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EXACT SOLUTIONS AND STABILITY OF SIXTH ORDER DIFFERENCE EQUATIONS

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Abstract. Some natural phenomena are sometimes modeled by using recursive equations. Therefore, extracting exact solutions of such equations plays a significant role in explaining the future pattern of these problems. It is difficult sometimes to establish the exact solutions of some difference equations. Consequently, this work investigates the equilibrium points, local stability, global stability and periodicity of two difference equations. This paper also aims to find the analytic solutions of the respective equations and plots some 2D figures for some obtained results. The used method can be easily applied for other high-order difference equations.

1. Introduction

The investigation of difference equations has become an active topic for some scholars. This can be mainly attributed to the fact that most difference equations are extracted from modeling some natural phenomena (for example, chemical, physical, biological, social, economical and engineering problems) or from discretising some differential equations. Elaydi [1] used difference equations to model various applications such as the trade model, the propagation of annual plants, the transmission of information model, the host–parasitoid systems, the business cycle model, the larval–pupal–adult (LPA) model and the Nicholson-Bailey model. Murray [2] studied a single species population growth in discrete steps. According to [2] difference equations describe wide spectrum of biomedical applications such as cancer growth, ageing, cell proliferation and genetics. Water waves are modeled by partial differential equations from which one can obtain difference equations solved by some specific methods.

Some properties of recursive equations such as equilibria, local stability, global stability, boundedness and periodicity are usually discussed theoretically. More specifically, some scientists studied the qualitative behavior of some difference equations. For example, Almatrafi and Alzubaidi [3] investigated the local and global

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attractivity, periodicity and the analytic solutions of the following difference equation:

$$x_{n+1} = c_1 x_{n-3} + \frac{c_2 x_{n-3}}{c_3 x_{n-3} - c_4 x_{n-7}}, \ n = 0, 1, ...$$

Elabbasy et al. [4] studied the qualitative properties of the difference equation

$$x_{n+1} = a x_n - \frac{b x_n}{c x_n - d x_{n-1}}, \ n = 0, 1, ...$$

Alayachi et al. [5] discussed the qualitative behaviors of the difference equation

$$y_{n+1} = A y_{n-1} + \frac{B y_{n-1} y_{n-3}}{C y_{n-3} + D y_{n-5}}, \ n = 0, 1, ...$$

The authors in [6] examined the local stability, global stability, periodicity and the solutions of the difference equation

$$u_{m+1} = a u_{m-1} + \frac{b u_{m-1} u_{m-4}}{c u_{m-4} - d u_{m-6}}, \quad m = 0, 1, ...$$

Garić-Demirović et al. [7] studied the stability of the difference equation

$$x_{n+1} = \frac{A x_n^2 + B x_n x_{n-1} + C x_{n-1}^2}{a x_n^2 + b x_n x_{n-1} + c x_{n-1}^2}, \ n = 0, 1, ...$$

The study in [8] concentrates on discussing the periodicity and the stability of the equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-k}}{B x_n + C x_{n-k}}.$$

More results about difference equations can be found in the refs. [10]-[19].

This work aims to investigate the equilibria, local stability, global attractivity and the exact solutions of the following difference equations

$$u_{n+1} = \alpha u_{n-1} + \frac{\beta u_{n-1} u_{n-5}}{\gamma u_{n-3} - \delta u_{n-5}}, \quad n = 0, 1, ..., \tag{1}$$

$$u_{n+1} = \alpha u_{n-1} - \frac{\beta u_{n-1} u_{n-5}}{\gamma u_{n-3} + \delta u_{n-5}}, \quad n = 0, 1, ..., \tag{2}$$

where the coefficients α, β, γ, and δ are positive real numbers and the initial conditions u_i *for all* $i = -5, -4, ..., 0$, are arbitrary non-zero real numbers. We also present the numerical solutions via some 2D graphs.

2. On the Equation $u_{n+1} = \alpha u_{n-1} + \frac{\beta u_{n-1} u_{n-5}}{\gamma u_{n-3} - \delta u_{n-5}}$

This section is devoted to study the qualitative behaviors of Eq. (1). The equilibrium point of Eq. (1) is given by

$$\bar{u} = \alpha \bar{u} + \frac{\beta \bar{u}^2}{\gamma \bar{u} - \delta \bar{u}},$$

which leads to $\bar{u} = 0$, if $(\gamma - \delta)(1 - \alpha) \neq \beta$.

3. Local Stability

In this section, we examine the local stability about the equilibrium point of Eq. (1). In order to discuss the stability of Eq. (1), we first define the function $g : (0,\infty)^3 \longrightarrow (0,\infty)$ by

$$g(x,y,z) = \alpha x + \frac{\beta xz}{\gamma y - \delta z}. \tag{3}$$

Then,

$$\frac{\partial g(x,y,z)}{\partial x} = \alpha + \frac{\beta z}{\gamma y - \delta z}, \tag{4}$$

$$\frac{\partial g(x,y,z)}{\partial y} = -\frac{\beta \gamma xz}{(\gamma y - \delta z)^2}, \tag{5}$$

$$\frac{\partial g(x,y,z)}{\partial z} = \frac{\beta \gamma xy}{(\gamma y - \delta z)^2}. \tag{6}$$

Now, we evaluate Eqs. (4), (5) and Eq. (6) at $\bar{u}$. That is

$$\frac{\partial g(\bar{u},\bar{u},\bar{u})}{\partial x} = \alpha + \frac{\beta \bar{u}}{\gamma \bar{u} - \delta \bar{u}} = \alpha + \frac{\beta}{\gamma - \delta} := -q_1,$$
$$\frac{\partial g(\bar{u},\bar{u},\bar{u})}{\partial y} = -\frac{\beta \gamma \bar{u}^2}{(\gamma \bar{u} - \delta \bar{u})^2} = -\frac{\beta \gamma}{(\gamma - \delta)^2} := -q_2,$$
$$\frac{\partial g(\bar{u},\bar{u},\bar{u})}{\partial z} = \frac{\beta \gamma \bar{u}^2}{(\gamma \bar{u} - \delta \bar{u})^2} = \frac{\beta \gamma}{(\gamma - \delta)^2} := -q_3.$$

Thus, the linearised equation of Eq. (1) about the equilibrium point is given by

$$v_{n+1} + q_1 v_{n-1} + q_2 v_{n-3} + q_3 v_{n-5} = 0.$$

Theorem 1 Assume that

$$|\alpha(\gamma - \delta) + \beta|\,|\gamma - \delta| < (\gamma - \delta)^2 - 2\beta\gamma.$$

Then, the equilibrium point $\bar{u} = 0$ is locally asymptotically stable.

Proof. Theorem A in [20] guarantees that the stability of the equilibrium point occurs if

$$|q_1| + |q_2| + |q_3| < 1. \tag{7}$$

Plugging $q_i,\ i = 1,2,3$, into Eq. (7) leads to

$$\left|-\left(\alpha + \frac{\beta}{\gamma - \delta}\right)\right| + \left|\frac{\beta\gamma}{(\gamma - \delta)^2}\right| + \left|-\frac{\beta\gamma}{(\gamma - \delta)^2}\right| < 1.$$

Hence,

$$\left|\alpha + \frac{\beta}{\gamma - \delta}\right| + \frac{2\beta\gamma}{(\gamma - \delta)^2} < 1,$$

which can be written as

$$|\alpha(\gamma - \delta) + \beta|\,|\gamma - \delta| < (\gamma - \delta)^2 - 2\beta\gamma.$$

4. Global Stability of the Equilibrium Point

The global stability is investigated in this section. We use Theorem B in [21] to show the stability.

Theorem 2 Assume that $\gamma y > \delta z$, then the equilibrium point of Eq. (1) is a global attractor if $\delta(\alpha - 1) \neq \beta$.

Proof. Assume that $p,\ q \in \mathbb{R}$ and let $g : [p,q]^3 \longrightarrow [p,q]$ is a function defined by Eq. (3). Since $\gamma y > \delta z$, the function g is increasing in x and z and decreasing in y. Let (ϕ, ψ) be a solution for the following system:

$$\phi = g(\phi, \psi, \phi),\ \psi = g(\psi, \phi, \psi).$$

Plugging this into Eq. (1) leads to

$$\phi = g(\phi, \psi, \phi) = \alpha\phi + \frac{\beta\phi^2}{\gamma\psi - \delta\phi},$$
$$\psi = g(\psi, \phi, \psi) = \alpha\psi + \frac{\beta\psi^2}{\gamma\phi - \delta\psi},$$

which can be rewritten as

$$\gamma\phi\psi - \delta\phi^2 = \alpha\gamma\phi\psi - \alpha\delta\phi^2 + \beta\phi^2, \tag{8}$$
$$\gamma\phi\psi - \delta\psi^2 = \alpha\gamma\phi\psi - \alpha\delta\psi^2 + \beta\psi^2. \tag{9}$$

Subtracting Eq. (9) from Eq. (8) leads to

$$[\delta(1-\alpha) + \beta](\psi^2 - \phi^2) = 0.$$

Hence, if $\delta(\alpha - 1) \neq \beta$, then $\phi = \psi$. Therefore, Theorem B in [21] ensures that the equilibrium point is a global attractor.

Theorem 3 Let $\alpha + \frac{\beta z}{\gamma y - \delta z} < 0$, then the equilibrium point of Eq. (1) is a global attractor if $\delta \neq \alpha\gamma$.

Proof. The proof is omitted.

5. Periodicity of the Solutions

This section is devoted to discuss the prime period two solutions of Eq. (1).

Theorem 4 Equation (1) has no prime period two solutions.

Proof. We will use contradiction to prove this theorem. Suppose that Eq. (1) has positive prime period two solutions given by

$$..., U_1, U_2, U_1, U_2, ...$$

Then,

$$U_1 = \alpha U_1 + \frac{\beta U_1^2}{\gamma U_1 - \delta U_1},$$
$$U_2 = \alpha U_2 + \frac{\beta U_2^2}{\gamma U_2 - \delta U_2}.$$

Or,

$$(1-\alpha)U_1 = \frac{\beta U_1}{\gamma - \delta},$$
$$(1-\alpha)U_2 = \frac{\beta U_2}{\gamma - \delta},$$

which implies that $U_1 = U_2$. This contradicts the fact that $U_1 \neq U_2$.

6. Exact Solution of Eq. (1) when $\alpha = \beta = \gamma = \delta = 1$

In this section, we investigate the exact solutions of the following rational difference equation

$$u_{n+1} = u_{n-1} + \frac{u_{n-1}u_{n-5}}{u_{n-3} - u_{n-5}}, \quad n = 0, 1, ..., \tag{10}$$

where the initial conditions are positive real numbers.

Theorem 5 Let $\{u_n\}_{n=-5}^{\infty}$ be a solution to Eq. (10) and suppose that $u_{-5} = a,\ u_{-4} = b,\ u_{-3} = c,\ u_{-2} = d,\ u_{-1} = e,\ u_0 = f$. Then, for $n = 0, 1, 2, ...$, the solutions of Eq. (10) are given by the following formulas:

$$\begin{aligned}
u_{8n-5} &= \frac{e^{2n}c^n}{a^{n-1}(c-e)^n(a-c)^n}, \\
u_{8n-4} &= \frac{f^{2n}d^n}{b^{n-1}(d-f)^n(b-d)^n}, \\
u_{8n-3} &= \frac{c^{n+1}e^{2n}}{a^n(a-c)^n(c-e)^n}, \\
u_{8n-2} &= \frac{d^{n+1}f^{2n}}{b^n(b-d)^n(d-f)^n}, \\
u_{8n-1} &= \frac{e^{2n+1}c^n}{a^n(a-c)^n(c-e)^n}, \\
u_{8n} &= \frac{f^{2n+1}d^n}{b^n(b-d)^n(d-f)^n}, \\
u_{8n+1} &= -\frac{c^{n+1}e^{2n+1}}{a^n(c-e)^n(a-c)^{n+1}}, \\
u_{8n+2} &= -\frac{d^{n+1}f^{2n+1}}{b^n(d-f)^n(b-d)^{n+1}}.
\end{aligned}$$

Proof. It can be easily observed that the solutions are true for $n = 0$. We now assume that $n > 0$ and that our assumption holds for $n - 1$. That is,

$$\begin{aligned}
u_{8n-13} &= \frac{e^{2n-2}c^{n-1}}{a^{n-2}(c-e)^{n-1}(a-c)^{n-1}}, \\
u_{8n-12} &= \frac{f^{2n-2}d^{n-1}}{b^{n-2}(d-f)^{n-1}(b-d)^{n-1}}, \\
u_{8n-11} &= \frac{c^n e^{2n-2}}{a^{n-1}(a-c)^{n-1}(c-e)^{n-1}},
\end{aligned}$$

$$
\begin{aligned}
u_{8n-10} &= \frac{d^n f^{2n-2}}{b^{n-1}(b-d)^{n-1}(d-f)^{n-1}},\\
u_{8n-9} &= \frac{e^{2n-1}c^{n-1}}{a^{n-1}(a-c)^{n-1}(c-e)^{n-1}},\\
u_{8n-8} &= \frac{f^{2n-1}d^{n-1}}{b^{n-1}(b-d)^{n-1}(d-f)^{n-1}},\\
u_{8n-7} &= -\frac{c^n e^{2n-1}}{a^{n-1}(c-e)^{n-1}(a-c)^{n}},\\
u_{8n-6} &= -\frac{d^n f^{2n-1}}{b^{n-1}(d-f)^{n-1}(b-d)^{n}}.
\end{aligned}
$$

Eq. (10) gives us that

$$
\begin{aligned}
u_{8n-5} &= u_{8n-7} + \frac{u_{8n-7}u_{8n-11}}{u_{8n-9}-u_{8n-11}}\\
&= -\frac{c^n e^{2n-1}}{a^{n-1}(c-e)^{n-1}(a-c)^{n}} + \frac{-\frac{c^n e^{2n-1}}{a^{n-1}(c-e)^{n-1}(a-c)^{n}}\frac{c^n e^{2n-2}}{a^{n-1}(a-c)^{n-1}(c-e)^{n-1}}}{\frac{e^{2n-1}c^{n-1}}{a^{n-1}(a-c)^{n-1}(c-e)^{n-1}} - \frac{c^n e^{2n-2}}{a^{n-1}(a-c)^{n-1}(c-e)^{n-1}}}\\
&= -\frac{c^n e^{2n-1}}{a^{n-1}(c-e)^{n-1}(a-c)^{n}} - \frac{e^{2n-2}c^n}{a^{n-1}(c-e)^{n-1}(a-c)^{n}\left(\frac{1}{c}-\frac{1}{e}\right)}\\
&= -\frac{c^n e^{2n-1}}{a^{n-1}(c-e)^{n-1}(a-c)^{n}} + \frac{e^{2n-1}c^{n+1}}{a^{n-1}(c-e)^{n}(a-c)^{n}}\\
&= \frac{e^{2n}c^n}{a^{n-1}(c-e)^{n}(a-c)^{n}}.
\end{aligned}
$$

Moreover, it can be seen from Eq. (10) that

$$
\begin{aligned}
u_{8n-4} &= u_{8n-6} + \frac{u_{8n-6}u_{8n-10}}{u_{8n-8}-u_{8n-10}}\\
&= -\frac{d^n f^{2n-1}}{b^{n-1}(d-f)^{n-1}(b-d)^{n}} + \frac{-\frac{d^n f^{2n-1}}{b^{n-1}(d-f)^{n-1}(b-d)^{n}}\frac{d^n f^{2n-2}}{b^{n-1}(b-d)^{n-1}(d-f)^{n-1}}}{\frac{f^{2n-1}d^{n-1}}{b^{n-1}(b-d)^{n-1}(d-f)^{n-1}} - \frac{d^n f^{2n-2}}{b^{n-1}(b-d)^{n-1}(d-f)^{n-1}}}\\
&= -\frac{d^n f^{2n-1}}{b^{n-1}(d-f)^{n-1}(b-d)^{n}} - \frac{d^n f^{2n-2}}{b^{n-1}(d-f)^{n-1}(b-d)^{n}\left(\frac{1}{d}-\frac{1}{f}\right)}\\
&= -\frac{d^n f^{2n-1}}{b^{n-1}(d-f)^{n-1}(b-d)^{n}} + \frac{d^{n+1}f^{2n-1}}{b^{n-1}(d-f)^{n}(b-d)^{n}}\\
&= \frac{f^{2n}d^n}{b^{n-1}(d-f)^{n}(b-d)^{n}}.
\end{aligned}
$$

Furthermore, Eq. (10) gives us

$$
\begin{aligned}
u_{8n-3} &= u_{8n-5} + \frac{u_{8n-5}u_{8n-9}}{u_{8n-7} - u_{8n-9}} \\
&= \frac{e^{2n}c^{n}}{a^{n-1}(c-e)^{n}(a-c)^{n}} + \frac{\frac{e^{2n}c^{n}}{a^{n-1}(c-e)^{n}(a-c)^{n}}\,\frac{e^{2n-1}c^{n-1}}{a^{n-1}(a-c)^{n-1}(c-e)^{n-1}}}{-\frac{c^{n}e^{2n-1}}{a^{n-1}(c-e)^{n-1}(a-c)^{n}} - \frac{e^{2n-1}c^{n-1}}{a^{n-1}(a-c)^{n-1}(c-e)^{n-1}}} \\
&= \frac{e^{2n}c^{n}}{a^{n-1}(c-e)^{n}(a-c)^{n}} - \frac{e^{2n}c^{n}}{a^{n}(c-e)^{n}(a-c)^{n-1}} \\
&= \frac{e^{2n}c^{n}}{(c-e)^{n}(a-c)^{n}}\left(\frac{a}{a^{n}} - \frac{a-c}{a^{n}}\right) \\
&= \frac{e^{2n}c^{n+1}}{a^{n}(c-e)^{n}(a-c)^{n}}.
\end{aligned}
$$

Also, Eq. (10) leads to

$$
\begin{aligned}
u_{8n-2} &= u_{8n-4} + \frac{u_{8n-4}u_{8n-8}}{u_{8n-6} - u_{8n-8}} \\
&= \frac{f^{2n}d^{n}}{b^{n-1}(d-f)^{n}(b-d)^{n}} + \frac{\frac{f^{2n}d^{n}}{b^{n-1}(d-f)^{n}(b-d)^{n}}\,\frac{f^{2n-1}d^{n-1}}{b^{n-1}(b-d)^{n-1}(d-f)^{n-1}}}{-\frac{d^{n}f^{2n-1}}{b^{n-1}(d-f)^{n-1}(b-d)^{n}} - \frac{f^{2n-1}d^{n-1}}{b^{n-1}(b-d)^{n-1}(d-f)^{n-1}}} \\
&= \frac{f^{2n}d^{n}}{b^{n-1}(d-f)^{n}(b-d)^{n}} - \frac{f^{2n}d^{n}}{b^{n}(d-f)^{n}(b-d)^{n-1}} \\
&= \frac{f^{2n}d^{n}}{b^{n}(d-f)^{n}(b-d)^{n}}\left(b - \frac{1}{(b-d)^{-1}}\right) \\
&= \frac{f^{2n}d^{n+1}}{b^{n}(d-f)^{n}(b-d)^{n}}.
\end{aligned}
$$

Moreover, Eq. (10) gives

$$
\begin{aligned}
u_{8n-1} &= u_{8n-3} + \frac{u_{8n-3}u_{8n-7}}{u_{8n-5} - u_{8n-7}} \\
&= \frac{c^{n+1}e^{2n}}{a^{n}(a-c)^{n}(c-e)^{n}} + \frac{\frac{c^{n+1}e^{2n}}{a^{n}(a-c)^{n}(c-e)^{n}}\left(-\frac{c^{n}e^{2n-1}}{a^{n-1}(c-e)^{n-1}(a-c)^{n}}\right)}{\frac{e^{2n}c^{n}}{a^{n-1}(c-e)^{n}(a-c)^{n}} + \frac{c^{n}e^{2n-1}}{a^{n-1}(c-e)^{n-1}(a-c)^{n}}} \\
&= \frac{c^{n+1}e^{2n}}{a^{n}(a-c)^{n}(c-e)^{n}} - \frac{c^{n}e^{2n}}{a^{n}(c-e)^{n-1}(a-c)^{n}\left(1+\frac{c-e}{c}\right)} \\
&= \frac{c^{n+1}e^{2n}}{a^{n}(a-c)^{n}(c-e)^{n}} - \frac{c^{n}e^{2n}}{a^{n}(c-e)^{n-1}(a-c)^{n}} \\
&= \frac{c^{n}e^{2n+1}}{a^{n}(a-c)^{n}(c-e)^{n}}.
\end{aligned}
$$

Finally, Eq. (10) gives

$$\begin{aligned}
u_{8n} &= u_{8n-2} + \frac{u_{8n-2}u_{8n-6}}{u_{8n-4} - u_{8n-6}} \\
&= \frac{d^{n+1}f^{2n}}{b^n(b-d)^n(d-f)^n} + \frac{\frac{d^{n+1}f^{2n}}{b^n(b-d)^n(d-f)^n}\left(-\frac{d^n f^{2n-1}}{b^{n-1}(d-f)^{n-1}(b-d)^n}\right)}{\frac{f^{2n}d^n}{b^{n-1}(d-f)^n(b-d)^n} + \frac{d^n f^{2n-1}}{b^{n-1}(d-f)^{n-1}(b-d)^n}} \\
&= \frac{d^{n+1}f^{2n}}{b^n(b-d)^n(d-f)^n} - \frac{d^n f^{2n}}{b^n(b-d)^n(d-f)^{n-1}} \\
&= \frac{d^n f^{2n+1}}{b^n(b-d)^n(d-f)^n}.
\end{aligned}$$

Other solutions can be similarly proved.

7. Numerical Examples

The above-mentioned theoretical results are plotted in this section. The values of the parameters are selected according to the above conditions.

Example 1. The local stability in the neighborhood of the equilibrium point is plotted in Figure 1 under the assumptions $\alpha = \beta = 0.5$, $\gamma = 1$, $\delta = 3$, $u_{-5} = -0.2$, $u_{-4} = 0.2$, $u_{-3} = -0.1$, $u_{-2} = 0.01$, $u_{-1} = 0.1$, $u_0 = -0.1$.

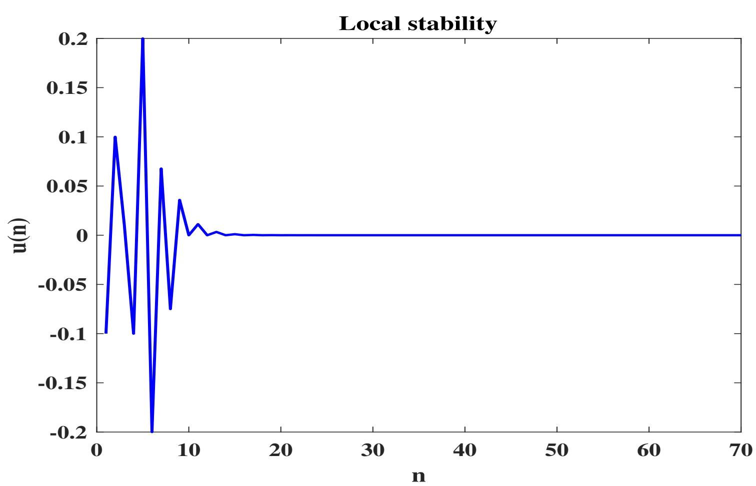

Figure 1. Local stability about the equilibrium.

Example 2. Figure 2 illustrates the global stability of the equilibrium point under the values $\alpha = \beta = 0.5$, $\gamma = 0.1$, $\delta = 3$, $u_{-5} = 5$, $u_{-4} = -5$, $u_{-3} = 4$, $u_{-2} = -3$, $u_{-1} = 2.5$, $u_0 = -2.5$.

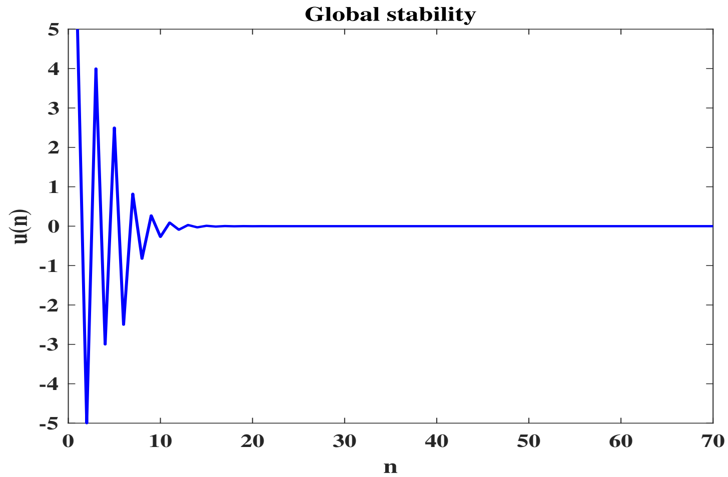

FIGURE 2. Global stability about the equilibrium.

Example 3. In Figure 3, we plot the analytical solution of Eq. (10) under the values $\alpha = \beta = \gamma = \delta = 1$, $u_{-5} = -0.752$, $u_{-4} = -1.3$, $u_{-3} = 5$, $u_{-2} = -0.5$, $u_{-1} = 0.1$, $u_0 = 1.3$.

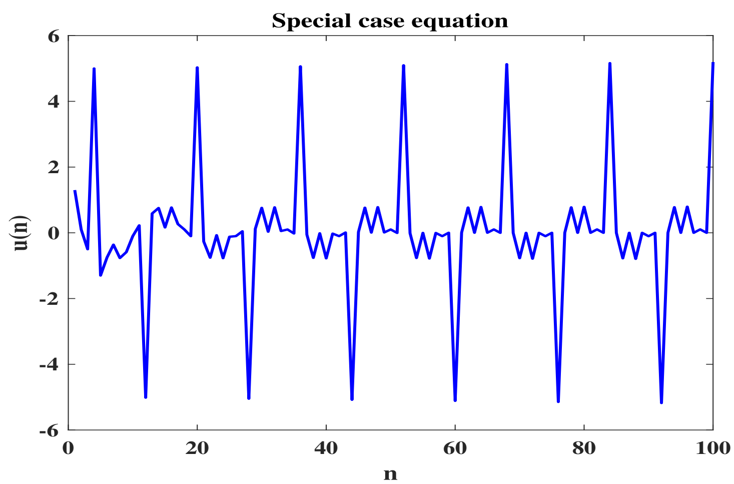

FIGURE 3. The behaviour of Eq.(10) at $u_{-5} = -0.752$, $u_{-4} = -1.3$, $u_{-3} = 5$, $u_{-2} = -0.5$, $u_{-1} = 0.1$, $u_0 = 1.3$.

8. ON THE EQUATION $u_{n+1} = \alpha u_{n-1} - \frac{\beta u_{n-1}u_{n-5}}{\gamma u_{n-3}+\delta u_{n-5}}$

The equilibrium point of Eq. (2) is given by

$$\bar{u} = \alpha\bar{u} - \frac{\beta\bar{u}^2}{\gamma\bar{u} + \delta\bar{u}}.$$

Hence, $\bar{u} = 0$, if $(\gamma + \delta)(\alpha - 1) \neq \beta$.

9. Local Stability

This section presents the local stability about the fixed point. In order to investigate the stability of Eq. (2), we define the function $h:(0,\infty)^3 \longrightarrow (0,\infty)$ by

$$h(x,y,z)=\alpha x-\frac{\beta xz}{\gamma y+\delta z}. \tag{11}$$

Then,

$$\frac{\partial h(x,y,z)}{\partial x} = \alpha-\frac{\beta z}{\gamma y+\delta z}, \tag{12}$$

$$\frac{\partial h(x,y,z)}{\partial y} = \frac{\beta\gamma xz}{(\gamma y+\delta z)^2}, \tag{13}$$

$$\frac{\partial h(x,y,z)}{\partial z} = -\frac{\beta\gamma xy}{(\gamma y+\delta z)^2}. \tag{14}$$

Calculating Eqs. (12), (13) and Eq. (14) at $\bar{u}$ yields

$$\begin{aligned}
\frac{\partial h(\bar{u},\bar{u},\bar{u})}{\partial x} &= \alpha-\frac{\beta\bar{u}}{\gamma\bar{u}+\delta\bar{u}}=\alpha-\frac{\beta}{\gamma+\delta}:=-p_1,\\
\frac{\partial h(\bar{u},\bar{u},\bar{u})}{\partial y} &= \frac{\beta\gamma\bar{u}^2}{(\gamma\bar{u}+\delta\bar{u})^2}=\frac{\beta\gamma}{(\gamma+\delta)^2}:=-p_2,\\
\frac{\partial h(\bar{u},\bar{u},\bar{u})}{\partial z} &= -\frac{\beta\gamma\bar{u}^2}{(\gamma\bar{u}+\delta\bar{u})^2}=-\frac{\beta\gamma}{(\gamma+\delta)^2}:=-p_3.
\end{aligned}$$

Consequently, the linearised equation of Eq. (2) about the equilibrium point is given by

$$v_{n+1}+p_1v_{n-1}+p_2v_{n-3}+p_3v_{n-5}=0.$$

Theorem 6 Assume that $\beta<\alpha(\gamma+\delta)$. Then, the equilibrium point $\bar{u}=0$ is locally asymptotically stable if

$$\beta(\gamma-\delta)\leq(1-\alpha)(\gamma+\delta)^2.$$

Proof. Theorem A in [20] guarantees that the stability of the equilibrium point occurs if

$$|p_1|+|p_2|+|p_3|<1. \tag{15}$$

Substituting $p_i,\ i=1,2,3$, into Eq. (15) gives

$$\left|-\left(\alpha-\frac{\beta}{\gamma+\delta}\right)\right|+\left|-\frac{\beta\gamma}{(\gamma+\delta)^2}\right|+\left|\frac{\beta\gamma}{(\gamma+\delta)^2}\right|<1.$$

Assume that $\beta<\alpha(\gamma+\delta)$, then

$$(\alpha(\gamma+\delta)-\beta)(\gamma+\delta)+2\beta\gamma<(\gamma+\delta)^2.$$

Hence,

$$\beta(\gamma-\delta)\leq(1-\alpha)(\gamma+\delta)^2.$$

10. Global Stability of the Equilibrium Point

The global stability is analyzed in this section. Theorem B in [21] is applied to obtain the condition under which the equilibrium point is a global stable.
Theorem 7 The equilibrium point of Eq. (2) is a global attractor if $\alpha \neq 1$.
Proof. Assume that $p,\ q \in \mathbb{R}$ and let $h : [p,q]^3 \longrightarrow [p,q]$ is a function defined by Eq. (11). If $\alpha > \frac{\beta z}{\gamma y+\delta z}$, then from Eqs. (12), (13) and Eq. (14), we observe that h is increasing in x and y and decreasing in z. Let (ϕ,ψ) be a solution for the following system:

$$\phi = g(\phi,\phi,\psi),\ \ \psi = g(\psi,\psi,\phi).$$

Hence,

$$\begin{aligned}\phi = g(\phi,\phi,\psi) = \alpha\phi - \frac{\beta\phi\psi}{\gamma\phi+\delta\psi},\\ \psi = g(\psi,\psi,\phi) = \alpha\psi - \frac{\beta\psi\phi}{\gamma\psi+\delta\phi},\end{aligned}$$

which can be rewritten as

$$\begin{aligned}\gamma\phi^2+\delta\phi\psi &= \alpha\gamma\phi^2+\alpha\delta\phi\psi-\beta\phi\psi, &(16)\\ \gamma\psi^2+\delta\phi\psi &= \alpha\gamma\psi^2+\alpha\delta\phi\psi-\beta\psi\phi. &(17)\end{aligned}$$

Subtracting Eq. (17) from Eq. (16) leads to

$$\gamma(1-\alpha)(\phi^2-\psi^2) = 0.$$

If $\alpha \neq 1$, then $\phi = \psi$. Therefore, Theorem B in [21] guarantees that the equilibrium point is a global attractor.
Theorem 8 Assume that $\alpha < \frac{\beta z}{\gamma y+\delta z}$. Then, the equilibrium point of Eq. (2) is a global attractor if $\gamma + \alpha\delta \neq \beta$.
Proof. The proof is omitted.

11. Exact Solution of Eq. (2) when $\alpha = \beta = \gamma = \delta = 1$

This section shows the exact solutions of the following equation:

$$u_{n+1} = u_{n-1} - \frac{u_{n-1}u_{n-5}}{u_{n-3}+u_{n-5}}, \quad n = 0,1,..., \tag{18}$$

where the initial conditions are selected to be positive real numbers.
Theorem 9 Let $\{u_n\}_{n=-5}^{\infty}$ be a solution to Eq. (18) and suppose that $u_{-5} = a,\ u_{-4} = b,\ u_{-3} = c,\ u_{-2} = d,\ u_{-1} = e,\ u_0 = f$. Then, for $n = 0,1,2,...$, the solutions of Eq. (18) are given by the following formulas:

$$
\begin{aligned}
u_{8n-5} &= \frac{e^{2n}c^{2n-1}}{\prod_{i=1}^{2n-1}(ie+c)(ic+a)},\\
u_{8n-4} &= \frac{f^{2n}d^{2n-1}}{\prod_{i=1}^{2n-1}(if+d)(id+b)},\\
u_{8n-3} &= \frac{c^{2n}e^{2n}}{\prod_{i=1}^{2n}(ic+a)\prod_{i=1}^{2n-1}(ie+c)},\\
u_{8n-2} &= \frac{d^{2n}f^{2n}}{\prod_{i=1}^{2n}(id+b)\prod_{i=1}^{2n-1}(if+d)},\\
u_{8n-1} &= \frac{e^{2n+1}c^{2n}}{\prod_{i=1}^{2n}(ie+c)(ic+a)},\\
u_{8n} &= \frac{f^{2n+1}d^{2n}}{\prod_{i=1}^{2n}(if+d)(id+b)},\\
u_{8n+1} &= \frac{c^{2n+1}e^{2n+1}}{\prod_{i=1}^{2n+1}(ic+a)\prod_{i=1}^{2n}(ie+c)},\\
u_{8n+2} &= \frac{d^{2n+1}f^{2n+1}}{\prod_{i=1}^{2n+1}(id+b)\prod_{i=1}^{2n}(if+d)}.
\end{aligned}
$$

Proof. It can be easily seen that the solutions are true for $n = 0$. We suppose that $n > 0$ and assume that our assumption holds for $n-1$. That is,

$$
\begin{aligned}
u_{8n-13} &= \frac{e^{2n-2}c^{2n-3}}{\prod_{i=1}^{2n-3}(ie+c)(ic+a)},\\
u_{8n-12} &= \frac{f^{2n-2}d^{2n-3}}{\prod_{i=1}^{2n-3}(if+d)(id+b)},\\
u_{8n-11} &= \frac{c^{2n-2}e^{2n-2}}{\prod_{i=1}^{2n-2}(ic+a)\prod_{i=1}^{2n-3}(ie+c)},\\
u_{8n-10} &= \frac{d^{2n-2}f^{2n-2}}{\prod_{i=1}^{2n-2}(id+b)\prod_{i=1}^{2n-3}(if+d)},\\
u_{8n-9} &= \frac{e^{2n-1}c^{2n-2}}{\prod_{i=1}^{2n-2}(ie+c)(ic+a)},\\
u_{8n-8} &= \frac{f^{2n-1}d^{2n-2}}{\prod_{i=1}^{2n-2}(if+d)(id+b)},\\
u_{8n-7} &= \frac{c^{2n-1}e^{2n-1}}{\prod_{i=1}^{2n-1}(ic+a)\prod_{i=1}^{2n-2}(ie+c)},\\
u_{8n-6} &= \frac{d^{2n-1}f^{2n-1}}{\prod_{i=1}^{2n-1}(id+b)\prod_{i=1}^{2n-2}(if+d)}.
\end{aligned}
$$

From Eq. (18), one can have

$$
\begin{aligned}
u_{8n-5} &= u_{8n-7} - \frac{u_{8n-7}u_{8n-11}}{u_{8n-9}+u_{8n-11}} \\
&= \frac{c^{2n-1}e^{2n-1}}{\prod_{i=1}^{2n-1}(ic+a)\prod_{i=1}^{2n-2}(ie+c)} - \frac{\frac{c^{2n-1}e^{2n-1}}{\prod_{i=1}^{2n-1}(ic+a)\prod_{i=1}^{2n-2}(ie+c)}\frac{c^{2n-2}e^{2n-2}}{\prod_{i=1}^{2n-2}(ic+a)\prod_{i=1}^{2n-3}(ie+c)}}{\frac{e^{2n-1}c^{2n-2}}{\prod_{i=1}^{2n-2}(ie+c)(ic+a)} + \frac{c^{2n-2}e^{2n-2}}{\prod_{i=1}^{2n-2}(ic+a)\prod_{i=1}^{2n-3}(ie+c)}} \\
&= \frac{c^{2n-1}e^{2n-1}}{\prod_{i=1}^{2n-1}(ic+a)\prod_{i=1}^{2n-2}(ie+c)} - \frac{c^{2n-1}e^{2n-2}}{\prod_{i=1}^{2n-1}(ic+a)\prod_{i=1}^{2n-2}(ie+c)\prod_{i=1}^{2n-3}(ie+c)\left(\frac{1}{\prod_{i=1}^{2n-2}(ie+c)}+\frac{1}{e\prod_{i=1}^{2n-3}(ie+c)}\right)} \\
&= \frac{c^{2n-1}e^{2n-1}}{\prod_{i=1}^{2n-1}(ic+a)\prod_{i=1}^{2n-2}(ie+c)} - \frac{c^{2n-1}e^{2n-2}}{\prod_{i=1}^{2n-1}(ic+a)\prod_{i=1}^{2n-2}(ie+c)\left(\frac{1}{(2n-2)e+c}+\frac{1}{e}\right)} \\
&= \frac{c^{2n-1}e^{2n-1}}{\prod_{i=1}^{2n-1}(ic+a)\prod_{i=1}^{2n-2}(ie+c)}\left(1-\frac{1}{e\left(\frac{1}{(2n-2)e+c}+\frac{1}{e}\right)}\right) \\
&= \frac{c^{2n-1}e^{2n-1}}{\prod_{i=1}^{2n-1}(ic+a)\prod_{i=1}^{2n-2}(ie+c)}\left(\frac{e}{(2n-1)e+c}\right) \\
&= \frac{e^{2n}c^{2n-1}}{\prod_{i=1}^{2n-1}(ie+c)(ic+a)}.
\end{aligned}
$$

Moreover, Eq. (18) gives us that

$$
\begin{aligned}
u_{8n-4} &= u_{8n-6} - \frac{u_{8n-6}u_{8n-10}}{u_{8n-8}+u_{8n-10}} \\
&= \frac{d^{2n-1}f^{2n-1}}{\prod_{i=1}^{2n-1}(id+b)\prod_{i=1}^{2n-2}(if+d)} - \frac{\frac{d^{2n-1}f^{2n-1}}{\prod_{i=1}^{2n-1}(id+b)\prod_{i=1}^{2n-2}(if+d)}\frac{d^{2n-2}f^{2n-2}}{\prod_{i=1}^{2n-2}(id+b)\prod_{i=1}^{2n-3}(if+d)}}{\frac{f^{2n-1}d^{2n-2}}{\prod_{i=1}^{2n-2}(if+d)(id+b)} + \frac{d^{2n-2}f^{2n-2}}{\prod_{i=1}^{2n-2}(id+b)\prod_{i=1}^{2n-3}(if+d)}} \\
&= \frac{d^{2n-1}f^{2n-1}}{\prod_{i=1}^{2n-1}(id+b)\prod_{i=1}^{2n-2}(if+d)} - \frac{d^{2n-1}f^{2n-2}}{\prod_{i=1}^{2n-1}(id+b)\prod_{i=1}^{2n-2}(if+d)\prod_{i=1}^{2n-3}(if+d)\left(\frac{1}{\prod_{i=1}^{2n-2}(if+d)}+\frac{1}{f\prod_{i=1}^{2n-3}(if+d)}\right)} \\
&= \frac{d^{2n-1}f^{2n-1}}{\prod_{i=1}^{2n-1}(id+b)\prod_{i=1}^{2n-2}(if+d)} - \frac{d^{2n-1}f^{2n-2}}{\prod_{i=1}^{2n-1}(id+b)\prod_{i=1}^{2n-2}(if+d)\left(\frac{1}{(2n-2)f+d}+\frac{1}{f}\right)} \\
&= \frac{d^{2n-1}f^{2n-1}}{\prod_{i=1}^{2n-1}(id+b)\prod_{i=1}^{2n-2}(if+d)} - \frac{d^{2n-1}f^{2n-1}((2n-2)f+d)}{\prod_{i=1}^{2n-1}(id+b)\prod_{i=1}^{2n-2}(if+d)\,((2n-1)f+d)} \\
&= \frac{d^{2n-1}f^{2n-1}}{\prod_{i=1}^{2n-1}(id+b)\prod_{i=1}^{2n-2}(if+d)}\left(1-\frac{(2n-2)f+d}{(2n-1)f+d}\right) \\
&= \frac{d^{2n-1}f^{2n}}{\prod_{i=1}^{2n-1}(id+b)\prod_{i=1}^{2n-2}(if+d)((2n-1)f+d)} \\
&= \frac{f^{2n}d^{2n-1}}{\prod_{i=1}^{2n-1}(if+d)(id+b)}.
\end{aligned}
$$

In addition, Eq. (18) leads to

$$\begin{aligned}
u_{8n-3} &= u_{8n-5} - \frac{u_{8n-5}u_{8n-9}}{u_{8n-7}+u_{8n-9}} \\
&= \frac{e^{2n}c^{2n-1}}{\prod_{i=1}^{2n-1}(ie+c)(ic+a)} - \frac{\frac{e^{2n}c^{2n-1}}{\prod_{i=1}^{2n-1}(ie+c)(ic+a)}\frac{e^{2n-1}c^{2n-2}}{\prod_{i=1}^{2n-2}(ie+c)(ic+a)}}{\frac{c^{2n-1}e^{2n-1}}{\prod_{i=1}^{2n-1}(ic+a)\prod_{i=1}^{2n-2}(ie+c)}+\frac{e^{2n-1}c^{2n-2}}{\prod_{i=1}^{2n-2}(ie+c)(ic+a)}} \\
&= \frac{e^{2n}c^{2n-1}}{\prod_{i=1}^{2n-1}(ie+c)(ic+a)} - \frac{c^{2n-2}e^{2n}}{\prod_{i=1}^{2n-1}(ie+c)(ic+a)\prod_{i=1}^{2n-2}(ic+a)\left(\frac{1}{\prod_{i=1}^{2n-1}(ic+a)}+\frac{1}{c\prod_{i=1}^{2n-2}(ic+a)}\right)} \\
&= \frac{e^{2n}c^{2n-1}}{\prod_{i=1}^{2n-1}(ie+c)(ic+a)} - \frac{c^{2n-2}e^{2n}}{\prod_{i=1}^{2n-1}(ie+c)(ic+a)\left(\frac{1}{(2n-1)c+a}+\frac{1}{c}\right)} \\
&= \frac{e^{2n}c^{2n-1}}{\prod_{i=1}^{2n-1}(ie+c)(ic+a)} - \frac{c^{2n-1}e^{2n}((2n-1)c+a)}{\prod_{i=1}^{2n-1}(ie+c)(ic+a)((2n)c+a)} \\
&= \frac{e^{2n}c^{2n-1}}{\prod_{i=1}^{2n-1}(ie+c)(ic+a)}\left(1-\frac{(2n-1)c+a}{(2n)c+a}\right) \\
&= \frac{e^{2n}c^{2n}}{\prod_{i=1}^{2n-1}(ie+c)(ic+a)((2n)c+a)} = \frac{c^{2n}e^{2n}}{\prod_{i=1}^{2n}(ic+a)\prod_{i=1}^{2n-1}(ie+c)}.
\end{aligned}$$

Similarly, one can prove other solutions.

12. Numerical Examples

This section is assigned to present some 2D figures for the above results.

Example 4. This example illustrates the local stability in the neighborhood of the equilibrium point under the values $\alpha = 0.5$, $\beta = \gamma = 1$, $\delta = 2$, $u_{-5} = 0.05$, $u_{-4} = -0.04$, $u_{-3} = 0.01$, $u_{-2} = 0.08$, $u_{-1} = -0.1$, $u_0 = 0.1$. See Figure 4.

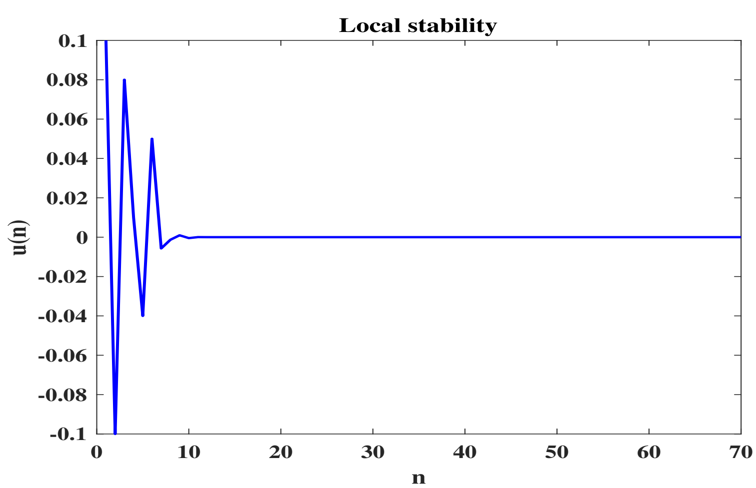

Figure 4. Local stability about the equilibrium.

Example 5. In Figure 5, we present the behavior of the solution of Eq. (2) about the equilibrium point under the selected values $\alpha = 0.8$, $\beta = 0.1$, $\gamma = 2$, $\delta = 10$, $u_{-5} = 2$, $u_{-4} = -4$, $u_{-3} = 1$, $u_{-2} = 5$, $u_{-1} = -6$, $u_0 = 6$. The solution is stable about the equilibrium point.

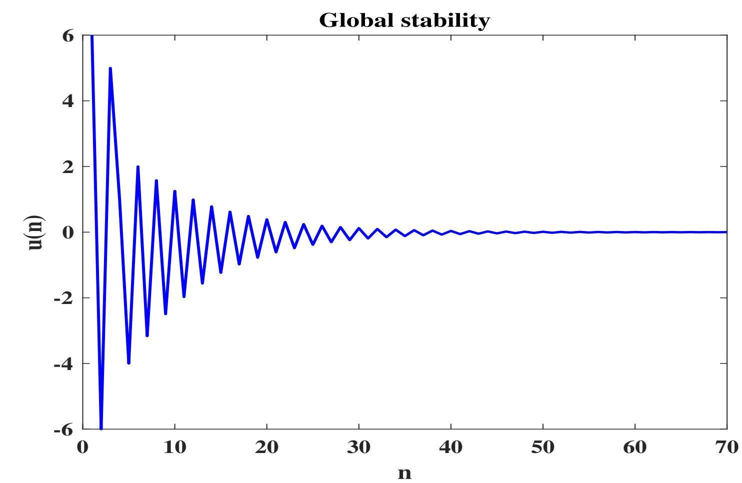

FIGURE 5. Global stability about the equilibrium.

Example 6. The analytic solutions of Eq. (18) are plotted in Figure 6 when we consider the values $u_{-5} = -0.3$, $u_{-4} = 0.5$, $u_{-3} = -0.4$, $u_{-2} = 0.3$, $u_{-1} = -0.2$, $u_0 = 0.2$.

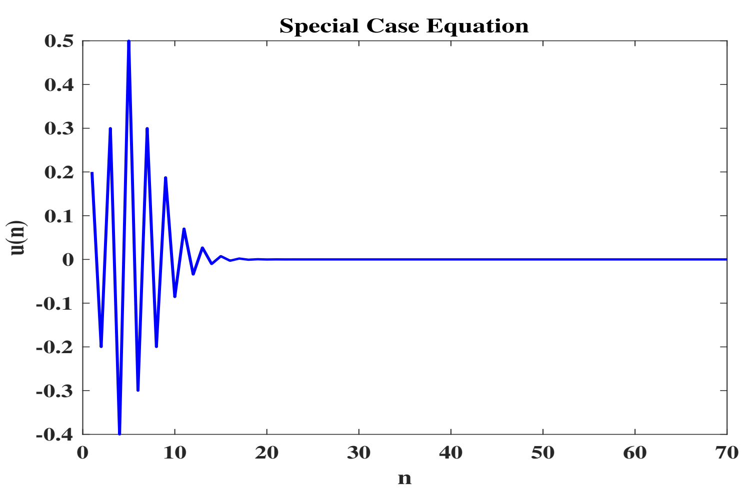

FIGURE 6. The behavior of Eq. (18) at $u_{-5} = -0.3$, $u_{-4} = 0.5$, $u_{-3} = -0.4$, $u_{-2} = 0.3$, $u_{-1} = -0.2$, $u_0 = 0.2$.

13. Conclusion

This work has investigated the local and global stability and periodicity of the solution of Eqs. (1) and (2). The exact solutions of Eqs. (10) and (18) have been also obtained. Theorem (1) shows a simple condition under which the equilibrium point of Eq. (1) is locally asymptotically stable while Theorem (2) illustrates that the equilibrium point is a global attractor if $\delta(\alpha-1) \neq \beta$. Moreover, we have proved that Eq. (1) has no prime period two solutions. Theorems (6) and (7) give simple conditions for the local and global stability of Eq. (2). The exact solutions of Eqs. (10) and (18) exist and have been obtained in various difference relations. Finally, Section 12 has presented some 2D figures to confirm the theoretical results given in this work. For example, Figures (1) and (2) show the local and global stability around the equilibrium point while Figure (3) illustrates the solutions of Eq. (10).

References

[1] S. Elaydi, An Introduction to Difference Equations, 3rd Ed., Springer, USA, 2005.

[2] J. D. Murray, Mathematical Biology: I. An Introduction, 3rd Ed., Springer-Verlag, New York, 2001.

[3] M. B. Almatrafi and M. M. Alzubaidi, Analysis of the Qualitative Behaviour of an Eighth-Order Fractional Difference Equation, Open Journal of Discrete Applied Mathematics, 2, 1, 41-47, 2019.

[4] E. M. Elabbasy, H. El-Metwally and E. M. Elsayed, On the Difference Equation $x_{n+1} = ax_n - (bx_n)/(cx_n - dx_{n-1})$, Advances in difference Equations, Volume 2006, Article ID 82579, 1-10.

[5] H. S. Alayachi, M. S. M. Noorani, A. Q. Khan and M. B. Almatrafi, Analytic Solutions and Stability of Sixth Order Difference Equations, Mathematical Problems in Engineering, Volume 2020, Article ID 1230979, 12 page.

[6] Mohammed B. Almatrafi and Marwa M. Alzubaidi, Qualitative analysis for two fractional difference equations, Nonlinear Engineering, 9, 265–272, 2020.

[7] M. Garić-Demirović, M. Nurkanović and Z. Nurkanović, Stability, Periodicity and Neimark-Sacker Bifurcation of Certain Homogeneous Fractional Difference Equations, International Journal of Difference Equations, 12, 1, 27-53, 2017.

[8] M. Saleh, N. Alkoumi and Aseel Farhat, On the Dynamic of a Rational Difference Equation $x_{n+1} = \alpha + \beta x_n + \gamma x_{n-k}/Bx_n + Cx_{n-k}$, Chaos, Solitons and Fractals, 96, 76–84, 2017.

[9] M. A. El-Moneam and E. M. E. Zayed, Dynamics of the Rational Difference Equation, Information Sciences Letters, 3, 2, 45-53, 2014.

[10] M. B. Almatrafi, Solutions Structures for Some Systems of Fractional Difference Equations, Open Journal of Mathematical Analysis, 3, 1, 51-61, 2019.

[11] M. B. Almatrafi, E.M. Elsayed and Faris Alzahrani, Qualitative Behavior of Two Rational Difference Equations, Fundamental Journal of Mathematics and Applications, 1, 2, 194-204, 2018.

[12] M. B. Almatrafi and E. M. Elsayed, Solutions And Formulae For Some Systems Of Difference Equations, MathLAB Journal , 1, 3, 356-369, 2018.

[13] M. B. Almatrafi, E. M. Elsayed and Faris Alzahrani, Qualitative Behavior of a Quadratic Second-Order Rational Difference Equation, International Journal of Advances in Mathematics, 2019, 1, 1-14, 2019.

[14] F. Belhannache, N. Touafek and R. Abo-zeid, On a Higher-Order Rational Difference Equation, J. Appl. Math. & Informatics, 34, 5-6, 369-382, 2016.

[15] A. Khaliq and Sk. S. Hassan, Dynamics of a Rational Difference Equation $x_{n+1} = ax_n + (\alpha + \beta x_{n-k})/(A + Bx_{n-k})$, International Journal of Advances in Mathematics, 2018, 1, 159-179, 2018.

[16] T. Khyat and M. R. S. Kulenović, The Invariant Curve Caused by Neimark-Sacker Bifurcation of a Perturbed Beverton-Holt Difference Equation, International Journal of Difference Equations, 12, 2, 267-280, 2017.

[17] Y. Kostrov and Z. Kudlak, On a Second-Order Rational Difference Equation with a Quadratic Term, International Journal of Difference Equations, 11, 2, 179-202, 2016.

[18] K. Liu, P. Li, F. Han and W. Zhong, Global Dynamics of Nonlinear Difference Equation $x_{n+1} = A + x_n/x_{n-1}x_{n-2}$, Journal of Computational Analysis and Applications, 24, 6, 1125-1132, 2018.

[19] S. Moranjkić and Z. Nurkanović, Local and Global Dynamics of Certain Second-Order Rational Difference Equations Containing Quadratic Terms, Advances in Dynamical Systems and Applications, 12, 2, 123-157, 2017.

[20] E. M. Elabbasy, H. El-Metawally and E. M. Elsayed, On The Difference Equation $x_{n+1} = (ax_n^2 + bx_{n-1}x_{n-k})/(cx_n^2 + dx_{n-1}x_{n-k})$, Sarajevo Journal of Mathematics, 4, 17, 1-10, 2008.

[21] E. M. Elsayed, Dynamics of Recursive Sequence of Order Two, Kyungpook Math. J., 50, 483-497, 2010.

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