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UNIQUENESS OF MEROMORPHIC FUNCTION WITH ITS LINEAR c-SHIFT AND LINEAR q_c-SHIFT OPERATORS

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Abstract. In this article, we have investigated the uniqueness problems of meromorphic function with its linear c-shift and linear q_c-shift operators. We establish some theorems which improves two recent results due to Zhen (J. Contemp. Math. Anal., 54(5)(2019), 296-301) and Banerjee-Bhattacharyya (Adv. Differ. Equ., 509(2019)). Some examples have been exhibited by us relevant to the content of the paper.

1. Introduction and Definitions

Throughout the paper by $\mathbb{C}$ and $\mathbb{N}$ we respectively mean the set of all complex numbers and natural numbers. We denote $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. By any meromorphic function f we always mean that it is defined on $\mathbb{C}$. For any non-constant meromorphic function $h(z)$ we define $S(r,h) = o(T(r,h)), (r \longrightarrow \infty, r \notin E)$ where, E denotes any set of positive real numbers having finite linear measure. We follow the standard notation of Nevanlinna theory as given in [8]. We recall that $T(r,f)$ denotes the Nevanlinna characteristic function of the non-constant meromorphic function and $N(r, \frac{1}{f-a}) = N(r,a;f)$ $(\overline{N}(r, \frac{1}{f-a}) = \overline{N}(r,a;f))$ denotes the counting function (reduced counting function) of a-points of f. For $a = \infty$, we use $N(r,f) = N(r,\infty;f)$ $\big((\overline{N}(r,f) = \overline{N}(r,\infty;f)\big)$ to denote counting (reduced counting) function of poles of f.

The following definitions are used in the paper.

Definition 1.1. [8] *For a constant value a, we denote the set of all a-points (counting multiplicities or CM) of f by $E(a,f)$, and all distinct a-points (ignoring multiplicities or IM) of f by $\overline{E}(a,f)$. For two non-constant meromorphic functions f and g, we say f and g share the value a CM, if $E(a,f) = E(a,g)$. On the other hand, if $\overline{E}(a,f) = \overline{E}(a,g)$ we say f and g share the value a IM.*

Definition 1.2. [8] *A meromorphic function f is said to be of order ρ if* $\limsup\limits_{r\to\infty} \dfrac{\log T(r,f)}{\log r} = \rho$.

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Definition 1.3. [8] *For a complex constant a, the deficiency of a is denoted by $\delta(a,f)$ and defined by* $\delta(a,f) = 1 - \limsup\limits_{r\to\infty} \dfrac{N(r,a;f)}{T(r,f)}$.

Let c be a non-zero complex constant, $k \geq 2$ be a natural number and $f(z)$ be a meromorphic function. The shift operator of $f(z)$ is denoted by $f(z+c)$. The difference and k-th difference operator are denoted by $\Delta_c f(z)$ and $\Delta_c^k f(z)$ respectively and defined as follows

$$\Delta_c f(z) = f(z+c) - f(z) \quad \text{and} \quad \Delta_c^k f(z) = \Delta_c^{k-1}\left(\Delta_c f(z)\right).$$

Recently in [1], Banerjee-Bhattacharyya introduced linear c-shift operator $L_c f(z)$ which is the generalized form of $\Delta_c^k f(z)$, defined as follows

$$L_c f(z) = \sum_{j=0}^{k} a_j f(z+jc),$$

where $a_j \in \mathbb{C}$ and $a_k \neq 0$. They also introduced reduced linear c-shift operator $L_c^r f(z)$ defined as follows

$$L_c^r f(z) = \sum_{j=0}^{k} b_j f(z+jc),$$

where $b_k = a_k, b_{k-1} = -a_{k-1}, \ldots, b_0 = (-1)^k a_0$ with $\sum\limits_{j=0}^{k}(-1)^{k-j} b_j = 0$. Inspired by the definition of $L_c f(z)$ here we introduce linear q_c-shift operator and define as follows

$$L_{q_c} f(z) = \sum_{j=0}^{k} a_j f(qz+jc),$$

where $a_j \in \mathbb{C}, a_k \neq 0$ and $q \in \mathbb{C}\setminus\{0\}$.

The analogue of famous Nevanlinna's theory for difference operator was first started by Hulburd-Korhonen [6], [7] and Chiang-Feng [5]. After that many important results ([3], [11], [14], [17]) came out in this direction, among which we recall a few.

In 2009, Heittokangas et. al. [9] investigated on relation between a meromorphic function and its shift operator when they share a, ∞ CM. Their result is the following.

Theorem A. [9] *Let $f(z)$ be a meromorphic function of order $\rho(f) < 2$ and $c \in \mathbb{C}$. If $f(z+c)$ and $f(z)$ share a, ∞ CM, where $a \in \mathbb{C}$, then for some constant τ,*

$$\frac{f(z+c) - a}{f(z) - a} = \tau.$$

Also in [10] Heittokangas et. al. studied on the relation between $f(z)$ and $f(z+c)$ when they share three small function CM and two small function CM, one small function IM. The case of two small function IM, one small function CM and three small function IM have been partially solved by Charak et. al. [2]. In [12], Huang-Zhang obtained similar type of result as in *Theorem A* for entire functions. Instead of shift operator and a, ∞ CM sharing they considered k-th order difference operator and 0 CM sharing to obtain the following result.

Theorem B. [12] *Let $f(z)$ be a transcendental entire function of order $\rho(f) < 2$. If $\Delta_c^k f(z)$ and $f(z)$ share 0 CM, where $k \in \mathbb{N}$ and $c \in \mathbb{C}\setminus\{0\}$ are such that $\Delta_c^k f(z) \not\equiv 0$ then*

$$\Delta_c^k f(z) \equiv \tau f(z)$$

for some constant τ.

Theorem 1.1. *Let $f(z)$ be a transcendental entire function of order $\rho(f) < 2$. If $L_c f(z)$ and $f(z)$ share 0 CM and $c \in \mathbb{C} \setminus \{0\}$ such that $L_c f(z) \not\equiv 0$, then $L_c f(z) \equiv \beta f(z)$, for some positive constant β.*

Notice that *Theorem 1.1* extend *Theorem B* to a large extent.

Example 1.1. *Consider $f(z) = \eta_1(z)\ 2^{z/c} + \eta_2(z)\ 3^{z/c}$, where $\eta_j(z+c) = \eta_j(z)$ for $j = 1, 2$ and $\rho(f) < 2$. Let*

$$L_c f(z) = \sum_{j=0}^{2} a_j f(z + jc),$$

choose $a_0 = 14, a_1 = -5, a_2 = 1$. Then we can check that $L_c f(z) = 8f(z)$. So $L_c f(z)$ and $f(z)$ share 0 CM and $L_c f(z) \equiv \beta f(z)$, where $\beta = 8$.

In 2013, Chen-Yi [4] proved the following theorem considering $\Delta_c f(z)$ and $f(z)$ share three distinct values a, b, ∞ CM.

Theorem C. [4] *Let $f(z)$ be a transcendental meromorphic function such that the order $\rho(f)$ is not an integer or infinite and $c \in \mathbb{C}$ be a constant such that $f(z+c) \not\equiv f(z)$. If $\Delta_c f(z)$ and $f(z)$ share three distinct values a, b, ∞ CM, then $f(z+c) \equiv 2f(z)$.*

The following example shows that the conclusion of *Theorem C* also holds for meromorphic function of integer order.

Example 1.2. *Consider $f(z) = e^{\frac{z \log 2}{c}}$. Notice that order of $f(z)$ is an integer. It is easy to see that $\Delta_c f(z)$ and $f(z)$ share three distinct values a, b, ∞ CM and $f(z+c) \equiv 2f(z)$.*

Recently Lü-Lü [16] removed the order restriction in *Theorem C* to prove the following theorem.

Theorem D. [16] *Let $f(z)$ be a transcendental meromorphic function of finite order and let $c \in \mathbb{C}$ be a constant such that $f(z+c) \not\equiv f(z)$. If $\Delta_c f(z)$ and $f(z)$ share three distinct values a, b, ∞ CM, then $f(z+c) \equiv 2f(z)$.*

The following example shows that for $L_c f(z)$, "3 CM" sharing can not be replaced by "2 CM+1 IM" sharing in *Theorem D.*

Example 1.3. *Let us consider $f(z) = (e^z - 1)^2 + 1$ and*

$$L_c f(z) = \sum_{j=0}^{3} a_j f(z + jc),$$

choose $a_0 = a_2 = -\frac{1}{8}$, $a_1 = a_3 = \frac{1}{8}$. Choose $c = \pi i$, then $L_c f(z) = e^z$. It is easy to see that $L_c f(z)$ and $f(z)$ share the value $2, \infty$ CM and 1 IM but $L_c f(z) \not\equiv f(z)$.

In 2019, Zhen [18] improved the last result by considering polynomial sharing instead of value sharing.

Theorem E. [18] *Let $f(z)$ be a transcendental meromorphic function of finite order and let $c(\neq 0)$ be a finite number. If $\Delta_c f(z)$ and $f(z)$ share three distinct polynomials P_1, P_2, ∞ CM, then $f = \Delta_c f(z)$.*

In 2019, Banerjee-Bhattacharyya [1] extend *Theorem D* in the following manner.

Theorem F. [1] *Let $f(z)$ be a non-constant transcendental meromorphic function of finite order which is not of period c and a, b be two distinct finite constants. Suppose $L_c^r f(z)$ and $f(z)$ share a, b, ∞ CM, then $L_c^r f(z) \equiv f(z)$.*

Our next two theorems are the improved version of two most recent results namely *Theorem E* and *Theorem F* for linear c-shift and linear q_c-shift operators.

Theorem 1.2. *Let $f(z)$ be a transcendental meromorphic function of finite order. Let $a(z)$ $(\not\equiv 0)$ be a small function of $f(z)$. If $L_c f(z)$ and $f(z)$ share $a(z)$, ∞ CM and $c \in \mathbb{C} \setminus \{0\}$ such that $L_c f(z) \not\equiv 0$, $\delta(0, f) > 0$, then $L_c f(z) \equiv f(z)$.*

Next corollary shows that in *Theorem 1.2*, if $f(z)$ be a transcendental entire function of finite order then the condition ∞ CM sharing is no longer needed.

Corollary 1.1. *Let $f(z)$ be a transcendental entire function of finite order. Let $a(z)$ $(\not\equiv 0)$ be a small function of $f(z)$. If $L_c f(z)$ and $f(z)$ share $a(z)$ CM and $c \in \mathbb{C} \setminus \{0\}$ such that $L_c f(z) \not\equiv 0$, $\delta(0, f) > 0$, then $L_c f(z) \equiv f(z)$.*

The following example satisfies *Corollary 1.1.*

Example 1.4. *Consider $f(z) = e^{\frac{z \log 4}{c}}$, then $\delta(0, f) = 1 > 0$ and order of $f(z)$ is finite. Let*

$$L_c f(z) = \sum_{j=0}^{2} a_j f(z + jc),$$

choose $a_0 = 1, a_1 = -4, a_2 = 1$. Then we can check that $L_c f(z)$ and $f(z)$ share any non-zero value a CM and $L_c f(z) \equiv f(z)$.

In the next example we see that if $\delta(0, f) = 0$ then the conclusion of *Theorem 1.2* cease to be hold, so the condition $\delta(0, f) > 0$ is essential.

Example 1.5. *Consider $f(z) = e^z + a$, where $a \neq 0$, then $\delta(0, f) = 0$ and order of $f(z)$ is finite. Let $c = \pi i$ and*

$$L_c f(z) = \sum_{j=0}^{2} a_j f(z + jc),$$

choose $a_0 = -1, a_1 = 1, a_2 = 1$. Then $L_c f(z) = -e^z + a$. Now it is easy to see that $L_c f(z)$ and $f(z)$ share the non-zero value a CM but $L_c f(z) \not\equiv f(z)$.

Theorem 1.3. *Let $f(z)$ be a transcendental meromorphic function of zero order. Let $a(z)$ $(\not\equiv 0)$ be a small function of $f(z)$. If $L_{q_c} f(z)$ and $f(z)$ share $a(z)$, ∞ CM and $q, c \in \mathbb{C} \setminus \{0\}$ such that $L_{q_c} f(z) \not\equiv 0$, $\delta(0, f) > 0$, then $L_{q_c} f(z) \equiv f(z)$.*

Corollary 1.2. *Let $f(z)$ be a transcendental entire function of zero order. Let $a(z)$ $(\not\equiv 0)$ be a small function of $f(z)$. If $L_{q_c} f(z)$ and $f(z)$ share $a(z)$ CM and $q, c \in \mathbb{C} \setminus \{0\}$ such that $L_{q_c} f(z) \not\equiv 0$, $\delta(0, f) > 0$, then $L_{q_c} f(z) \equiv f(z)$.*

2. Lemmas

Lemma 2.1. [6] *Let $f(z)$ be a non-constant meromorphic function of finite order, let c_1, c_2 be two arbitrary complex numbers. Then we have*

$$m\left(r, \frac{f(z+c_1)}{f(z+c_2)}\right) = S(r,f).$$

Lemma 2.2. [15] *Let $f(z)$ be a non-constant meromorphic function of order zero and $c \in \mathbb{C},\ q \in \mathbb{C}\setminus\{0\}$. Then*

$$m\left(r, \frac{f(qz+c)}{f(z)}\right) = S(r,f)$$

on a set of logarithmic density 1.

Lemma 2.3. [5] *Let $A_0(z), A_1(z), \ldots, A_n(z)$ be entire functions such that there exists an integer l, $0 \le l \le n$, such that*

$$\rho(A_l) > \max_{\substack{1 \le j \le n,\\ j \ne l}} \{\rho(A_j)\}.$$

If $f(z)(\not\equiv 0)$ is a meromorphic solution of

$$A_n(z)f(z+n) + A_{n-1}(z)f(z+n-1) + \ldots + A_0(z)f(z) = 0,$$

then $\rho(f) \ge \rho(A_l) + 1$.

Lemma 2.4. [13] *Let h is a non-constant meromorphic function satisfying*

$$\overline{N}(r,h) + \overline{N}\left(r, \frac{1}{h}\right) = S(r,h).$$

Let $f = a_0h^p + a_1h^{p-1} + \ldots + a_p$ and $g = b_0h^q + b_1h^{q-1} + \ldots + b_q$ be polynomial in h with coefficients $a_0, a_1, \ldots, a_p$; $b_0, b_1, \ldots, b_p$ being small function of h and $a_0b_0a_p \not\equiv 0$. If $q \le p$, then $m\left(r, \frac{g}{f}\right) = S(r,h)$.

3. Proofs of the theorems

Proof of Theorem 1.1. Since $L_cf(z)$ and $f(z)$ share 0 CM then there exists a polynomial $P(z)$ such that

$$\frac{L_cf(z)}{f(z)} = e^{P(z)}. \tag{3.1}$$

If $P(z)$ is constant then it is easy to see that there exists a positive constant β such that $L_cf(z) = \beta f(z)$. If $P(z)$ is non constant polynomial then from definition of $L_cf(z)$ and using (3.1) we get

$$\sum_{j=1}^{k} a_j f(z+jc) - (e^{P(z)} - a_0)f(z) = 0. \tag{3.2}$$

Using *Lemma 2.3* in (3.2) we obtain $\rho(f) \ge \rho(e^{P(z)} - a_0) + 1 \ge 2$, this contradicts the given condition $\rho(f) < 2$. □

Proof of Theorem 1.2. As $L_cf(z)$ and $f(z)$ share $a(z), \infty$ CM and $f(z)$ is transcendental meromorphic function of finite order then there exists a polynomial $P(z)$ such that

$$\frac{L_cf(z)-a(z)}{f(z)-a(z)} = e^{P(z)}. \tag{3.3}$$

Note that the equation (3.3) can be written as

$$\begin{aligned} -L_cf(z) + e^{P(z)}f(z) &= \left(e^{P(z)}-1\right)a(z) \\ \implies -\sum_{j=0}^{k} a_jf(z+jc) + e^{P(z)}f(z) &= \left(e^{P(z)}-1\right)a(z) \end{aligned} \tag{3.4}$$

Now dividing (3.4) by $\left(e^{P(z)}-1\right)a(z)f(z)$ we get

$$-\frac{1}{\left(e^{P(z)}-1\right)a(z)}\frac{\sum_{j=0}^{k} a_jf(z+jc)}{f(z)} + \frac{e^{P(z)}}{\left(e^{P(z)}-1\right)a(z)} = \frac{1}{f(z)}. \tag{3.5}$$

Let us assume $e^{P(z)} \not\equiv 1$.

Case 1: If $P(z)$ is constant such that $e^{P(z)} \not\equiv 1$, then from (3.5) and *Lemma 2.1* we get

$$\begin{aligned} m\left(r, \frac{1}{f(z)}\right) &= m\left(r, -\frac{1}{\left(e^{P(z)}-1\right)a(z)}\frac{\sum_{j=0}^{k} a_jf(z+jc)}{f(z)} + \frac{e^{P(z)}}{\left(e^{P(z)}-1\right)a(z)}\right) \\ &\leq m\left(r, \frac{\sum_{j=0}^{k} a_jf(z+jc)}{f(z)}\right) + 2m\left(r, \frac{1}{a(z)}\right) + O(1) \\ &\leq 2T\left(r, a(z)\right) + S(r,f) = S(r,f). \end{aligned}$$

Thus

$$N\left(r, \frac{1}{f(z)}\right) = T\left(r, \frac{1}{f(z)}\right) - m\left(r, \frac{1}{f(z)}\right) = T\left(r, f(z)\right) + S(r,f). \tag{3.6}$$

It is clear that (3.6) implies $\delta(0,f) = 0$, this contradicts the given condition $\delta(0,f) > 0$.

Case 2: Let $P(z)$ is non constant such that $e^{P(z)} \not\equiv 1$. Now using Nevanlinna's fundamental theorem we get

$$
\begin{aligned}
m\left(r, \frac{1}{f(z)-a(z)}\right) &\leq T\left(r, \frac{1}{f(z)-a(z)}\right) \\
&\leq T\left(r, f(z)-a(z)\right) + S(r,f) \\
&\leq T(r, f(z)) + T(r, a(z)) + S(r,f) \\
&\leq T(r, f(z)) + S(r,f).
\end{aligned} \tag{3.7}
$$

Now using (3.3), (3.7) and *Lemma 2.1* we get

$$
\begin{aligned}
T\left(r, e^{P(z)}\right) &= m\left(r, \frac{L_c f(z) - a(z)}{f(z)-a(z)}\right) \\
&= m\left(r, \frac{\sum_{j=0}^{k} a_j (f-a)(z+jc)}{(f-a)(z)} + \frac{\sum_{j=0}^{k} a_j a(z+jc) - a(z)}{f(z)-a(z)}\right) \\
&\leq \sum_{j=0}^{k} m\left(r, \frac{a_j (f-a)(z+jc)}{(f-a)(z)}\right) + m\left(r, \sum_{j=0}^{k} a_j a(z+jc) - a(z)\right) \\
&\quad + m\left(r, \frac{1}{f(z)-a(z)}\right) \\
&\leq T\left(r, \sum_{j=0}^{k} a_j a(z+jc) - a(z)\right) + T(r, f(z)) + S(r,f) \\
&\leq T\left(r, f(z)\right) + S(r,f).
\end{aligned} \tag{3.8}
$$

So from (3.8) it is clear that $S\left(r, e^{P(z)}\right)$ can be replace by $S(r,f)$.
Now by *Lemma 2.4* we get

$$
m\left(r, \frac{1}{e^{P(z)}-1}\right) = S\left(r, e^{P(z)}\right) = S(r,f). \tag{3.9}
$$

Now from (3.5), (3.9) and *Lemma 2.1* we get

$$
\begin{aligned}
& m\left(r, \frac{1}{f(z)}\right) \qquad (3.10)\\
= \quad & m\left(r, -\frac{1}{\left(e^{P(z)}-1\right)a(z)}\frac{\sum_{j=0}^{k} a_j f(z+jc)}{f(z)} + \frac{e^{P(z)}}{\left(e^{P(z)}-1\right)a(z)}\right)\\
\leq \quad & m\left(r, \frac{\sum_{j=0}^{k} a_j f(z+jc)}{f(z)}\right) + m\left(r, \frac{1}{e^{P(z)}-1}\right)\\
& +m\left(r, \frac{e^{P(z)}}{e^{P(z)}-1}\right) + 2m\left(r, \frac{1}{a(z)}\right) + S(r,f)\\
\leq \quad & m\left(r, 1+\frac{1}{e^{P(z)}-1}\right) + 2T(r,a(z)) + S(r,f) = S(r,f).
\end{aligned}
$$

Thus

$$N\left(r, \frac{1}{f(z)}\right) = T\left(r, f(z)\right) + S(r,f). \tag{3.11}$$

It is clear that (3.11) implies $\delta(0,f) = 0$, this contradicts the given condition $\delta(0,f) > 0$.

Thus from Case 1 and case 2 it is clear that $e^{P(z)} \equiv 1$. Therefore from (3.3) we obtain $L_c f(z) \equiv f(z)$. □

Proof of Theorem 1.3. As $L_{q_c} f(z)$ and $f(z)$ share $a(z), \infty$ CM and $f(z)$ is transcendental meromorphic function of order zero then there exists a constant γ such that

$$\frac{L_{q_c} f(z) - a(z)}{f(z) - a(z)} = \gamma. \tag{3.12}$$

Let us assume $\gamma \neq 1$. The equation (3.12) can be written as

$$
\begin{aligned}
& -L_{q_c} f(z) + \gamma f(z) = (\gamma - 1)\, a(z)\\
\Longrightarrow \; & -\sum_{j=0}^{k} a_j f(qz+jc) + \gamma f(z) = (\gamma - 1)\, a(z) \qquad (3.13)
\end{aligned}
$$

Now dividing (3.13) by $(\gamma - 1)\, a(z) f(z)$ we get

$$-\frac{1}{(\gamma-1)\, a(z)}\frac{\sum_{j=0}^{k} a_j f(qz+jc)}{f(z)} + \frac{\gamma}{(\gamma-1)\, a(z)} = \frac{1}{f(z)}. \tag{3.14}$$

From (3.14) and using *Lemma 2.2* we get

$$\begin{aligned} m\left(r,\frac{1}{f(z)}\right) &= m\left(r,-\frac{1}{(\gamma-1)\,a(z)}\frac{\sum_{j=0}^{k}a_jf(qz+jc)}{f(z)}+\frac{\gamma}{(\gamma-1)\,a(z)}\right)\\ &\leq m\left(r,\frac{\sum_{j=0}^{k}a_jf(qz+jc)}{f(z)}\right)+2m\left(r,\frac{1}{a(z)}\right)+O(1)\\ &\leq 2T\left(r,a(z)\right)+S(r,f)=S(r,f). \end{aligned}$$

Thus

$$N\left(r,\frac{1}{f(z)}\right)=T\left(r,\frac{1}{f(z)}\right)-m\left(r,\frac{1}{f(z)}\right)=T\left(r,f(z)\right)+S(r,f). \tag{3.15}$$

It is clear that (3.15) implies $\delta(0,f)=0$, this contradicts the given condition $\delta(0,f)>0$.

Thus $\gamma=1$. Therefore $L_{q_c}f(z)\equiv f(z)$. □

4. Application

The main focus of this paper is to study the uniqueness of $L_cf(z)$ and $f(z)$. Actually we have determined the sufficient conditions under which the uniqueness of $L_cf(z)$ and $f(z)$ happens. Also the same conclusion yields the following difference equation

$$L_cf(z)-f(z)=0. \tag{4.1}$$

So it will be natural to find the form of the functions which satisfy the equation. In other words, we will try to find the form of a solutions of the difference equation (4.1). For $j=1,2,\ldots,k$, consider $\gamma_j(\neq 1)$ be the roots of the equation $\sum_{j=0}^{k}a_jz^j=1$. We claim that one of the solutions of (4.1) will be of the form

$$f(z)=\eta_1(z)\gamma_1^{z/c}+\eta_2(z)\gamma_2^{z/c}+\ldots+\eta_k(z)\gamma_k^{z/c}, \tag{4.2}$$

where η_j $(j = 1, 2, \ldots, k)$ are periodic functions of period c. Now we verify our claim in the following manner:

$$\begin{aligned}
& L_c f(z) \\
= \;& a_k f(z+kc) + \ldots + a_1 f(z+c) + a_0 f(z) \\
= \;& a_k \left\{ \eta_1(z+kc)\gamma_1^{\frac{z+kc}{c}} + \ldots + \eta_k(z+kc)\gamma_k^{\frac{z+kc}{c}} \right\} + \ldots \\
& + a_1 \left\{ \eta_1(z+c)\gamma_1^{\frac{z+c}{c}} + \ldots + \eta_k(z+c)\gamma_k^{\frac{z+c}{c}} \right\} + a_0 \left\{ \eta_1(z)\gamma_1^{\frac{z}{c}} + \ldots + \eta_k(z)\gamma_k^{\frac{z}{c}} \right\} \\
= \;& \left\{ a_k\gamma_1^k + \ldots + a_1\gamma_1 + a_0 \right\} \eta_1(z)\gamma_1^{\frac{z}{c}} + \ldots + \left\{ a_k\gamma_k^k + \ldots + a_1\gamma_k + a_0 \right\} \eta_k(z)\gamma_k^{\frac{z}{c}} \\
= \;& \eta_1(z)\gamma_1^{\frac{z}{c}} + \ldots + \eta_k(z)\gamma_k^{\frac{z}{c}} \\
= \;& f(z),
\end{aligned}$$

as γ_j $(j = 1, 2, \ldots, k)$ be the roots of the equation $\sum_{j=0}^{k} a_j z^j = 1$. Thus (4.2) gives a form of a solution of the difference equation (4.1).

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