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A STUDY FOR A CLASS OF ENTIRE DIRICHLET SERIES IN n - VARIABLES

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Abstract. Let L represents a class of entire functions represented by Dirichlet series in n - variables of the form $f(s) = \sum_{m=1}^{\infty} a_m\, e^{\lambda_{n_{m_n}} s}$ whose coefficients belong to the set of complex numbers $\mathbb{C}$. L which becomes a complete Banach space is thereby proved to be a complex FK-space and a Frechet space.

1. Introduction

Let

$$f(s_1, s_2, \ldots, s_n) = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} a_{m_1,m_2,\ldots,m_n}\, e^{(\lambda_{1_{m_1}} s_1 + \lambda_{2_{m_2}} s_2 + \ldots + \lambda_{n_{m_n}} s_n)} \tag{1}$$

be a n-tuple Dirichlet series where $s_j = \sigma_j + it_j,\ j \in \{1, 2, \ldots, n\}$ and $a_{m_1,m_2,\ldots,m_n} \in \mathbb{C}$. Also

$$0 < \lambda_{p_1} < \lambda_{p_2} < \ldots < \lambda_{p_k} \to \infty \text{ as } k \to \infty \ \text{ for } \ p = 1, 2, \ldots, n.$$

To simplify the form of n-tuple Dirichlet series, we have the following notations

$$\begin{aligned} s &= (s_1, s_2, \ldots, s_n) \in \mathbb{C}^n, \\ m &= (m_1, m_2, \ldots, m_n) \in \mathbb{C}^n \text{ and} \\ \lambda_{n_{m_n}} &= (\lambda_{1_{m_1}}, \lambda_{2_{m_2}}, \ldots, \lambda_{n_{m_n}}) \in \mathbb{R}^n. \\ \lambda_{n_{m_n}} s &= \lambda_{1_{m_1}} s_1 + \lambda_{2_{m_2}} s_2 + \ldots + \lambda_{n_{m_n}} s_n \\ |\lambda_{n_{m_n}}| &= \lambda_{1_{m_1}} + \lambda_{2_{m_2}} + \ldots + \lambda_{n_{m_n}} \\ |m| &= m_1 + m_2 + \ldots + m_n. \end{aligned}$$

Thus the series (1) can be written as

$$f(s) = \sum_{m=1}^{\infty} a_m\, e^{\lambda_{n_{m_n}} s}. \tag{2}$$

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Janusauskas in [4] showed that if there exists a tuple $p > \bar{0} = (0, 0, \ldots, 0)$ such that

$$\limsup_{|m|\to\infty} \frac{\sum_{k=1}^{\infty} \log m_k}{p\lambda_{n_{m_n}}} = 0, \tag{3}$$

then the domain of absolute convergence of (2) coincides with its domain of convergence. Sarkar in [1] proved that the necessary and sufficient condition for series (2) satisfying (3) to be entire is that

$$\lim_{|m|\to\infty} \frac{\log |a_m|}{|\lambda_{n_{m_n}}|} = -\infty. \tag{4}$$

Consider L as the set of series (2) satisfying (3) and (4) for which

$$(|\lambda_{n_{m_n}}|)^{c_1|\lambda_{n_{m_n}}|}\, e^{c_2|m|\,(|\lambda_{n_{m_n}}|)}\, |a_m|$$

is bounded. Then every element of L represents an entire function. Define the binary operations in L as

$$\begin{aligned} f(s) + h(s) &= \sum_{m=1}^{\infty} (a_m + b_m)\, e^{\lambda_{n_{m_n}} s}, \\ \xi f(s) &= \sum_{m=1}^{\infty} (\xi a_m)\, e^{\lambda_{n_{m_n}} s}, \\ f(s).h(s) &= \sum_{m=1}^{\infty} (|\lambda_{n_{m_n}}|)^{c_1|\lambda_{n_{m_n}}|}\, e^{c_2|m|\,(|\lambda_{n_{m_n}}|)}\, a_m b_m\, e^{\lambda_{n_{m_n}} s}. \end{aligned}$$

The norm in L is defined as

$$\|f\| = \sup_{|m|\geq 1} (|\lambda_{n_{m_n}}|)^{c_1|\lambda_{n_{m_n}}|}\, e^{c_2|m|\,(|\lambda_{n_{m_n}}|)}\, |a_m|. \tag{5}$$

Definition 1 A space L is called an FK-space if the following conditions are satisfied

(1.a) L is a linear space over the field of complex numbers (or real numbers) and elements of L are sequences of complex numbers (or real numbers).

(1.b) L is a locally convex topological linear space in which the topology is given by a countable family of semi-norms.

(1.c) L is metrizable and is a complete metric space.

(1.d) If $\{\alpha_m\}$ is a base for L such that for $l \in L$,

$$l = \sum_{m=1}^{\infty} \theta_m(l)\alpha_m$$

then $\theta_m(l)\,(|m| \geq 1)$ are continuous linear functionals. If the field for L is complex numbers then L is called a complex FK-space.

During the last two decades a lot of research has been carried out in the field of Dirichlet series and many important results have been proved where few of them may be found in [2] - [3]. Kumar and Manocha in [5] considered the condition $(\lambda_n)^{c_1(\lambda_n)}\ e^{\{c_2 n - c_1\}}(\lambda_n)\ \|a_n\|$ of weighted norm for a Dirichlet series in one variable and established some results on it.Recently in [6] results were established on Dirichlet series with complex frequencies. Until now a lot work has been done for the Dirichlet series in one variable. The purpose of this paper is to give a wider view to the study of Dirichlet series in n-variables. In this section main results have been proved.For the definitions of terms used refer [7, 8]. **Theorem 1** With respect to the usual addition and multiplication, a topology is defined such that L becomes a complex FK-space.

Proof. Let for $f(s), h(s) \in L$ define addition of $f(s)$ and $g(s)$ as $(f+h)(s) = f(s) + h(s)$ and scalar multiplication of $f(s)$ as $(\tau f)(s) = \tau f(s)$.

Let us now define the zero element of L defined by 0^* as the entire function which is zero that is $f = 0^*$ implies $\sum_{m=1}^{\infty} a_m e^{\lambda_{n_{m_n}} s} = 0$ which further implies $a_m = 0$ for all $|m| \geq 1$ and conversely.

Clearly L forms an infinite dimensional linear space over the field of complex numbers and hence one gets basis for L namely Schauder basis as

$$\delta_{m_1, m_2, \ldots, m_n} = e^{(\lambda_{1_{m_1}} s_1 + \lambda_{2_{m_2}} s_2 + \ldots + \lambda_{n_{m_n}} s_n)}$$

or

$$\delta_m = e^{\lambda_{n_{m_n}} s}.$$

Also

$$\begin{aligned} L_{m_1} &= (1, 0, 0, \ldots) \\ L_{m_2} &= (0, 1, 0, \ldots) \\ &\cdot \\ &\cdot \\ &\cdot \\ L_{m_n} &= (0, 0, \ldots, 1, 0, \ldots) \end{aligned}$$

where 1 in L_{m_n} is at the m_n-th place. It therefore implies that if $x(s) \in L$ then

$$x(s) = (a_1(x), a_2(x), \ldots, a_m(x), \ldots)$$

where

$$\lim_{|m| \to \infty} \frac{\log |a_m|}{|\lambda_{n_{m_n}}|} = -\infty$$

and this shows that L satisfies (1.a).

Define $H = \{L_{m_1}, L_{m_2}, \ldots, L_{m_n}, \ldots\}$. For each $L_{m_n} \in H$ define the norm as

$$\|f, L_{m_n}\| = \sup_{|m| \geq 1} (|\lambda_{n_{m_n}}|)^{c_1 |\lambda_{n_{m_n}}|} e^{c_2 |m| (|\lambda_{n_{m_n}}|)} |a_m|$$

where

$$f(s) = \sum_{m=1}^{\infty} a_m e^{\lambda_{n_{m_n}} s} \in L$$

is an entire function. Then as

$$\frac{\log |a_m|^{-1}}{|\lambda_{n_{m_n}}|} > v$$

for $v > c_2 |m|$ implies

$$|a_m| < e^{-v |\lambda_{n_{m_n}}|} \text{ for } |m| \geq |m'|,$$

where $m' = (m'_1, m'_2, \ldots, m'_n)$. Therefore

$$\|f, L_{m_n}\| < \sup_{|m| < |m'|} (|\lambda_{n_{m_n}}|)^{c_1 |\lambda_{n_{m_n}}|} e^{c_2 |m| (|\lambda_{n_{m_n}}|)} |a_m| + \sup_{|m| \geq |m'|} (|\lambda_{n_{m_n}}|)^{c_1 |\lambda_{n_{m_n}}|} e^{(c_2 |m| - v)(|\lambda_{n_{m_n}}|)} |a_m|$$

Thus

$$\|f, L_{m_n}\| < \infty$$

for any fixed $L_{m_n} \in H$. Hence $\|f, L_{m_n}\|$ is defined for each $L_{m_n} \in H$. Let L_{m_n} be fixed, then

$$\|f, L_{m_n}\| = f(s)$$

$$\begin{aligned} &\Leftrightarrow \quad |a_m| = 0 \text{ for } |m| \geq 1 \\ &\Leftrightarrow \quad f(s) = 0 \text{ for all } |s| \\ &\Leftrightarrow \quad f = 0^*. \end{aligned}$$

Since

$$h(s) = \sum_{m=1}^{\infty} b_m \, e^{\lambda_{n m_n} s}$$

Then

$$|a_m + b_m| \leq |a_m| + |b_m|$$

implies

$$\|f + h, L_{m_n}\| \leq \|f, L_{m_n}\| + \|h, L_{m_n}\|.$$

Again if υ is any complex number then

$$\|\upsilon f, L_{m_n}\| = |\upsilon| \|f, L_{m_n}\|.$$

Thus $\| \ldots, L_{m_n}\|$ defines a norm for each $L_{m_n} \in H$.
Hence L becomes a locally convex linear topological space as there exists a sequence $\{\| \ldots, L_{m_n}\| : |n| = 1, 2, 3, \ldots\}$ of enumerable number of norms on L. Let

$$\|f\| = \sup_{|m| \geq 1} \frac{\|f, L_{m_n}\|}{1 + \|f, L_{m_n}\|}$$

and

$$e(f, h) = \|f - h\|$$

Then e is a metric on L. It can be easily verified that the topology induced by e on L is the same as induced by the sequence $\{\| \ldots, L_{m_n}\|\}$. In fact if Y is open in the topology induced by the family of norms then Y is also open in the e-metric topology of L. Now let Y be open in the e-metric topology of L. Then for each $g(s) \in Y$ we have $\epsilon > 0$ such that

$$K = \{g(s) : g \in B(f; \epsilon)\} \subset Y \text{ for } 0 < \epsilon < 1$$

where $B(f; \epsilon)$ is an open ball centered at $f(s)$ and is of radius ϵ.
We find M such that

$$\sup_{|m| \geq |M|+1} \frac{1}{2^m} \frac{\|k - g, L_{m_n}\|}{1 + \|k - g, L_{m_n}\|} < \frac{\epsilon}{2},$$

where $k - g$ is a vector in the neighbourhood of 0. Let

$$F = \{x(s) : \|x, L_1\| < \epsilon_1\} \bigcap \ldots \bigcap \{x(s) : \|x, L_M\| < \epsilon_M\}$$

where

$$\epsilon_m < \frac{\epsilon}{2} \, (|m| = 1, 2, \ldots, |M|).$$

Let $k(s) \in g(s)+F$. Then $k(s) = g(s)+x(s)$ where $x(s) \in F$. Then

$$\begin{aligned} e(k,g) &= \sup_{1\le|m|\le|M|} \frac{1}{2^m}\frac{\|k-g, L_{m_n}\|}{1+\|k-g, L_{m_n}\|} + \sup_{|m|\ge|M|+1} \frac{1}{2^m}\frac{\|k-g, L_{m_n}\|}{1+\|k-g, L_{m_n}\|} \\ &< \sup_{1\le|m|\le|M|} \frac{1}{2^m}\frac{\epsilon_m}{1+\epsilon_m} + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Therefore $e(k,g) < \epsilon$ implies that $k(s) \in B(g\,;\epsilon)$ that is $k(s) \in K$ which further implies $k(s) \in Y$. Thus $g(s)+F \subset Y$ which establishes that Y is open in the topology induced by the family of norms. Hence L is metrizable.

Now we show that L is complete with respect to the metric e. It is known that a space is complete if and only if every nested sequence of closed balls whose radii tend to zero has non empty intersection.

Let $\{f_m : m \in M\}$ be a cauchy sequence in L. For each $m \in M$, let $W_m = \{x_k : k \ge m\}$ be m-th tail of sequence and s_m be twice the diameter of W_m. Also let B_m be a closed ball centered at f_m of radius $r_m = 2s_m$. Then

$$W_m \subseteq B_m.$$

Since the sequence is cauchy therefore $\lim_{m\to\infty} s_m = 0$. Now let $m \in M$ be arbitrary. Therefore there exists $k > m$ such that

$$s_k < \frac{1}{2}s_m.$$

Suppose $g(s) \in B_k$ then

$$\begin{aligned} e(g,f_m) &\le e(g,f_k) + e(f_k,f_m) \\ &\le r_k + s_m \\ &= 2s_k + s_m \\ &< 2s_m = r_m. \end{aligned}$$

Therefore $g(s) \in B_m$ and hence $B_k \subseteq B_m$. In the like manner we construct a nested sequence of the closed balls $\{B_m : m \in M\}$. Then from hypothesis nested sequence of closed balls has a non empty intersection say f. Let $\{f_{r_1}\}$ be a cauchy sequence in L where

$$f_{r_1}(s) = \sum_{t=1}^{\infty} a_t^{(r_1)}\, e^{\lambda_{n_{t_n}} s}.$$

Now

$$e(f_{r_1}, f_{r_2}) < \epsilon \text{ for all } r_1, r_2 \ge |M|$$

implies

$$\sup_{|m|\ge1} \frac{1}{2^m}\frac{\|f_{r_1}-f_{r_2}, L_{m_n}\|}{1+\|f_{r_1}-f_{r_2}, L_{m_n}\|} < \epsilon \text{ for } r_1, r_2 \ge |M|.$$

Thus

$$(1-2^m\epsilon)\,\|f_{r_1}-f_{r_2}, L_{m_n}\| < 2^m\epsilon \text{ for } r_1, r_2 \ge |M|\ , |m| = 1,2,\ldots$$

$$(1-2^m\epsilon)\sup_{|t|\ge1}\,(|\lambda_{n_{t_n}}|)^{c_1|\lambda_{n_{t_n}}|}\, e^{c_2|t|\,(|\lambda_{n_{t_n}}|)}\,|a_t^{(r_1)}-a_t^{(r_2)}| < 2^m\epsilon \text{ for } r_1, r_2 \ge |M|\ , |m| = 1,2,\ldots$$

$$(1-2^m\epsilon)\,|a_t^{(r_1)}-a_t^{(r_2)}| < 2^m\epsilon\,\{(|\lambda_{n_{t_n}}|)^{c_1|\lambda_{n_{t_n}}|}\, e^{c_2|t|\,(|\lambda_{n_{t_n}}|)}\}^{-1} \text{ for } r_1, r_2 \ge |M|\ , |t|\ge1\ , |m| = 1,2,\ldots$$

and

$$\lim_{r_2\to\infty} a_t^{(r_2)} \;=\; a_t \;\; , |t| = 1, 2, \ldots$$

implies

$$(1-2^m\epsilon)\,|a_t^{(r_1)}-a_t| \;<\; 2^m\epsilon\,\{(|\lambda_{n_{t_n}}|)^{c_1|\lambda_{n_{t_n}}|}\,e^{c_2|t|\,(|\lambda_{n_{t_n}}|)}\}^{-1} \text{ for } r_1 \;\geq\; |M| \;, |t|, |m| = 1, 2, \ldots$$

If $2^m\epsilon < \theta < 1$ then

$$|a_t^{(r_1)} - a_t| \;<\; \frac{\theta}{1-\theta}\,\{(|\lambda_{n_{t_n}}|)^{c_1|\lambda_{n_{t_n}}|}\,e^{c_2|t|\,(|\lambda_{n_{t_n}}|)}\}^{-1}$$

that is

$$|a_t| \;<\; |a_t^{(r_1)}| \;+\; \frac{\theta}{1-\theta}\,\{(|\lambda_{n_{t_n}}|)^{c_1|\lambda_{n_{t_n}}|}\,e^{c_2|t|\,(|\lambda_{n_{t_n}}|)}\}^{-1}$$

and since

$$\lim_{|t|\to\infty} \frac{\log|a_t^{(r_1)}|}{|\lambda_{n_{t_n}}|} \;=\; -\infty.$$

Hence it follows

$$\lim_{|t|\to\infty} \frac{\log|a_t|}{|\lambda_{n_{t_n}}|} \;=\; -\infty.$$

Thus

$$f(s) = \sum_{t=1}^{\infty} a_t\, e^{\lambda_{n_{t_n}} s}$$

represents an entire function such that

$$\|f_{r_1} - f, L_{m_n}\| \;<\; \epsilon \text{ where } r_1 \;\geq\; |M|\,, |m| = 1, 2, \ldots$$

Therefore

$$\|f_{r_1} - f, L_{m_n}\| \;\to\; 0 \text{ as } r_1 \;\to \infty$$

or

$$e(f_{r_1}, f) \;\to\; 0 \;\text{ as } r_1 \;\to \infty$$

This proves (1.c) of Definition (**??**).
Next we need to prove the condition (1.d). Let therefore

$$\beta \;=\; \sum_{m=1}^{\infty} \theta_m(\beta)\beta_m \;\; ; \beta \in L$$

$$\beta_m \equiv \gamma_m \;\text{ and }\; \gamma_m \;=\; e^{\lambda_{n_{m_n}} s}.$$

Then we show $\theta_m(\beta)$ is a continuous linear functional of β in L for each $|m| \geq 1$. Clearly θ_m is linear and since L is endowed with the topology given by the metric e and is a topological vector space. Therefore it is sufficient to prove that $(\theta_m(\beta))$ is continuous.
Let $\{\mu_s\} \subset L$ and suppose $e(\mu_s, 0) \;<\; \epsilon$ for $|s| \geq |s_o|$ where $|s| \geq 1$, then

$$\mu_s \;=\; \sum_{m=1}^{\infty} \theta_m(\mu_s)\beta_m.$$

Again if

$$\mu_s^{(M)} \;=\; \sum_{m=1}^{M} \theta_m(\mu_s)\beta_m$$

then $e(\mu_s^{(M)}, \mu_s) < \epsilon$ for $|M| \geq |M_o|$. Hence

$$\begin{aligned} e(\mu_s^{(M)}, 0) &\leq e(\mu_s^{(M)}, \mu_s) + e(\mu_s, 0) \\ &< 2\epsilon \text{ for all } |M| \geq |M_o|, |s| \geq |s_o|. \end{aligned}$$

Also

$$\|\mu_s^{(M)}, L_{m_n}\| - \|\mu_s^{(M-1)}, L_{m_n}\| = (|\lambda_{n_{M_n}}|)^{c_1|\lambda_{n_{M_n}}|} e^{c_2|M|\,(|\lambda_{n_{M_n}}|)} |\theta_M(\mu_s)|$$

where

$$\|\mu_s^{(M)}, L_{m_n}\| = \sup_{|m|>1} (|\lambda_{n_{m_n}}|)^{c_1|\lambda_{n_{m_n}}|} e^{c_2|m|(|\lambda_{n_{m_n}}|)} |\theta_m(\mu_s)|.$$

But

$$\|\mu_s^{(M)}, L_{m_n}\| < \epsilon \text{ for } |M| \geq |M_o|, |s| \geq |s_o|, |m| \geq 1$$

Therefore

$$|\theta_m(\mu_s)| < \epsilon \text{ for } |s| \geq |s_o|, |m| \geq 1.$$

Hence the theorem.

Linear Functionals: In this section continuous linear functionals on the space L have been characterized when L is endowed with the topology given by the norms $\{\|\ldots, L_{m_n}\| : n = 1, 2, \ldots\}$. **Theorem 2** Every continuous linear functional θ on the normed linear space
$(L, \|\ldots, L_{m_n}\|; |n| = 1, 2, \ldots)$ is of the form

$$\theta(f) = \sum_{m=1}^{\infty} a_m \mu_m ; \; f(s) = \sum_{m=1}^{\infty} a_m e^{\lambda_{n_{m_n}} s}$$

where

$$\{|\mu_m|/(|\lambda_{n_{m_n}}|)^{c_1|\lambda_{n_{m_n}}|} e^{c_2|m|\,(|\lambda_{n_{m_n}}|)}\}$$

is bounded.

Proof. Let θ be a continuous linear functional on the normed linear space $(L, \|\ldots, L_{m_n}\|; |n| = 1, 2, \ldots)$ and so there exists a positive constant G such that

$$|\theta(f)| \leq G\|f, L_{m_n}\| \text{ for all } f(s) \in L.$$

Let

$$f_M(s) = \sum_{m=1}^{M} a_m e^{\lambda_{n_{m_n}} s}$$

then

$$\|f - f_M, L_{m_n}\| = \sup_{|m|\geq|M|+1} (|\lambda_{n_{m_n}}|)^{c_1|\lambda_{n_{m_n}}|} e^{c_2|m|\,(|\lambda_{n_{m_n}}|)} |a_m|.$$

The above expression can be made as small as we want by making M large enough, one gets

$$\|f - f_M, L_{m_n}\| \to 0 \text{ as } |M| \to \infty.$$

Thus

$$\theta(f) = \lim_{M\to\infty} \theta(f_M) = \lim_{M\to\infty} \left(\sum_{m=1}^{M} a_m \mu_m \right)$$

where $\mu_m = \theta(e^{\lambda_{n_{m_n}} s})$. Now

$$|\mu_m| = |\theta(e^{\lambda_{n_{m_n}} s})| \leq G\|e^{\lambda_{n_{m_n}} s}, L_{m_n}\|$$

that is

$$|\mu_m| \leq G(|\lambda_{n_{m_n}}|)^{c_1|\lambda_{n_{m_n}}|} e^{c_2|m|\,(|\lambda_{n_{m_n}}|)}$$

Therefore

$$\frac{|\mu_m|}{(|\lambda_{n_{m_n}}|)^{c_1|\lambda_{n_{m_n}}|}\, e^{c_2|m|\,(|\lambda_{n_{m_n}}|)}} \le G. \tag{6}$$

Hence

$$\theta(f) = \sum_{m=1}^{\infty} a_m \mu_m \tag{7}$$

is convergent where μ_m is given by (6). This completes the proof of the theorem.
Theorem 3 If $\{\gamma_m\}$ forms a base for L that is for $\gamma \in L$,

$$\gamma = \sum_{m=1}^{\infty} \theta_m(\gamma)\gamma_m.$$

Let us define a metric $\zeta(\gamma, \gamma^{'})$ as follows

$$\zeta(\gamma, \gamma^{'}) = \sup \|(\theta_1(\gamma) - \theta_1(\gamma^{'}))\gamma_1 + \ldots + (\theta_m(\gamma) - \theta_m(\gamma^{'}))\gamma_m\|.$$

Then L is complete with respect to the metric ζ.
Proof. Let $\{\lambda_r\}$ be a sequence of entire functions in L such that $\zeta(\lambda_r, \lambda_s) < \epsilon$ for $|r|, |s| \ge |r_o|$. That is $\{\lambda_r\}$ is a ζ - cauchy sequence in L. Hence for each given $\epsilon > 0$ there exists $|r_o| = |r_o(\epsilon)|$ such that

$$\sup \|\sum_{i=1}^{|m|}(\phi_i(\lambda_r) - \phi_i(\lambda_s))\gamma_i\| < \epsilon \text{ for } |r|, |s| \ge |r_o|.$$

This implies $\|(\phi_i(\lambda_r) - \phi_i(\lambda_s))\gamma_i\| < \epsilon$ for $|r|, |s| \ge |r_o|$, $|i| \ge 1$.
Since $\gamma_i \ne 0$ for $|i| \ge 1$,

$$\|\phi_i(\lambda_r) - \phi_i(\lambda_s)\| < \epsilon \text{ for } |r|, |s| \ge |r_o|.$$

Therefore $\{\phi_i(\lambda_r)\}$ being a cauchy sequence in the usual topology of the complex plane tends to ϕ_i as $|r| \to \infty$.

$$\|\sum_{i=1}^{|m|}(\phi_i(\lambda_r) - \phi_i)\gamma_i\| < \epsilon \text{ for } |r| \ge |r_o|.$$

Now for $|r| = |r_o|$ and $\gamma = \lambda_{r_o}$,

$$\|\sum_{i=1}^{|m|}\phi_i(\lambda_{r_o})\gamma_i - \sum_{i=1}^{|n|}\phi_i(\lambda_{r_o})\gamma_i\| < \epsilon \text{ for } |n|, |m| \ge |n_o|.$$

Therefore

$$\|\sum_{i=1}^{|m|}\phi_i\gamma_i - \sum_{i=1}^{|n|}\phi_i\gamma_i\| \le \|\sum_{i=1}^{|m|}(\phi_i - \phi_i(\lambda_{r_o}))\gamma_i\| +$$

$$\|\sum_{i=1}^{|n|}(\phi_i - \phi_i(\lambda_{r_o}))\gamma_i\| + \|\sum_{i=1}^{|m|}\phi_i(\lambda_{r_o})\gamma_i - \sum_{i=1}^{|n|}\phi_i(\lambda_{r_o})\gamma_i\|$$

This implies

$$\|\sum_{i=1}^{|m|}\phi_i\gamma_i - \sum_{i=1}^{|n|}\phi_i\gamma_i\| < 3\epsilon \text{ for } |n|, |m| \ge |n_o|.$$

Hence $\{\sum_{i=1}^{|m|} \phi_i \gamma_i\}$ converges to λ as L is complete with respect to the metric e. Thus $\phi_i = \phi_i(\lambda)$. Therefore $\zeta(\lambda_r, \lambda) < \epsilon$, $|r| \geq |r_o|$. Hence $\{\lambda_r\}$ converges to λ where $\lambda \in L$ which proves the theorem. **Theorem 4** The space L_e is a Frechet space where e is the metric defined on L.

Proof. L_e is a normed linear metric space. In above theorem it has been proved that L_e is complete with respect to the metric e. Thus L_e is a Frechet space.

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