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ON GENERALIZED EULER-MASCHERONI CONSTANTS

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ABSTRACT. In this study, we establish two new representations of the Euler-Mascheroni constant and provide an elementary proof for the classical Euler-Mascheroni constant related to the Riemann zeta function. New representations for the Euler-Mascheroni constant are also derived.

1. INTRODUCTION

[8] introduced a generalized gamma function $\Gamma_k(z)$ for $k \in \mathbb{N}_0$ which relates to the constant γ_k as $\Gamma(z)$ does to γ. The generalized Euler-Mascheroni constant is defined by

$$\gamma_k = \lim_{n\to\infty}\left(-\frac{\ln^{k+1} n}{k+1} + \sum_{j=1}^{n}\frac{\ln^k j}{j}\right), k = 0, 1, 2, ... \tag{1.1}$$

and are coefficients of the Laurent expansion of the Riemann zeta function $\zeta(s)$ about $s = 1$:

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} A_k (s-1)^k, Re(s) \geq 0,$$

where $A_k = \frac{(-1)^k}{k!}\gamma_k$.
The constant, A_k, was first defined by Stieltjes in 1885 and has been studied extensively by other authors. It is worth noting that $\gamma_0 = \gamma$ is the Euler-Mascheroni constant.
[3] observed that the limit for evaluating (1.1) using Euler-Maclaurin summation is when $k = 35$ and thus established an integral representation to compute the first 2000 Euler-Mascheroni constants. A seris expansion for γ_k was derived in [8] as follows:

$$\gamma_k = -\frac{1}{k+1}\ln^{k+1}\left(\frac{3}{2}\right) + (-1)^k k! \sum_{n=1}^{\infty}\frac{1}{4^n(2n+1)} \cdot \sum_{j=1}^{\infty}\frac{(-1)^j s(2n+1,j)}{(k+1-j)!}\zeta^{(k+1-j)}(2n+1), \tag{1.2}$$

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where $k \in \mathbb{N}$.
[7] studied the distribution of a family $\{\gamma(P)\}$ of generalized Euler-Mascheroni constant for finite set of primes P and established a connection between the distribution of $\gamma(P_r) - \exp(-\gamma)$ and the Riemann hypothesis where P_r is the set of the first r primes.
In [10], sharp upper and lower bounds for the generalized Euler-Mascheroni constant were established as well as improvements for previous bounds of the classical Euler-Mascheroni constant. A one parameter generalization of the Euler-mascheroni constant was examined in [11] and various representations for γ derived. [13] presented a generalized Euler-Mascheroni constant function $\gamma(z)$, extented two Euler's zeta function series involving γ to polylogarithm series for $\gamma(z)$ and further generalized Somo's quadratic recurrence constant.

2. Materials and Methods

The Riemann zeta is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, Re(s) > 1 \tag{2.1}$$

$$= \prod_{p} \left(1 - p^{-s}\right)^{-1}, Re(s) > 1 \tag{2.2}$$

where p is a prime number.
The derivatives of the Riemann zeta function is given by

$$\zeta^k(s) = (-1)^k \sum_{n=1}^{\infty} \frac{\ln^k n}{n^s}. \tag{2.3}$$

The digamma function is given by the logarithmic derivative of the gamma function:

$$\psi(z) = \frac{d \ln \Gamma(z)}{dz} = \frac{\Gamma"(z)}{\Gamma(z)}. \tag{2.4}$$

The digamma function is defined by

$$\psi(z+1) + \gamma = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+z}\right), z \in \mathbb{C}, \tag{2.5}$$

where γ is the Euler-Mascheroni constant.
For $|z| < 1$,

$$\ln \Gamma(z+1) = -\gamma z + \sum_{k=2}^{\infty} \frac{(-1)^k \zeta(k)}{k} z^k, \tag{2.6}$$

$$= -\ln(1+z) - (\gamma - 1)z + \sum_{k=2}^{\infty} \frac{(-1)^k (\zeta(k) - 1)}{k} z^k. \tag{2.7}$$

To expand $\ln \Gamma(z+1)$ in a power series about $z = 0$, stirling numbers are utilized. The stirling numbers of the first kind,$s(m,j)$, is defined by the generating function as

$$\ln^j \left(1 + \frac{z}{n}\right) = \sum_{n=j}^{\infty} \frac{j!}{m!} s(m,j) \left(\frac{z}{n}\right)^m, \tag{2.8}$$

where $|z| < 1$.
Alternatively, stirling numbers of the first kind are also defined as

$$\frac{1}{j}\ln^j(1+t) = \sum_{m=j}^{\infty} s(n,j)\frac{t^n}{n!}, \tag{2.9}$$

or

$$\ln^j(1-t) = \sum_{n=j}^{\infty} \frac{(-1)^n}{n!} s(n,j)t^n. \tag{2.10}$$

From the above, we see that $s(n,1) = (-1)^{n-1}(n-1)!$, $s(n,n) = 1$,
$s(n,2) = (-1)^n(n-1)!\sum_{j=1}^{n-1}\frac{1}{j}$ and
$s(n,3) = \frac{1}{2}(-1)^{n+1}\left(H_{n-1}^2 - H_{n-1}^{(2)}\right)$,
where H_{n-1} is the $(n-1)$th harmonic number and $H_{n-1}^{(2)}$ is a harmonic number of order 2.

[2] established a generalized gamma function and some identities as follows:

Definition 2.1. For $z \in \mathbb{C} \setminus \mathbb{Z}^- U\{0\}$ and $k \in \mathbb{N}_0$,

$$\Gamma_k(z) = \lim_{n\to\infty} \frac{\prod_{j=1}^{n} \exp\left(\frac{1}{k+1}\ln^{k+1}\left(1+\frac{1}{j}\right)^z\right)}{\exp\left(\frac{1}{k+1}\ln^{k+1} z\right)\prod_{j=1}^{n}\exp\left(\frac{1}{k+1}\ln^{k+1}\left(1+\frac{z}{j}\right)\right)} \tag{2.11}$$

and a functional equation

$$\Gamma_k(z+1) = \exp\left(\frac{1}{k+1}\ln^{k+1} z\right)\Gamma_k(z). \tag{2.12}$$

The identity was established as

$$\frac{1}{\Gamma_k(z)} = e^{\gamma_k z}\exp\left(\frac{1}{k+1}\ln^{k+1} z\right)\prod_{j=1}^{\infty}\exp\left(\frac{1}{k+1}\ln^{k+1}\left(1+\frac{z}{j}\right)\right)\exp\left(\frac{-z}{j}\ln^k j\right). \tag{2.13}$$

where γ_k is the generalized Euler-Mascheroni constant.
We also find the following identities:

$$\Gamma_k(z)\Gamma_k(1-z) = \frac{1}{t_k(z)} \tag{2.14}$$

and

$$\Gamma_k(1+z)\Gamma_k(1-z) = \frac{1}{\prod_{j=1}^{\infty}\exp\left(\frac{1}{k+1}\ln^{k+1}\left(1-\frac{z^2}{j^2}\right)\right)} \tag{2.15}$$

where

$$t_k(z) = \exp\left(\frac{1}{k+1}\ln^{k+1} z\right)\prod_{j=1}^{\infty}\exp\left(\frac{1}{k+1}\ln^{k+1}\left(1-\frac{z^2}{j^2}\right)\right). \tag{2.16}$$

and established the identity

$$\ln\Gamma_k(z+1) = -\gamma_k z - (-1)^k k! \sum_{m=2}^{\infty}\frac{z^m}{m!}\sum_{n=1}^{m}\frac{s(m,n)}{(k+1-n)!}\zeta^{(k+1-n)}(m) \tag{2.17}$$

for $-1 < z \leq 1$.
If f_n is a sequence in a measurable function space L^+, by the monotone convergence theorem we have (see [9])

$$\int \sum_n f_n = \sum_n f_n \int f_n. \tag{2.18}$$

By Ramanujan ([4]), we have

$$\gamma = 1 - \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(k+1)(2k+1)}. \tag{2.19}$$

Also in [12] we have

$$\gamma = \ln 2 - \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(2k+1)^{2k}}. \tag{2.20}$$

Euler discovered a formula for calculating $\zeta(2k)$ as

$$\zeta(2k) = \frac{(-1)^k (2\pi)^{2k} B_{2k}}{2(2k)!}, \tag{2.21}$$

where $k \in \mathbb{N}$.
[1] also established that

$$\zeta(2k+1) = \frac{(-1)^{1-k}(2\pi)^{2k+1}}{2(2k+1)!} \int_0^1 B_{2k+1}(t) \cot(\pi t) dt, \tag{2.22}$$

where $k \in \mathbb{N}$.
The generalized Clausen function is defined by

$$Cl_{2k+1}(t) = \sum_{n=1}^{\infty} \frac{\cos(nt)}{n^{2k+1}} Cl_{2k+1}(t) \tag{2.23}$$

or

$$Cl_{2k}(t) = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n^2 k}. \tag{2.24}$$

[6] established a formular for $\zeta(2k+1)$ involving the Clausen function as

$$\zeta(2k+1) = \frac{2^{4k+1}}{1-2^{2k}} Cl_{2k+1}\left(\frac{\pi}{2}\right). \tag{2.25}$$

In [2], we have

$$\ln^{k+1}(j+1) - \ln^{k+1} j = \frac{1}{j}(k+1)\ln^k j + \sum_{m=2}^{\infty} \left(\frac{1}{j}\right)^m \frac{1}{m!} \sum_{n=1}^{m} \frac{(k+1)!}{(k+1-n)!} s(m,n) \ln^{k+1-n} j, \tag{2.26}$$

for $|z| < 1$.

3. Results and Discussion

We begin this section by establishing a lemma which will be useful in proving one of the theorems.

Lemma 3.1. *Let* $z = \frac{1}{2}$. *Then*

$$\ln \Gamma_k \left(\frac{3}{2}\right) = \frac{1}{k+1} \ln^{k+1} \frac{1}{2} - \ln t_k \left(\frac{1}{2}\right). \tag{3.1}$$

Proof. Applying logarithm on (2.12), we get

$$\ln \Gamma_k(z+1) = \frac{1}{k+1} \ln^{k+1} z + \ln \Gamma_k(z), \tag{3.2}$$

and substituting $z = \frac{1}{2}$ into it yields

$$\ln \Gamma_k \left(\frac{3}{2}\right) = \frac{1}{k+1} \ln^{k+1} \frac{1}{2} + \ln \Gamma_k \left(\frac{1}{2}\right). \tag{3.3}$$

Also, by substituting $z = \frac{1}{2}$ into (3.17), we obtain

$$\ln \Gamma_k \left(\frac{3}{2}\right) = \frac{1}{k+1} \ln^{k+1} \frac{1}{2} - \ln t_k \left(\frac{1}{2}\right) - \ln \Gamma_k \left(\frac{1}{2}\right). \tag{3.4}$$

By making $\ln \Gamma_k \left(\frac{1}{2}\right)$ the subject from (3.3) and substituting into (3.4), we obtain

$$\ln \Gamma_k \left(\frac{3}{2}\right) = \frac{1}{k+1} \ln^{k+1} \frac{1}{2} - \ln t_k \left(\frac{1}{2}\right), \tag{3.5}$$

which completes the proof. □

Theorem 3.2. *Let* $k = 0, 1, 2, ...$ *Then*

$$\gamma_k = - \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{n=1}^{m} \frac{k!}{(k+1-n)!} s(m,n) \zeta^{(k+1-n)}(m). \tag{3.6}$$

Proof. Taking logarithm on both sides of (2.13), we obtain

$$\gamma_k z = - \ln \Gamma_k(z) - \frac{1}{k+1} \ln^{k+1} z - \sum_{j=1}^{\infty} \frac{1}{k+1} \ln^{k+1} \left(1 + \frac{z}{j}\right) + \sum_{j=1}^{\infty} \frac{z}{j} \ln^k j. \tag{3.7}$$

Taking again logarithm on both sides of (2.11), we obtain

$$\ln \Gamma_k(z) = \sum_{j=1}^{\infty} \frac{z}{k+1} \ln^{k+1} \left(1 + \frac{1}{j}\right) - \frac{1}{k+1} \ln^{k+1} z - \sum_{j=1}^{\infty} \frac{1}{k+1} \ln^{k+1} \left(1 + \frac{z}{j}\right). \tag{3.8}$$

Substituting (3.8) into (3.7) yields

$$\gamma_k = \sum_{j=1}^{\infty} \frac{1}{j} \ln^k j - \sum_{j=1}^{\infty} \frac{1}{k+1} \ln^{k+1} \left(1 + \frac{1}{j}\right). \tag{3.9}$$

By (2.26), we get

$$\gamma_k = \sum_{j=1}^{\infty} \frac{1}{j} \ln^k j - \frac{1}{k+1} \left(\sum_{j=1}^{\infty} \frac{(k+1)}{j} \ln^k j + \sum_{m=2}^{\infty} \left(\frac{1}{j}\right)^m \frac{1}{m!} \sum_{n=1}^{m} \frac{(k+1)k!}{(k+1-n)!} s(m,n) \ln^{k+1-n}(m) \right).$$

Simplifying, we obtain

$$\gamma_k = -\sum_{m=2}^{\infty}\left(\frac{1}{j}\right)^m \frac{1}{m!}\sum_{n=1}^{m}\frac{k!}{(k+1-n)!}s(m,n)\ln^{k+1-n}(m)$$
$$= -\sum_{m=2}^{\infty}\frac{1}{m!}\sum_{n=1}^{m}\frac{k!}{(k+1-n)!}s(m,n)(-1)^{k+1-n}\sum_{m=2}^{\infty}\frac{\ln^{k+1-n}(m)}{j^m}.$$

By (2.3), the proof is complete. □

Remark 3.1. For $k=0$, (3.6) becomes

$$\gamma = -\sum_{m=2}^{\infty}\frac{1}{m!}\sum_{n=1}^{m}\frac{s(m,n)}{(1-n)!}\zeta^{(1-n)}(m),$$
$$= -\sum_{m=2}^{\infty}\frac{(-1)^m(-1)}{m}\zeta(m),$$
$$= \sum_{m=2}^{\infty}\frac{(-1)^m}{m}\zeta(m), \tag{3.10}$$

which is a known result due to Euler.

Remark 3.2. If $k=1$ in (3.6), we obtain

$$\gamma_1 = -\left(\sum_{m=2}^{\infty}\frac{1}{m!}\sum_{n=1}^{m}\frac{s(m,n)}{(2-n)!}\zeta^{(2-n)}(m)\right)$$
$$= -\left(\sum_{m=2}^{\infty}\frac{1}{m!}\left(\frac{s(m,1)}{1!}\zeta^{'}(m)+\frac{s(m,2)}{0!}\zeta(m)\right)\right).$$

By (2.8) yields

$$\gamma_1 = \sum_{m=2}^{\infty}\frac{(-1)^m}{m}\zeta^{'}(m) - \sum_{m=2}^{\infty}\left(\frac{(-1)^m}{m}\zeta(m)\left(\sum_{i=1}^{m-1}\frac{1}{i}\right)\right)$$
$$= \sum_{m=2}^{\infty}\frac{(-1)^m}{m}\zeta^{'}(m) - \gamma\sum_{m=2}^{\infty}\left(\sum_{i=1}^{m-1}\frac{1}{i}\right)$$
$$= \sum_{m=2}^{\infty}\frac{(-1)^m}{m}\zeta^{'}(m) - \gamma\sum_{m=2}^{\infty}H_{m-1}. \tag{3.11}$$

Remark 3.3. If $k=2$ in (3.6), we get

$$\gamma_2 = \sum_{m=2}^{\infty}\frac{(-1)^m}{m}\zeta^{''}(m) + 2\sum_{m=2}^{\infty}\frac{(-1)^m}{m}H_{m-1}\zeta^{'}(m) + \gamma\sum_{m=2}^{\infty}\left(H_{m-1}^2 - H_{m-1}^{(2)}\right), \tag{3.12}$$

where γ is the Euler-Mascheroni constant,H_{n-1} is the $(n-1)$th harmonic number and H_{n-1}^2 is a harmonic number of order 2.

Theorem 3.3. *Let* $k=0,1,2,...$ *Then*

$$\gamma_k = \frac{1}{k+1}\ln^{k+1}2 - (-1)^k k!\sum_{m=2}^{\infty}\frac{1}{4^m(2m+1)!}\sum_{n=1}^{m}\frac{s(2m+1,n)}{(k+1-n)!}\zeta^{(k+1-n)}(2m+1). \tag{3.13}$$

Proof. Replacing z with $-z$ in (2.17), we get

$$\ln \Gamma_k(1-z) = \gamma_k z + (-1)^k k! \sum_{m=2}^{\infty} \frac{z^m}{m!} \sum_{n=1}^{m} \frac{s(m,n)}{(k+1-n)!} \zeta^{(k+1-n)}(m). \tag{3.14}$$

Combining (2.14) and (2.15), we obtain

$$\Gamma_k(1+z)\Gamma_k(1-z) = \frac{\exp\left(\frac{1}{k+1} \ln^{k+1} z\right)}{t_k(z)}. \tag{3.15}$$

Applying logarithm to (3.15), we get

$$\ln \Gamma_k(1+z) + \ln \Gamma_k(1-z) = \frac{1}{k+1} \ln^{k+1} z - \ln t_k(z), \tag{3.16}$$

which further gives

$$\ln \Gamma_k(1+z) = \frac{1}{k+1} \ln^{k+1} z - \ln t_k(z) - \ln \Gamma_k(1-z). \tag{3.17}$$

substituting (3.14) into (3.17) yields

$$\ln \Gamma_k(z+1) = \frac{1}{k+1} \ln^{k+1} z - \ln t_k(z) - \gamma_k z - (-1)^k k! \sum_{m=2}^{\infty} \frac{z^m}{m!} \sum_{n=1}^{m} \frac{s(m,n)}{(k+1-n)!} \zeta^{(k+1-n)}(m). \tag{3.18}$$

Adding (2.17) and (3.18) yields

$$\ln \Gamma_k(z+1) = \frac{1}{2(k+1)} \ln^{k+1} z - \frac{1}{2} \ln t_k(z) - \gamma_k z - (-1)^k k! \sum_{m=2}^{\infty} \frac{z^m}{m!} \sum_{n=1}^{m} \frac{s(m,n)}{(k+1-n)!} \zeta^{(k+1-n)}(m). \tag{3.19}$$

Thus,

$$\ln \Gamma_k(z+1) = \frac{1}{2(k+1)} \ln^{k+1} z - \frac{1}{2} \ln t_k(z) - \gamma_k z - (-1)^k k! \sum_{m=1}^{\infty} \frac{z^{2m+1}}{(2m+1)!} \sum_{n=1}^{m} \frac{s(2m+1,n)}{(k+1-n)!} \zeta^{(k+1-n)}(2m+1). \tag{3.20}$$

Substituting $z = \frac{1}{2}$ into (3.20) gives

$$\begin{aligned} \ln \Gamma_k\left(\frac{3}{2}\right) &= \frac{1}{2(k+1)} \ln^{k+1} \frac{1}{2} - \frac{1}{2} \ln t_k \frac{1}{2} \\ &- \frac{1}{2}\gamma_k - (-1)^k k! \sum_{m=1}^{\infty} \frac{1}{2^{2m+1}(2m+1)!} \sum_{n=1}^{m} \frac{s(2m+1,n)}{(k+1-n)!} \zeta^{(k+1-n)}(2m+1). \end{aligned} \tag{3.21}$$

By Lemma 3.1 we have

$$\gamma_k = \frac{-2}{k+1} \ln^{k+1} \frac{1}{2} + \frac{1}{k+1} \ln^{k+1} \frac{1}{2} - (-1)^k k! 2 \sum_{m=1}^{\infty} \frac{1}{2^{2m+1}(2m+1)!} \sum_{n=1}^{m} \frac{s(2m+1,n)}{(k+1-n)!} \zeta^{(k+1-n)}(2m+1). \tag{3.22}$$

Thus,

$$\gamma_k = \frac{1}{k+1}\ln^{k+1}2 - (-1)^k k! \sum_{m=1}^{\infty} \frac{1}{4^m(2m+1)!} \sum_{n=1}^{m} \frac{s(2m+1,n)}{(k+1-n)!} \zeta^{(k+1-n)}(2m+1), \tag{3.23}$$

which completes the proof. □

Remark 3.4.

1 For $k = 0$ in (3.13), we obtain

$$\gamma = \ln 2 - \sum_{m=1}^{\infty} \frac{1}{4^m(2m+1)} \zeta(2m+1). \tag{3.24}$$

2 For $k = 1$ in (3.13), we get

$$\gamma_1 = \frac{1}{2}\ln^2 2 - \sum_{m=1}^{\infty} \frac{1}{2^{2m-1}(2m+1)} \left(\zeta^{'}(2m+1) - H_{2m}\zeta(2m+1) \right). \tag{3.25}$$

Now, we restate Euler-Mascheroni constant (3.10) and provide a proof.

Theorem 3.4. *(Euler-Mascheroni)*

$$\gamma = \sum_{m=2}^{\infty} \frac{(-1)^m}{m} \zeta(m), \tag{3.26}$$

where $\zeta(m)$ is the Riemann zeta function.

Proof. By integrating (2.5) and applying (2.18), we get

$$\begin{aligned} \ln\Gamma(z+1) + \gamma z &= \sum_{m=1}^{\infty} \left(\frac{z}{m} - \ln(m+z) + \ln(m) \right) \qquad (3.27) \\ &= \sum_{m=1}^{\infty} \left(\frac{z}{m} - \ln\left(1 + \frac{z}{m}\right) \right). \end{aligned}$$

By (2.26) and (2.8), we obtain

$$\ln\Gamma(z+1) + \gamma z = \sum_{m=2}^{\infty} \frac{(-1)^m z^m}{m} \zeta(m). \tag{3.28}$$

and letting $z = 1$ completes the proof. □

We further give new series representations for the Euler-Mascheroni constant involving Bernoulli numbers, Bernoulli polynomials and generalized clausen functions.

Theorem 3.5.

$$\gamma = \sum_{k=1}^{\infty} \frac{(-1)^k (2\pi)^{2k} B_{2k}}{4k(2k)!} - \sum_{k=1}^{\infty} \frac{(-1)^{1-k}(2\pi)^{2k+1}}{2(2k+1)(2k+1)!} \int_0^1 B_{2k+1}(t)\cot(\pi t)dt, \tag{3.29}$$

where B_{2k} and $B_{2k+1}(t)$ are Bernoulli numbers and polynomials respectively.

Proof. For even values of m in (3.10), we

$$\gamma = \sum_{m=2}^{\infty} \frac{\zeta(m)}{m} - \sum_{m=2}^{\infty} \frac{\zeta(m+1)}{m+1}. \tag{3.30}$$

If $m = 2k$ for $k \in \mathbb{N}$, we get

$$\gamma = \sum_{k=1}^{\infty} \frac{\zeta(2k)}{2k} - \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{2k+1}. \tag{3.31}$$

Substituting (2.21) and (2.22) into (3.31) completes the proof. □

The remark below is a deduction on the Euler-Mascheroni constant associated with the generalized Clausen function.

Remark 3.5. Substituting (2.25) into (2.19) and (2.20) yields

$$\gamma = 1 - \sum_{k=1}^{\infty} \frac{2^{k+1}}{(1-2^{2k})(k+1)(2k+1)} Cl_{2k+1}\left(\frac{\pi}{2}\right), \tag{3.32}$$

$$= \ln 2 - \sum_{k=1}^{\infty} \frac{2^{k+1}}{(1-2^{2k})(2k+1)} Cl_{2k+1}\left(\frac{\pi}{2}\right), \tag{3.33}$$

where $Cl_{2k+1}(t)$ is a generalized Clausen function.

4. Conclusions

New generalized Euler-Mascheroni constants have been established. For $k = 0$ a new seies representation for the classical Euler-Mascheroni constant is established. In addition, we derived expressions for γ involving Bernoulli constants, Bernoulli polynomials and the Clausen function.

References

[1] G. Abe-I-kpeng, M. M. Iddrisu and K. Nantomah, Some identities and inequalities related to the Riemann zeta function, Moroccan Journal of Pure and Applied Analysis, 2, 179–185, 2019

[2] G. Abe-I-kpeng, M. M. Iddrisu and K. Nantomah, On a generalized gamma function and its properties, Journal of Mathematical and Computational Science, 11(5), 5916–5930, 2021

[3] O. R. Ainsworth and L. W. Howell, An integral representation of the generalized Euler-Mascheroni constant, NASA Technical Report 2456,1985.

[4] R. P. Brent, Ramanujan and Euler's constant, American Mathematical Society, 1985.

[5] J. Bonnar, The gamma function, Treasure Trove of Mathematics, 2004.

[6] D. F. Connon, Some infinite series involving the Riemann zeta function, International Journal of Mathematics and Computer Science, 7(1), 11–83, 2012.

[7] H. G. Diamond and K. Ford, Generalized Euler's constants, Cambridge Philosophical Society, 145(1), 1–14, 2008.

[8] K. Dilcher, On generalized gamma functions related to the Laurent coefficients of the Riemann zeta function, Aequationes Mathematicae, 48, 55–85, 1994.

[9] G. B. Folland, A Guide to Advanced Real Analysis, The Mathematical Association of America, 2004.

[10] T. Huang, B. Han, X. Ma, and Y. Chu, Optimal bounds for the generalized Euler-Mascheroni constant, Journal of Inequalities and Applications, 2018.

[11] S. Kaczkowski, On a generalization of Euler's constant, Surveys in Mathematics and its Applications, 10, 259–274, 2021.

[12] J. S. Lagarias, Euler's constant: work and modern developments, Bulletin of the American Mathematical Society, 50(4), 527–628, 2013.

[13] J. Sondow and P. Hadjicostas, The generalized Euler constant function $\gamma(z)$ and a generalization of Somo's quadratic recurrence constant,Journal of Mathematical Analysis and Applications, 332, 292–314, 2007.

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