
$$\leq \frac{1}{\alpha}\int_0^r\left[\frac{1}{s}\int_0^s\left(\frac{1+Ct}{1+Dt}\right)(1+At)(1+Bt)^{\frac{A-2B}{B}}dt\right]ds; \quad (29)$$

for $0<\alpha\leq 1, B=0$,

$$\frac{1}{A\alpha}\int_0^r\left[\frac{1}{s}\int_0^s\left(\frac{1-Ct}{1-Dt}\right)e^{-At}(1-At)dt\right]ds \leq |f(z)|$$
$$\leq \frac{1}{A\alpha}\int_0^r\left[\frac{1}{s}\int_0^s\left(\frac{1+Ct}{1+Dt}\right)e^{At}(1+At)dt\right]ds. \quad (30)$$

Estimates are sharp.

Proof. From (22), we have

$$|(1-\alpha)f'(z)+\alpha(zf'(z))'| = |g'(z)|\left|\frac{1+Cw(z)}{1+Dw(z)}\right|, w(z)\in U. \quad (31)$$

From (13), we have

$$\frac{1-Cr}{1-Dr} \leq \left|\frac{1+Cw(z)}{1+Dw(z)}\right| \leq \frac{1+Cr}{1+Dr}. \quad (32)$$

Let $F'(z)=(1-\alpha)f'(z)+\alpha(zf'(z))'$.
As $h(z)\in S^*(A,B)$, so from Lemma 3, we have

$$\begin{cases}(1-Ar)(1-Br)^{\frac{A-2B}{B}}\leq|g'(z)|\leq(1+Ar)(1+Br)^{\frac{A-2B}{B}}, & \text{if } B\neq 0;\\ e^{-Ar}[1-Ar]\leq|g'(z)|\leq e^{Ar}[1+Ar], & \text{if } B=0.\end{cases}\tag{33}$$

Using (32) and (33) in (31), it yields

$$\begin{cases}\left(\dfrac{1-Cr}{1-Dr}\right)(1-Ar)(1-Br)^{\frac{A-2B}{B}}\leq|F'(z)|\\ \leq\left(\dfrac{1+Cr}{1+Dr}\right)(1+Ar)(1+Br)^{\frac{A-2B}{B}}, & \text{if } B\neq 0;\\ \left(\dfrac{1-Cr}{1-Dr}\right)e^{-Ar}[1-Ar]\leq|F'(z)|\leq\left(\dfrac{1+Cr}{1+Dr}\right)e^{Ar}[1+Ar], & \text{if } B=0.\end{cases}\tag{34}$$

On integrating, (34) gives

$$\begin{cases}\int_0^r\left(\dfrac{1-Ct}{1-Dt}\right)(1-At)(1-Bt)^{\frac{A-2B}{B}}dt\leq|F(z)|\\ \leq\int_0^r\left(\dfrac{1+Ct}{1+Dt}\right)(1+At)(1+Bt)^{\frac{A-2B}{B}}dt, & \text{if } B\neq 0;\\ \int_0^r\left(\dfrac{1-Ct}{1-Dt}\right)e^{-At}[1-At]dt\leq|F(z)|\leq\int_0^r\left(\dfrac{1+Ct}{1+Dt}\right)e^{At}[1+At]dt, & \text{if } B=0.\end{cases}\tag{35}$$

Therefore, we have

$$\begin{cases}\int_0^r\left(\dfrac{1-Ct}{1-Dt}\right)(1-At)(1-Bt)^{\frac{A-2B}{B}}dt\leq|(1-\alpha)f(z)+\alpha zf'(z)|\\ \leq\int_0^r\left(\dfrac{1+Ct}{1+Dt}\right)(1+At)(1+Bt)^{\frac{A-2B}{B}}dt, & \text{if } B\neq 0;\\ \int_0^r\left(\dfrac{1-Ct}{1-Dt}\right)e^{-At}[1-At]dt\leq|(1-\alpha)f(z)+\alpha zf'(z)|\\ \leq\int_0^r\left(\dfrac{1+Ct}{1+Dt}\right)e^{At}[1+At]dt, & \text{if } B=0.\end{cases}\tag{36}$$

For $\alpha=0$, the resluts (27) and (28) are obvious from (36).
Also for $0<\alpha\leq 1$ and on integrating (36), the results (29) and (30) are obvious.
Sharpness follows for the function $f_n(z)$ defined in (26).

Remark 5
(i) On putting $A=1$, $B=-1$ in Theorem 1, we obtain the result proved by Selvaraj and Thirupathi [15].
(ii) For $\alpha=1$, Theorem 1 gives the result proved by Singh and Singh [16].
(iii) On putting $\alpha=1,A=1,B=-1,C=(2\alpha-1)\beta,D=\beta$, Theorem 1 agrees with the result due to Selvaraj and Stelin [13].
(iv) For $\alpha=1,A=1,B=-1$, Theorem 1 gives the result proved by Xiong and Liu [21].

Theorem 6 Let $F'(z)=(1-\alpha)f'(z)+\alpha(zf'(z))'$, where $f(z)\in Q^*_\alpha(A,B;C,D)$, then

$$Re\frac{(zF'(z))'}{F'(z)} \geq \begin{cases} \dfrac{1-Ar}{1-Br}-\dfrac{(A-B)r}{1-B^2r^2}-\dfrac{(C-D)r}{(1-Cr)(1-Dr)}, & \text{if } R_1 \leq R_2, \\ \dfrac{1-Ar}{1-Br}-\dfrac{(A-B)r}{1-B^2r^2}+\dfrac{C+D}{C-D} & \\ +2\dfrac{\sqrt{(1-D)(1-C)(1+Cr^2)(1+Dr^2)}-(1-CDr^2)}{(C-D)(1-r^2)}, & \text{if } R_1 \geq R_2, \end{cases} \tag{37}$$

where R_1 and R_2 are defined in Lemma 6.
Proof. As $f(z)\in Q^*_\alpha(A,B;C,D)$, we have

$$(1-\alpha)f'(z)+\alpha(zf'(z))'=g'(z)\left(\frac{1+Cw(z)}{1+Dw(z)}\right)=g'(z)P(z).$$

Here $F'(z)=(1-\alpha)f'(z)+\alpha(zf'(z))'$. So on differentiating it logarithmically, we get

$$\frac{(zF'(z))'}{F'(z)}=\frac{(zg'(z))'}{g'(z)}+\frac{zP'(z)}{P(z)}. \tag{38}$$

Now for $g\in S^*(A,B)$, we have

$$Re\left(\frac{(zg'(z))'}{g'(z)}\right)\geq\frac{1-Ar}{1-Br}-\frac{(A-B)r}{1-B^2r^2}. \tag{39}$$

So using Lemma 6 and inequality (39) in equation (38), the result (37) is obvious. Sharpness follows for the function $f_n(z)$ to be same as in (26).

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