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COMPARISON OF SOME NEW APPROXIMATIONS WITH THE LEIBNIZ FORMULA REGARDING THEIR CONVERGENCE TO THE ARCHIMEDES' CONSTANT

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Abstract. In this paper we present the Archimedes' Constant through numerical series and give two comparisons (inequalities), in which the Euler (e) and the Archimedes (π) constants are included. We compute the approximations to the Archimedes' Constant and compare them with the Leibniz formula using Matlab.

1. Introduction

The number π is commonly defined as the ratio of the circumference of a circle to its diameter. It appears in many formulas in all areas of mathematics, physics and engineering. The best-known approximations to π dating before the Common Era were accurate to two decimal places, extended to an accuracy of seven decimal places by the mid-first millennium by Chinese mathematicians. After this, no further progress was made until the late medieval period.

It has been represented by the Greek letter π since the mid-18th century, and is spelled out as "pi". It is also referred to as the Archimedes' Constant, since Archimedes is credited with the first theoretical calculation of π. The algorithm for calculating the value of π was a geometrical approach using polygons, devised around 250 BC, which dominated for over 1000 years after the death of Archimedes.

In 1761 Lambert proved that π was irrational, that is, that it cannot be written as a ratio of integer numbers. In 1882 Lindeman proved that π was transcendental, that is, that π is not the root of any algebraic equation with rational coefficients. Calculating the digits of π has proven to be a fascination for mathematicians throughout history, where some spent their lives making efforts to find the fastest and most accurate approximations of π.

The development of technology and computers in the mid-20th century again revolutionized the hunt for digits of π. Mathematicians John Wrench and Levi Smith reached 1,120 digits in 1949 using a desk calculator. Millions of digits have been calculated, with the record held (as of September 1999) by a supercomputer at the University of Tokyo that calculated 206,158,430,000 digits. Nowadays, number

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π has been computed to 50 trillion digits, and even this gargantuan record will likely soon be broken.

The calculation of π was revolutionized by the development of infinite series techniques in the 16th and 17th centuries. An infinite series is the sum of the terms of an infinite sequence. Infinite series allowed mathematicians to compute π with much greater precision than Archimedes and others who used geometrical techniques.

In this paper we will present some estimations of the Archimedes' Constant through numerical series, which are compared with the well-known approximations obtained by the Leibniz formula for the calculation of π.

2. Auxiliary facts

In order to obtain our results, we need the following well-known facts. In mathematics, the Leibniz formula for π, named after Gottfried Leibniz, is given as follows

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4} \tag{1}$$

Lemma 2.1 $e^{\pi} > \pi^{e}$

Lemma 2.2 for $a > b > e$, $b^a > a^b$

Lemma 2.3 $arctanx + \arctan y = \arctan \frac{x+y}{1-xy}$, $xy < 1$

Lemma 2.4 $\pi = \frac{S}{r^2}$, where S and r represent the area and the radius of the circle, respectively.

Proofs of these lemmas, can be found in most of the textbooks on Mathematical analysis.

3. Main Results

Proposition 3.1

$$\pi^{e^{1+\pi}} > e^{\pi^{1+e}} \tag{2}$$

Proof. Using Lemma 2.1 and making the substitutions $a = e^{\pi}$ and $b = \pi^{e}$ in Lemma 2.2 we get:

$$(\pi^{e})^{e^{\pi}} > (e^{\pi})^{\pi^{e}} \tag{3}$$

$$\implies \pi^{e^{1+\pi}} > e^{\pi^{1+e}} \tag{4}$$

Proposition 3.2

$$\pi^{e^{1+\pi+\pi^{1+e}}} > e^{\pi^{1+e+e^{1+\pi}}} \tag{5}$$

Proof. Using Lemma 2.1, and making the substitutions $a = e^{\pi^{1+e}}$ and $b = \pi^{e^{1+\pi}}$ in Lemma 2.2, we get:

$$(\pi^{e^{1+\pi}})^{e^{\pi^{1+e}}} > (e^{\pi^{1+e}})^{\pi^{e^{1+\pi}}} \tag{6}$$

$$\pi^{e^{1+\pi+\pi^{1+e}}} > e^{\pi^{1+e+e^{1+\pi}}} \tag{7}$$

Proposition 3.3

$$\frac{\pi}{4}=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(3^{-(2n+1)}+2^{-(2n+1)}\right) \tag{8}$$

Proof. In Lemma 2.3, substituting $y=1$, we get:

$$\arctan x+\frac{\pi}{4}=\arctan\left(\frac{1+x}{1-x}\right) \tag{9}$$

$$\arctan\left(\frac{1+x}{1-x}\right)-\arctan x=\frac{\pi}{4} \tag{10}$$

For

$$-1<\left(\frac{1+x}{1-x}\right)<1 \tag{11}$$

the following relation is valid

$$\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1+x}{1-x}\right)^{2n+1}}{2n+1}-\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}=\frac{\pi}{4} \tag{12}$$

Taking $x=-\frac{1}{2}$ we will obtain

$$\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{3}\right)^{2n+1}}{2n+1}+\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{2}\right)^{2n+1}}{2n+1}=\frac{\pi}{4} \tag{13}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\left(\frac{1}{3}\right)^{2n+1}+\left(\frac{1}{2}\right)^{2n+1}\right)=\frac{\pi}{4} \tag{14}$$

The Matlab code for the sum $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\left(\frac{1}{3}\right)^{2n+1}+\left(\frac{1}{2}\right)^{2n+1}\right)$ is given in Figure 1.

```
min = 0;
max = 1000;
sum = 0;
n = min:max;
term_vector = zeros(1, length(n));
sum_vector = zeros(1, length(n));
term_vector = (((-1).^n)./(2 .*n+1)).*(3 .^(-(2 .*n+1))+2
    .^(-(2 .* n+1)));
for i = 1: (max - min + 1)
    sum = sum + term_vector(i);
    sum_vector(i) = sum;
end
disp(sum)
```

```
>> code1

 0.78540
```

FIGURE 1. Sum of the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\left(\frac{1}{3}\right)^{2n+1}+\left(\frac{1}{2}\right)^{2n+1}\right)$ for its first 1000 terms

and we obtain this result sum $= 0.78540$.

Proposition 3.4

$$\pi = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\binom{\frac{1}{2}}{n} \tag{15}$$

Proof. Using Lemma 2.4, and taking $r = 1$, we get:

$$\begin{aligned}\pi &= 4\int_0^1\sqrt{1-x^2}dx = 4\int_0^1\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(-x^2)^n dx = \\ &= \int_0^1\sum_{n=0}^{\infty}(-1)^n\binom{\frac{1}{2}}{n}x^{2n}dx = 4\sum_{n=0}^{\infty}(-1)^n\binom{\frac{1}{2}}{n}\frac{1}{2n+1} = \\ &= 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\binom{\frac{1}{2}}{n}\end{aligned} \tag{16}$$

The Matlab code for the sum $\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\binom{\frac{1}{2}}{n}$ is given in Figure 2.

```
min = 0;
max = 100;
format long g
sum_vector = [];
sum = 0;

for i = min: max
    term = 4*(((-1)^i)/(2*i+1))*((gamma(1/2+1))/(gamma(i
        +1)*gamma(-i+1/2+1)));
    sum = sum + term;
    sum_vector = [sum_vector sum];
end
disp(sum)
```

```
>> code2

3.141965696275884
```

FIGURE 2. Sum of the series $4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\binom{\frac{1}{2}}{n}$ for its first 100 terms

and we obtain this result sum = 3.141965696275884.

Proposition 3.5

$$\pi - 3 = \sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{1}{(n+1)(2n+1)} \tag{17}$$

Proof. Considering the fact

$$\arcsin x = x + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!(2n+1)}x^{2n+1} \tag{18}$$

then we get,

$$\int_0^1 \arcsin x dx = \int_0^1 x dx + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!(2n+1)}\int_0^1 x^{2n+1}dx \tag{19}$$

$$\left(x\arcsin x + \sqrt{1-x^2}\right)\Big|_0^1 = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{1}{(2n+1)(2n+2)} \tag{20}$$

$$\frac{\pi}{2} = \frac{3}{2} + \sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{1}{(2n+1)(2n+2)} \tag{21}$$

$$\pi - 3 = \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{(2n+1)(n+1)} \tag{22}$$

The Matlab code for the sum $\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{(2n+1)(n+1)}$ is given in Figure 3.

```
min = 1;
max = 100;
format long g
sum_vector = [];
sum = 0;

for i = min: max
    term = ((DFactorial(2*i-1))/(DFactorial(2*i)))*(1/((i
       +1)*(2*i+1)));
    sum = sum + term;
    sum_vector = [sum_vector sum];
end
disp(sum)

function [DFact] = DFactorial(n)
    if n==0
        DFact = 1;
        else if n==1
        DFact = 1;
        else
        DFact = n*DFactorial(n-2);
        end
    end
end
```

```
>> code3
         0.141407790543118
```

FIGURE 3. Sum of the series $\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{(n+1)(2n+1)}$ for its first 100 terms

and we obtain this result sum = 0.1414077905431182.

Figure 4 shows the comparison of the obtained series with the Leibniz series. The black color represents the Leibniz series, the red one represents the series obtained in Proposition 3.3, the blue one represents the series obtained in Proposition 3.4 and green represents the series obtained in Proposition 3.5.

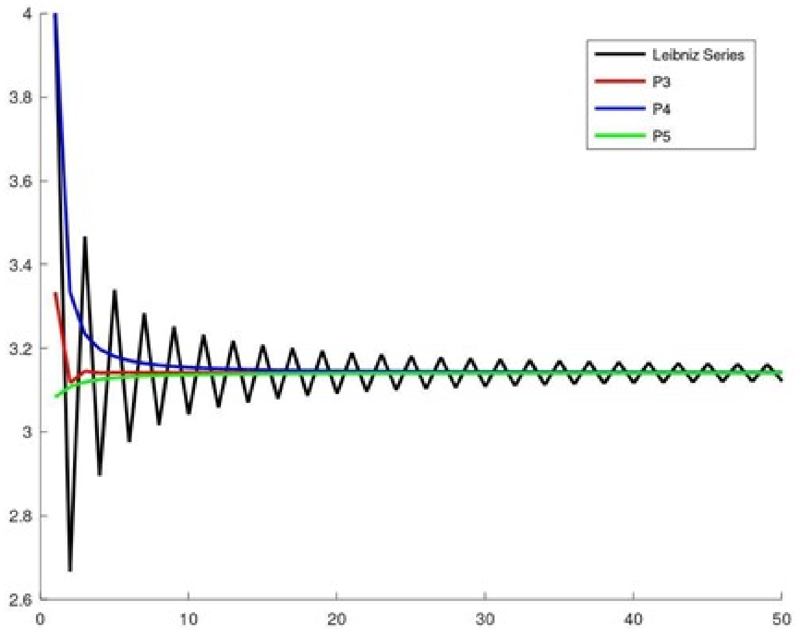

FIGURE 4. Comparison of the obtained series with the Leibniz series

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